Chapter 5

Historical formation and student understanding of mathematics

Luis Radford

with Maria G. Bartolini Bussi, Otto Bekken, Paolo Boero, Jean-Luc Dorier, Victor Katz, Leo Rogers, Anna Sierpinska, Carlos Vasco

Abstract: *The use of history of mathematies in the teaching and learning of mathematics requires didactical reflection. A crucial area to explore and analyse is the relation between how students achieve understanding in mathematics and the historical construction of mathematical thinking.*

5.1 Introduction

Luis Radford

The history of mathematics may be a useful resource for understanding the processes of formation of mathematical thinking, and for exploring the way in which such understanding can be used in the design of classroom activities.

It is in this spirit that in the last decades some mathematics educators have had recourse to the history of mathematics. However, such a task demands that mathematics educators be equipped with a clear and rich theoretical framework accounting for the general formation of mathematical knowledge. In addition to offering a clear epistemological stance, the theoretical framework has to ensure a fruitful articulation of the historical and psychological domains as well as to support a coherent and fecund methodology (see figure 5.1).

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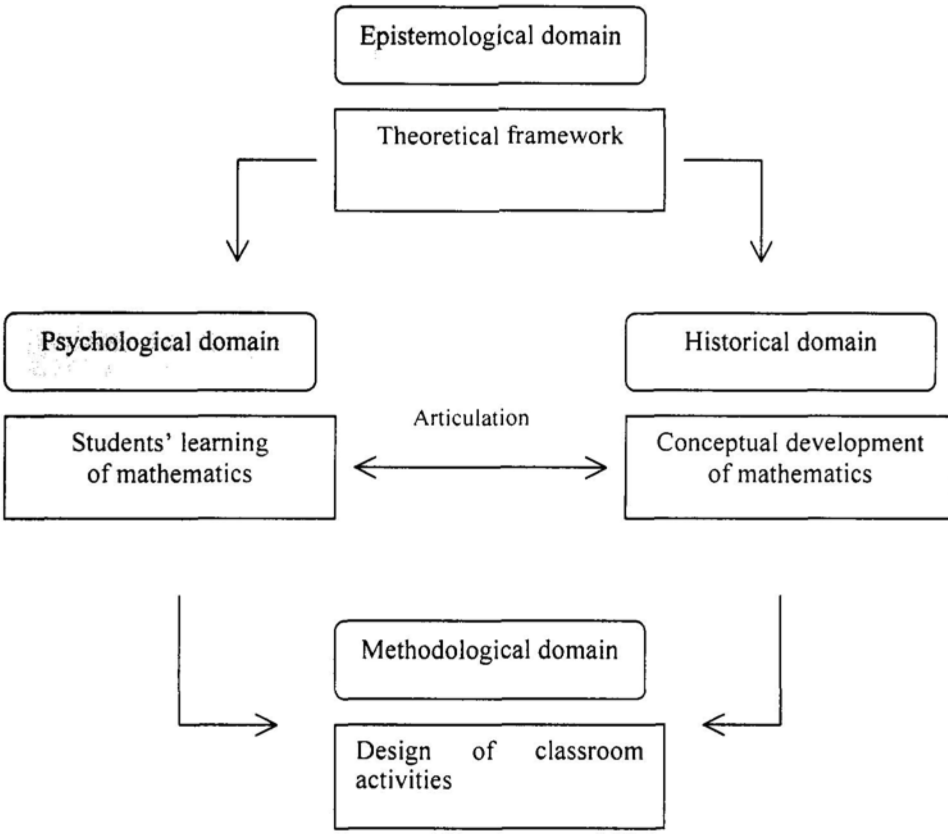

Figure 5.1: Theoretical framework allowing an articulation between the account of students' learning of mathematics and the account of the historical development of mathematics, and supporting a methodology for the design of historically based classroom activities.

The lack of such a suitable framework often leads to oversimplifying views about the way in which mathematical concepts have developed historically (see 'historical domain' in figure 5.1). Indeed, even though new historiographic paradigms have emerged in the past few years (see Gillies 1992, Høyrup 1995, Lizcano 1993, among others), the history of mathematics is all too often read in an *unhistorical* way. That is, narratives are presented which implicitly assume that past mathematicians were *essentially* dealing with our modern concepts, but just did not have our modern notations at their disposal. Reading history like this, in what might be called a *teleological* way, the historian seems to assume, in effect, that there was a course that the historical developments just had to take. In making this assumption, a *normative* dimension is introduced into the account, through which the historian endows other cultures and mathematicians of other epochs with rationalities and conceptualisations that were completely alien to them.

Besides this problem of conveniently framing the historical conceptual development of mathematics, the link between historical developments in mathematical thinking and the students' learning of mathematics (see horizontal arrow in Figure 5.1) has often been done in terms of a naïve psychological version of biological *recapitulationism.* Briefly stated, biological recapitulationism, an idea introduced at the end of the last century, following Darwin's writings on the evolution of species, posits that the development of the individual *(ontogenesis)* recapitulates the development of mankind *(phylogenesis).* The German biologist Ernst Haeckel seems to have been the first to transfer this 'biological' law to the psychological domain. He said that "the psychic development of the child is but a brief repetition of the phylogenetic evolution" (quoted by Mengal 1993, 94).

The concept of *genetic development* was partly elaborated in the 1970s, in the work of the psychologists Jean Piaget and Rolando Garcia, as a reaction to this simplistic psychological version of recapitulationism. In their book *Psychogenesis and the history of science* (1989—a book that has had a significant influence on mathematics educators interested in the use of the history of mathematics — they presented a different perspective. They argued that we should try to understand the problem of knowledge in terms of the intellectual instruments and mechanisms allowing its acquisition. According to them, the first of those mechanisms is a general process which accounts for the individual's assimilation and integration of what is new on the basis of his or her previous knowledge. (This is a view that runs against the positivist view that knowledge simply accumulates in a straightforward way.) But then there is an apparent dilemma. On the one hand, in gaining knowledge the individual is seen as selecting, transforming, adapting and incorporating the elements provided by the external world to his or her own cognitive structures (Piaget and Garcia 1989, 246); while, on the other hand, there can be no assimilation of 'pure' objects divorced from their context, insofar as objects always have a social signification (p. 247). This paradox led Piaget and Garcia to discuss the influence of the social environment on the evolution of knowledge in the individual.

Pursuing this further led Piaget and Garcia to ask whether two different social environments could lead to two different psychogenetic developments. Since the works of Bachelard, Kuhn and Feyerabend had stressed the significant role played by social settings in the formation of conceptual systems and theoretical knowledge, Piaget and Garcia's question was hardly inevitable. The question has become even more urgent nowadays in the light of recent cognitive, anthropological and sociological discussions about the mind. In an interview given in the mid 1970s, when their book was still in preparation, Piaget clearly stated that one of the problems that led him to write the book was to investigate if there is only one possible line of evolution in the development of knowledge or if there are many, and he replied (Bringuier 1980, 100):

> Garcia, who is quite familiar with Chinese science, thinks that they have travelled a route very different from our own. So I decided to see whether it is possible to imagine a psychogenesis different from our own, which would be that of the Chinese child during the greatest period of Chinese science, and I think that it is possible.

However, in their book the problem was dealt with in terms of the difference between the individual's acquisition of knowledge and the 'epistemic paradigm' in which the individual finds him or herself subsumed. By epistemic paradigm they meant "a conception [of science] that has become part of accepted knowledge and is transmitted along with it, as naturally as oral or written language is transmitted from one generation to the next" (Piaget and Garcia 1989, 252). This concept was explicitly presented as an epistemological alternative to Kuhn's concept of paradigm and—in particular—its socially imposed norms. Thus, the 'failure' of Greek and Mediaeval thinkers to conceive the principle of inertia in physics, and the success of the Chinese in conceiving such a principle—which they apparently considered "as obvious as the fact that a cow is not a horse" (p. 253)—was explained in terms of the different epistemic paradigms in which Greek and Chinese science were couched (p. 254). Although the individual was seen as being in dialectical interaction with the object of knowledge, and it was recognised that society provides objects with specific meanings, Piaget and Garcia traced a clear frontier dividing the social and the individual. For them, a distinction must be made between mechanisms to acquire knowledge and the way in which objects are conceived by the subject. In a concise and clear phrase, they said: "Society can modify the latter, but not the former." (p. 267).

In their approach to the relations between ontogenesis and phylogenesis, Piaget and Garcia did not seek for a parallelism of contents between historical and psychogenetical developments but for the mechanisms of passage from one historical period to the following. They tried to show that those mechanisms are analogous to those of the passage from one psychogenetic stage to the next. In addition to the assimilation mechanism previously mentioned, they identified a second mechanism of passage. This was described as a process that leads from the *intra-object,* or analysis of objects, to the *inter-object,* or analysis of the transformations and relations of objects, to the *trans-object,* or construction of structures. The two mechanisms were considered as invariable and omnipresent, not only in time but in space too. That is, we do not have to specify what they are in a certain geographical space at a particular time since it is considered that they do not change from place to place and from time to time.

The Russian psychologist Lev Vygotsky was also concerned with the relationship between ontogenesis and phylogenesis, but—starting from a distinct conception of the mind—took a different approach. Instead of posing the problem in terms of some invariable mechanisms of acquisition of knowledge, he felt that thinking developed as the result of two lines or processes of development: a biological (or natural) process and a historical (or cultural) one. One of his fundemental differences with Piaget and Garcia's approach lies in the epistemological role of culture. For Piaget and Garcia, culture cannot modify the essential instruments of knowledge acquisition, for they saw these instruments as originating in the biological realm of the individual (Piaget and Garcia 1989, 184). In Vygotsky's approach, though, culture not only provides the specific forms of scientific concepts and methods of scientific inquiry but overall modifies the activity of mental functions through the use of tools -of whatever type, be they artefacts used to write as clay tablets in ancient Mesopotamia, or computers in contemporary societies, or intellectual artefacts such as words, language, or inner speech (Vygotsky 1994).

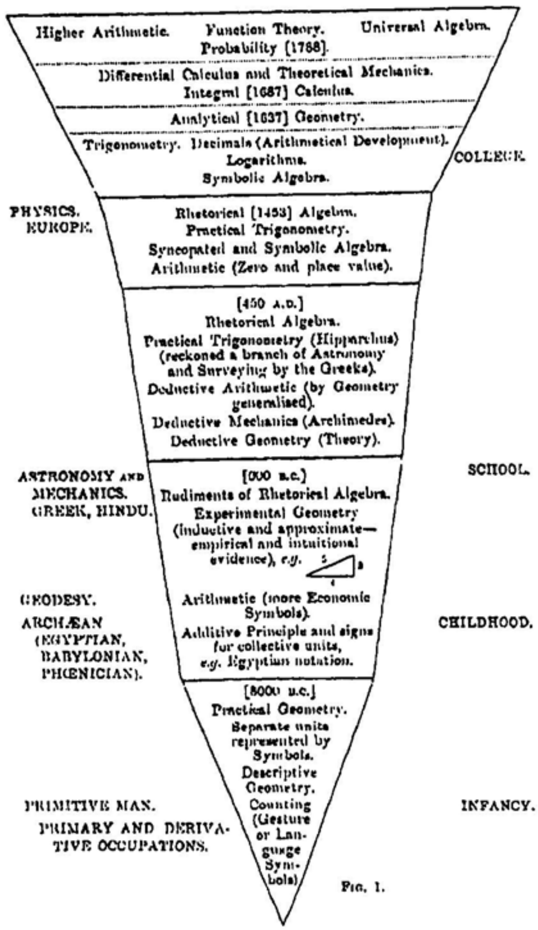

Figure 5.2: Comparison ofphylogenesis and ontogenesis have been made since the late 19th century, as seen in this 'Diagram of the development of mathematical experience in the race and in the individual' by Miss Barvell in the Mathematical Gazette *of1913.*

It is this cultural line of development in Vygotsky's account that renders any recapitulationism impossible. For instance, in one of the many passages in which he dealt with this topic, he discusses the development of higher mental functions in history and in the child, and goes on to say that "we do not mean to say that ontogenesis in any form or degree repeats or produces phylogenesis or is its parallel." (Vygotsky 1997, 19). One ofthe reasons is the variability introduced by the sociohistorical conditions, which are different in each period of the history. In this view, ontogenesis runs, so to speak, underpinned by biological phylogenesis *and* the sociohistorical conditions where ontogenesis takes place (pp. 19-20);

> The growing of the normal child into civilisation usually represents a single merging with the process of his organic maturation. Both planes of development -the natural and the cultural-coincide and merge. Both orders of changes mutually penetrate each other and form in essence a single order of social-biological formation of child personality.

The examples of Piaget and Garcia, and of Vygotsky, uncover the complexity of the problem of the relationship between phylogenesis and ontogenesis and the importance of working towards a clear theoretical framework.

This chapter summarises different ways in which the history of mathematics contributes to a better understanding of the student processes of learning mathematics and the design and analysis of teaching activities. In reference to the different domains mentioned in Figure 5.1, the sections presented in this chapter may be described as follows. In section 5.2, Victor Katz and his colleagues sketch some case studies dealing with the relations between the historical and psychological domains. More specifically, they give some examples from the history of mathematics where we see mathematicians struggling with problems that appear to present difficulties analogous to those faced by our students today, when they tackle the contemporary version of those problems in their school curriculum. They emphasise the importance of teachers having some knowledge of the history of mathematics, as it may help them to help their students overcome some important difficulties which arise in the mathematics classroom.

In section 5.3, Maria Bartolini Bussi and Anna Sierpinska present some sophisticated methodological approaches recently developed by mathematics educators. In these approaches, one of the goals is to study the historical conditions which made possible the emergence of a certain type or domain of mathematical knowledge (historical domain) and to adapt and integrate those conditions into the design of classroom activities (methodological domain) and the analysis of students' forms of mathematical thinking (psychological domain).

In section 5.4, Luis Radford, Paolo Boero and Carlos Vasco focus on the epistemological assumptions (epistemological domain) which underline three current teaching/research approaches using the history of mathematics: Brousseau's epistemological obstacles, Radford's socio-cultural perspective and Boero's Voices and Echoes Games. They make it evident that the interpretation of the conceptual development of mathematics (historical domain), and the investigation of the psychological processes underlying the learning of mathematics (psychological domain), as well as the linking of these phenomena with the design of classroom activities (methodological domain), will all depend upon the chosen framework.

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5.2 The role of historical analysis in predicting and interpreting students' difficulties in mathematics

Victor Katz, Jean-Luc Dorier, Otto Bekken and Anna Sierpinska

As noted in the introduction of this chapter, Piaget and Garcia (1989, 27-28) claim that

> the advances made in the course of the history of scientific thought from one period to the next, do not, except in rare instances, follow each other in random fashion, but can be seriated, as in psychogenesis, in the form of sequential 'stages.'... [and] the mechanisms mediating transitions from one historical period to the next are analogous to those mediating the transition from one psychogenetic stage to the next.

Anna Sfard has noted (private communication) that this analogy "is particularly striking at those special junctures where in order to assimilate or create or learn a new concept, the already constructed knowledge has to undergo a complete reorganisation, and the whole epistemological foundation has to be reconstructed as well." The claim of Piaget, which is supported by Sfard, needs of course to be supported by research into students' shifts in understanding mathematical difficulties. This research has been done in several specific cases of student difficulty, where there was a historical reason to believe that such a difficulty might exist. We summarise the results of some of these research studies below.

A first example of this phenomenon of students finding difficulties analogous to those of past mathematicians is familiar to most calculus teachers: the concept of a 'limit' in analysis. Teachers are aware that it is generally difficult to explain the formal notion of limit at the beginning of an elementary calculus class, where it 'logically' belongs. Students certainly 'know' that the limit of $2x+3$ as x approaches 7 is 17, but resist trying to prove such an obvious result using epsilons and deltas. They cannot comprehend why such a proof would be necessary.

To set this in context, historians are aware that the formal idea of a limit was not developed until a century and a half after the basic concepts of the calculus were invented by Newton and Leibniz. During that period, from about 1670 to 1820, many mathematicians used the concept of limit with great understanding — and could calculate limits in many important cases — but they did not have a definition which would enable the statement "the limit of $f(x)$ as x approaches a is L" to be proved with the rigor of classical Greek mathematics. Analysing the historical conditions and reasons why the shift from an intuitive to a formal understanding of limits took mathematicians so long to accomplish gives us valuable information which can help us both predict and interpret our students' difficulties in accomplishing this shift in a few short weeks (see Cornu 1991, Sierpinska 1988, Bum 1993).

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Besides the difficulty related to the passage from an intuitive to a rigorous understanding and use of the concept of limit, other difficulties arise from this concept in the comprehension of curvilinear area, tangent line and instantaneous flow. An intensive historical search on the development of calculus allowed M. Schneider (1988) to demonstrate that these difficulties surface from the same epistemological obstacle: the absence of separation, in the mind of students, between mathematics and an illusory 'sensible' world of magnitudes. This investigation provided Schneider with a research methodology to render such learning difficulties apparent: for example, the reactions of students in learning about Cavalieri's principles, indivisibles and related paradoxes reveal mental shifts in meaning from the world of magnitudes to their measures.

Jean-Luc Dorier (1998), in his studies of how best to teach the concepts of linear dependence and linear independence in linear algebra, has noted that although students entering university often have certain conceptions of these notions in concrete situations, they have difficulty in understanding the connection of the formal definition with these earlier situations. A historical analysis of the development of these concepts provides help in understanding the students' difficulties.

The twin concepts of linear dependence and independence emerged historically in the context of linear equations and, in particular, in Euler's analysis of Cramer's paradox dealing with the number of intersection points of two algebraic curves. Euler found that the paradox was based on the 'fact' that *n* linear equations determine exactly n unknown values, but realised that this latter statement is not always true. He discussed several examples in which systems of *n* equations in *n* unknowns do not have a single *n*-fold solution and realised that in certain cases the actual constraints imposed on the unknowns by the equations are fewer than *n*. That is, Euler stated that certain of the equations are "contained" in the others; this is his notion of what we can call inclusive dependence. After Euler's work, many mathematicians considered this problem of dependence and tried to determine conditions on the determinant of a dependent system which would show the nature of the set of solutions. But it was not until 1875 that Georg Frobenius pointed out the similarity of dependence of a set of equations to dependence of a set of n-tuples. He could then give a formal definition of the concept of 'linear dependence' and show how the notion of 'rank' of a system enabled one to determine the dimension of the set of solutions.

The teaching experiment reported by Dorier, based on a historical analysis of the development of the concept of rank, was designed to help the students understand the power of linear dependence as a formal and unifying concept. Indeed, from their secondary school practice of solving equations, students entering university usually have an Eulerian 'inclusive dependence' idea of equations. But at the university level, it is necessary for the students to move to the stage where they understand the formal concept of dependence in a global context. That is, they need to understand that the equations, and not just *n*-tuples, must be regarded as objects in their own right and that there needs to be a definition of linear dependence which applies to both of these cases, as well as in even more general contexts. Thus it was necessary to devise a teaching strategy to meet these needs.

On a more elementary level, students often have trouble making the shift from solving concrete problems using words and numbers to the more abstract problem of using letters to designate unknown quantities. Again, we know that, historically, it was a difficult conceptual switch. In order to help students understand the role of letters as representing unknowns, Radford and Grenier (1996a, 1996b) designed a teaching sequence in which students were asked to solve some word problems using manipulatives. These manipulatives were conceived in such a way that the unknown quantity was modelled by a hidden number of candies in a bag or a hidden number of hockey cards in an envelope, and so On. The teaching sequence was structured to allow the students to master two important rules of Islamic algebra, those of *al-muqabala* and *al-jabr*. In the second step of the teaching sequence, instead of using manipulatives, the students had to make drawings (e.g. of a bag containing an unknown number of candies) and, in the third step, the students had to use letters instead of drawings. The teaching sequence was inspired by a historical analysis of medieval Italian algebra (Radford 1995, 1997), in particular by an idea of the fourteenth century mathematician Antonio de Mazzinghi, who explained the concept of unknown as a 'hidden' quantity.

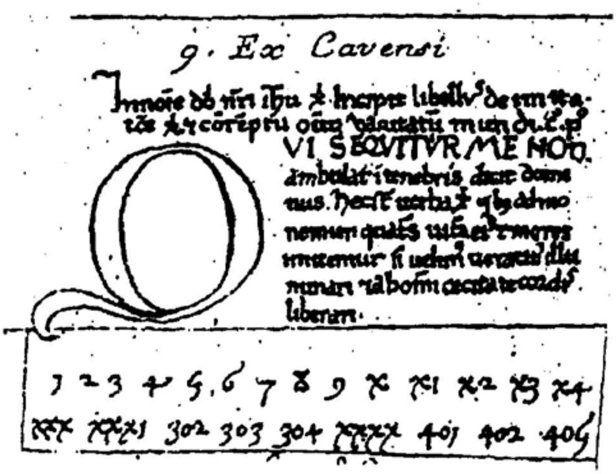

Figure 5.3: Not only 'hidden quantities' are hard to understand. The Hindu-Arabic numerals themselves were difficultfor early European users, as this medieval Italian manuscript testifies. The scribe has rendered as "xxx xxx1 302 303 . . . " what we would write as "30 31 32 33". Such a text helps today's teachers to appreciate how difficult it isfor pupils to learn positional notation.

Anna Sfard (1995) found furthermore that even if high school students could solve linear equations or systems of linear equations with numerical coefficients, it was still difficult for them to make the jump to solving systems with literal coefficients. She notes that at first she was "quite insensitive to the huge conceptual difference between equations with numerical coefficients and equations with parameters." And it took several weeks of hard work before the students could cope with such equations in a reasonable manner. Sfard found that colleagues had encountered similar difficulties. Again, a historical analysis shows that this difficulty is not surprising. Even though by the late medieval period, letters and other abbreviations were being used in algebra to designate unknowns and their powers, the rules for solving equations were always stated in terms of concrete examples.

Thus one could solve $x^2 + 10x = 39$, but not $x^2 + bx = c$. It was François Viète in the late sixteenth century who first introduced letters to designate known values (parameters) and in this way brought a great conceptual change to algebra. It was Viète's work that enabled formulas to be written to solve quadratic and cubic equations, for example, and that led, in general, to structural manipulations in algebra rather than purely operational ones. The historical difficulties in this shift from numerical to purely symbolic algebra again leads us to believe that teachers must be aware of the conceptual difficulties their students may have in making the same shift.

Lisa Hefendehl-Hebeker (1991) analysed the always difficult task of helping students understand the meaning of a negative number, and the reasons for the rules governing operations with these numbers. Negative numbers have, of course, been used for two millennia in China, but mathematicians in the West have always been suspicious of them, even though the rules for operation on them were known by the sixteenth century. Even as late as the nineteenth century, there were some English mathematicians who tried to reformulate algebra without the use of negative numbers, because they believed that they were nonsensical. The question, in fact, became whether negative numbers were 'quantities' and then what it meant for a 'quantity' to be less than zero. There were, of course, numerous attempts throughout the centuries to justify negative numbers, either by using them to model a particular idea (debt, for example) or by deriving the rules of operation by arguments based on the "principle of permanence of equivalent forms" (Peacock 1830), in particular the distributive and associative laws. Hefendehl-Hebeker shows in her article how modern students' confusions about these laws are mirrored in confusions of such authors as Stendhal and d'Alembert in the 18th century. A teacher would do well to study these 'confusions' to see why his or her own students could be confused. But Hefendehl-Hebeker also notes that Hermann Hankel in the mid-19th century advocated a change in point of view by looking at negatives as an extension of the number system rather than as quantities in their own right. That is, he urged that these numbers be introduced in a purely formal manner, without worrying about what kind of quantity they represent. Again, this history shows how one might try to introduce and justify negative numbers in the classroom.

Another set of numbers which often causes difficulties for students is the complex numbers. At one time in school they are told that negative numbers do not have square roots, and later they are told that in fact they do have square roots. Why have the rules changed? A historical analysis here shows again that there was a long period of development between the first discovery of complex numbers by Cardano and Bombelli in their studies of solutions of cubic equations in the fifteenth century and the general acceptance of these numbers into mathematics in the nineteenth. As in the case of negatives, it took centuries for mathematicians to give up the idea that 'number' must represent the measure of a quantity. The final acceptance of these numbers came only through their geometric interpretation, that is, on their modelling in a well-understood area of mathematics. Again, many textbooks today seem to violate this historical analysis by simply defining the square root of -1 by fiat, without any motivation whatsoever.

Non-Euclidean geometry was developed by three mathematicians early in the nineteenth century. Carl Friedrich Gauss, who developed it first, declined to publish anything on this topic, because he did not want to deal with the controversies he was sure would erupt. But two less famous mathematicians, Janos Bolyai in Hungary and Nikolai Lobachevsky in Russia, both published their studies in this field around 1830. Nevertheless, it proved very difficult for mathematicians to give up the very strong conviction that geometry describes a unique reality and, as such, can not admit a plurality of axiom systems, It was not until several mathematicians showed how non-Euclidean geometry could be modelled in Euclidean geometry that the mathematical community began to accept the validity of non-Euclidean geometry. So again, we should not be surprised when there is difficulty for students to understand that Euclidean geometry may not in fact be the 'best' geometry to describe the space in which we live.

A final common student difficulty involves the transition to abstraction. As a typical example, many instances of what today are called groups were known in the first eight decades of the nineteenth century — and some were known even earlier. Yet it was not until 1882 that the first complete formal definition of this abstract concept was given. Nevertheless, many current textbooks in abstract algebra begin by giving a formal definition of a group before the student has experienced many of these examples. It is not surprising that students have difficulties making the leap to abstraction; too little attention has been paid to the necessary steps that historically preceded this leap.

As these examples demonstrate — and there are numerous others — a teacher who is knowledgeable in the history of mathematics will anticipate student difficulties in areas where, historically, much work was needed to overcome significant difficulties. Thus the teacher can be prepared with appropriate teaching strategies for these situations, ones which may well be in accord with the historical developments and which will help the students overcome these obstacles to understanding. And as some of the research results in this area demonstrate, these strategies may well be effective. Yet the knowledge of history of mathematics is not sufficient to develop teaching strategies; if the analysis of historical conditions of the emergence of a concept is an important source of information to predict and analyse students' difficulties, teachers still must take into account the reality of teaching at a certain level with a certain type of student. There is no automatic transfer from history to teaching. First, the knowledge of history must be as complete as possible, involving primary sources whenever feasible. Second, there must exist a preliminary didactical investigation about students' difficulties. Finally, the confrontation of the historical and didactical situations must be made with great care, taking into account the conditions and constraints of the two different environments, the historical and the classroom.

Such work needs competence both in history and in mathematics education research and shows interesting possible interactions between these two fields for the future.

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5.3 The relevance of historical studies in designing and analysing classroom activities

Maria G. Bartolini Bussi and Anna Sierpinska

With contributions by Paolo Boero, Jean Luc Dorier, Ernesto Rottoli, Maggy Schneider, and Carlos Vasco

When a mathematics educator draws on the history of the domain in designing activities for the students he or she may be looking for facts: Who were the authors of that particular piece of mathematics? When did they live? What were their lives?

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By introducing historical anecdotes in his or her classes he or she may increase the students' motivation to learn mathematics. But a historical study may have other goals as well: looking for geneses of mathematical ideas or contexts of emergence of mathematical thinking, in the aim of defining conditions which have to be satisfied in order for the students to develop these ideas and thinking in their own minds.

5.3.1 Bringing historical texts into the classroom: the 'voices and echoes' games

For example, Boero *et al.* (1997, 1998) concerned themselves with the conditions of emergence of theoretical knowledge. Mathematical thinking is theoretical par excellence, and without developing this special attitude of mind in the students there is less opportunity for deepening their understanding of mathematics. A historico-epistemological analysis was, for these authors, a basis for an analytical definition of theoretical knowledge which included parameters such as organisation, coherence and systematic character, the role played by definitions and proofs, the speech genre characteristic of theoretical discourse, and the ways of viewing the objects of the theory. This definition became subsequently a basis for a didactic theory: indeed, Boero et al. have designed and implemented an innovative educational methodology in the classroom called the 'voices and echoes game', which draws on the Vygotskian distinction between everyday and scientific concepts and the Bakhtinian construct of 'voice'.

The main hypothesis of this methodology is the introduction, into the classroom, of 'voices' from the history of mathematics (in the form of selected primary sources, with commentaries). This might, by means of well chosen tasks, develop into a 'voices and echoes game' suitable for the mediation of some important elements of theoretical knowledge. The chosen examples of theoretical knowledge are conceptual leaps in the cultural history of mankind: the theory of falling bodies of Galileo and Newton, Mendel's probabilistic model of the transmission of hereditary traits, mathematical proof and algebraic language. All these feature aspects of a counterintuitive character. The authors claim that the 'new' manners of viewing and the methodological requirements are expressed by the 'voices' of the protagonists themselves in the speech genre that belongs to their cultural tradition. Such voices act as voices belonging to real people with whom an imaginary dialogue can be conducted beyond space and time. The voices are continuously regenerated in response to changing situations: They are not passively listened to but actively appropriated through an effort of interpretation. The authors describe a number of teaching experiments whereby they introduce some analytical tools (i.e. different types of echoes) which, on the one hand, are used to interpret classroom processes and, on the other, are used to design classroom activity. For instance, a *'mechanical echo'* consists in a precise paraphrasing of a verbal voice, whilst an *'assimilation echo'* refers to the transfer of the content/method conveyed by a voice to other problem situations. A *'resonance'* is a student's appropriation of a voice as a way of reconsidering and representing his or her experience. The most delicate issue in this methodology is, certainly, the selection of historical sources capable of conveying the crucial ideas of a scientific revolution in a concise manner, so as to comply with

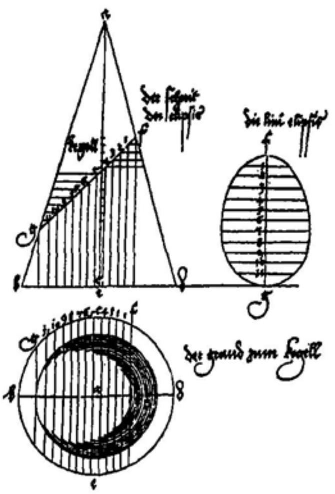

Figure 5, 4: Whether a section of a cone is the same as a section of a cylinder, and whether either is egg-shaped, has long been debated, Here Dürer's discussion of the ellipse (Underweysung der Messung, 1525)

the space and time constraints of institutionalised teaching. Boero's published experiments concern mainly grade 8 secondary school students, but studies presently in progress (with voices taken from Plato's dialogues) have given evidence that similar processes can be implemented also in primary schools and with pupils from a range of socio-cultural backgrounds (Garuti *et al* 1999). We analyse below (§5.4) the epistemological assumptions of this methodology.

This approach is consistent with the approach of Bartolini Bussi *et al.* (1996, 1999) who also introduced a guided reading of historical sources in primary school, in two long-term teaching experiments concerning perspective drawing and gears. Even if no explicit voices and echoes game was introduced in the classroom, the guided reading and interpretation of well selected historical sources had been used to institutionalise the pieces of knowledge built in the classroom by shifting them to a theoretical level. In both experiments the appropriation of the theoretical dimension of mathematical knowledge had led the pupils to produce theorems, i. e. statements with proofs inside a reference theory (Mariotti *et al.* 1997). The above experiments concern early grades of school (4-8).

Other experiments have been carried out successfully in the 11th grade (Ernesto Rottoli, personal communication), using original texts of Greek authors and excerpts from historical studies, in order to integrate the knowledge acquired during philosophy lessons and the knowledge acquired during mathematics lessons. The aim was to organise a deeper level of knowledge. The design was based on the awareness that in ancient times mathematics and philosophy were strictly linked to each other and some traces of this link are still present in highly organised and culturally rooted linguistic patterns.

5.3.2 Indirect use of historical and epistemological studies in the design of activities for students

In the research projects described above, elements of the history of science (fragments of original texts) were used in an explicit manner in the teaching sequences, and historico-epistemological studies were directly linked to the contents of teaching. The links between the historical studies and the teaching design can be much more implicit and indirect, and the relevance of these studies for the didactic activity somewhat less obvious.

5.3.3 The example of linear algebra

This is certainly the case of the research projects on the teaching and learning of linear algebra conducted, independently, by Dorier and Sierpinska.

The motivation of these research projects has been students' commonly stated difficulty with the axiomatic approach used in undergraduate linear algebra courses. This difficulty is often hard for mathematicians to understand, for whom the axiomatic approach is indeed the royal road to linear algebra, at last allowing the subject to be presented in a simple, neat and coherent way. The questions that naturally arise in this situation are: why is it difficult to understand a simple axiomatic theory? What are the conditions of coming to construct or understand this or that particular concept of this theory? What can be done to facilitate the understanding of this theory by the students? Some answers to these questions make no reference to history. For example, one may say that the axiomatic theories that constitute linear algebra are simple only in appearance. A slightly deeper mathematical analysis of the basic concepts of linear algebra shows their inner complexity (see, e.g. Sierpinska, Dreyfus, Hillel, 1999). This complexity may not be accessible to an undergraduate student, and therefore, he or she will have to accept the teacher's word that, for example, it makes sense to accept this definition rather than a different one. This happens so often in a linear algebra course, that many students end up developing what is called 'the obstacle of formalism' (Dorier *et al* 1997). It may not have been necessary to refer to history to answer these questions. But it proved useful and inspiring, both in explaining students' difficulties and in designing activities for them.

For example, a look at the history of linear algebra from a very broad perspective of currents of thought allowed the identification of three interacting modes of reasoning, labelled 'synthetic-geometric', 'analytic-arithmetic', and 'analytic-structural' (Sierpinska *et al.* 1997). These modes of reasoning are linked to different theoretical perspectives and imply different meanings of concepts. They are not equally accessible to beginning linear algebra students, and the students tend to be inflexible in using them in different contexts. An awareness of these modes of reasoning and their role in linear algebra helps in both designing activities for students and reacting to the students' responses to them in a teaching situation.

A more fact-focused look at the history of linear algebra allowed the identification of the contexts in which the basic linear algebra concepts emerged: analytic geometry, vector algebra, vector analysis and applications in physics; linear equations and determinants, linear differential and functional equations, abstraction of vector structures in functional analysis (Dorier 1995a, 1997). Specific contexts have been used in the design of history-inspired classroom activities. For example, instead of simply giving the definition of a linearly independent set of vectors and following it by a series of exercises, Dorier (1998a, 1998b) proposed to anchor the students' understanding in their experience of the Gaussian elimination method for solving systems of equations, which is introduced in secondary schools in France. The task for the students was to discuss and analyse this method. In this research, history was a source of inspiration and a means of control in the building of the didactic experiment, but the experiment did not aim at a reconstruction in the classroom of the historical development or even at commenting on historical texts.

Case study: Fermat as an inspiration for work with Cabri

The reference to history is also implicit in recent research by Sierpinska, Hillel & Dreyfus (submitted), which focuses on the students' understanding of the notion of vector and its coordinates in a basis. This research involved designing and evaluating a teaching sequence in the Cabri dynamic geometry environment. What emerged was the striking difference between the way in which Fermat approached the problem of finding a canonical equation of a conic in his *Ad locos planos et solidos isagoge* (c.1635) and the algorithmic procedure which is normally used in present day linear algebra courses. This triggered an understanding of the difference between geometric and arithmetic spaces, and a coherent explanation in these terms of the students' difficulties and conceptions. A brief outline of this explanation follows.

Elements of an n-dimensional arithmetic space are n-tuples of real numbers. By defining operations of addition and scalar multiplication on the n-tuples in a coordinate-wise fashion one obtains a vector space structure usually denoted by $\mathbb{R}^n$. There is a long-standing tradition of referring to the elements of the arithmetic spaces as 'points', and of using the language of Euclidean geometry to refer to their subsets such as straight lines and planes. This is what we do in linear algebra classes, without, however, discussing with the students the status, in the theory, of the geometric objects thus evoked. There are important differences between the 'arithmetic spaces' underlying vector spaces $\mathbb{R}^n$ and the 'geometric spaces' of Euclidean geometry. The objects of the arithmetic spaces are sets of n-tuples of real numbers defined by conditions (in the form of equations, inequalities, etc.) on the terms of the n-tuples belonging to the sets. These objects can be represented by geometric figures like lines or surfaces. The representations will depend on the choice of a coordinate system.

A set $\{(x, y) : x^2 + y^2 = 1)$, for example, will be represented by a geometric circle in an orthonormal coordinate system, and by a geometric ellipse in a non-orthonormal coordinate system. (Here *geometric circle* means the locus of points equidistant from a given point.) In geometric spaces, the roles of objects and representations are reversed. Objects, given by relations between their parts, can be represented by sets of n-tuples defined by conditions on their terms, e.g. by equations. These equations will be different depending on the choice of the coordinate system.

Fermat and Descartes worked with geometric spaces, and for them, equations were representations of geometric objects: they were introducing a system of coordinates into a pre-existing geometric space. But, in a process which started by the end of the 17th century with the work of Newton and other creators of calculus, representations started to play the role of objects: "Before Descartes, the solution of an algebraic equation was nothing but a tool to solve other problems. After Descartes and particularly at the end of the 17th century, to give an equation or a symbolic expression was just to give a curve, and to give an integral was just to give an area, even if the curve and the area are geometric objects that we can perfectly characterise without mentioning any equation or integral." (Panza 1996, 245). This process led to the replacement of the geometric space with, as it were, a system of coordinates without an underlying geometric space.

The geometric language and drawings of lines and planes in today's linear algebra textbooks are used as mere didactic aids in the introduction of the $\mathbb{R}^n$ spaces, illustrations which play no role in the building of the theory. But thinking of vectors as n-tuples leads, notoriously, to students' difficulties with the notions of 'change of basis' and 'coordinates of a vector in a basis', especially when these notions are introduced in the context of $\mathbb{R}^n$ spaces (Hillel & Sierpinska 1994). Indeed, for a student who is thinking in terms of arithmetic spaces, the notion of change of coordinates may not make sense. Insofar as an arithmetic space is nothing but a system of coordinates, changing the system means changing the space, so one should maybe speak of transformations of the space. The very notion of coordinates of a vector does not seem to make sense in the arithmetic frame of mind, where a vector is nothing but coordinates. In our courses we often try to give some meaning to the notion of change of basis by introducing the topic of canonical equations of conics. But in doing this, without warning the student, we revert to thinking in terms of geometric spaces: conics are again geometric objects which can be represented by different equations depending on the choice of the coordinate system. This only adds to the confusion in the students' minds. The notions of coordinates of a vector in a basis and change of basis make more sense for the students when they start working with vector spaces other than $\mathbb{R}^n$ (especially with function spaces) but, at an early stage in the teaching of linear algebra, it seems useful to restore the geometric genesis of the $\mathbb{R}^n$ spaces. This was the guiding idea of the teaching design and an important part of the rationale behind the choice of the computer environment, namely the preference of a Dynamic Geometry Software over a Computer Algebra System.

A posteriori, it is clear that it was not necessary to study Fermat's Isagoge to come to this understanding of the students' difficulties. But it helped a lot in clarifying ideas and making distinctions between blurred concepts. The simple reason for this can be that understanding ideas gains much from analyzing contrasting ways of thinking, from having access to their articulated exposition, and from following their evolution over long periods of time. All this is made possible in a historical study.

5.3.4 The example of calculus

Another example of the use of historical studies in understanding students' difficulties and designing activities for them is found in a research project conducted by Schneider (details in §8.2.2). This is a project concerned with calculus, which takes into account the order and choice of historical contexts, the historical forms of the central concepts, and the analysis of the evolution of these concepts in terms of epistemological obstacles (Schneider 1988). Activities for the students are designed with the intention of allowing the students to put to test, individually and collectively, their previous beliefs and to become aware of the limitations of these. The problem situations generated in these activities are expected to give rise to cognitive and socio-cognitive conflicts and to create favourable conditions for students to reach a better understanding.

Although the project is framed by a constructivist view, it is not assumed that the students construct theoretical knowledge only as described by the constructivist model. Indeed, in this project, students' understanding is seen as dependent, to a certain extent, on the didactic mediation of the teacher. For example, a game of 'voices and echoes' (in Boero's sense, see above) between Berkeley's text and the students about instantaneous velocity, with a meta-level type of intervention of the teacher (see Dorier 1995b), makes the students better aware of their own perception of mathematics and of the connections of this discipline with the perceptible phenomena of the physical world. In this project, the theory of epistemological obstacles and the constructivist approach are conceived of as hypotheses whose efficiency should be tested case by case, taking into account the specificity of the mathematical contents, the socio-cultural origin of students, the problem situations as described by some precise didactic variables, each situation having to be studied didactically (for an example of a didactic study of a situation related to instantaneous flow see Schneider 1992).

5.3.5 Research on the methodology of history-based design of activities for students

In neither of the examples of research given in this section was the methodology of history-based design and analysis of student activities an object of explicit discussion. Other research in mathematics education is concerned with this particular question, especially in the context of the theory of epistemological obstacles (e.g. Schneider 1988, 15-16; Sierpinska 1994, 120-125). Here, let us mention in more detail only a methodology proposed by Vasco (1995), which is not related to the framework of epistemological obstacles. The heuristics proposed in this work, called 'forward and backward heuristics', are aimed at helping to find hypotheses for potentially optimal sequencing of mathematics curricula. The 'forward heuristics' are meant to propose efficient ways of reviewing the phylogenesis of the particular mathematical subject, in order to optimise the ontogenetic mastery of that conceptual field. The 'backward heuristics' propose ways to trim, compress, and even alter the sequences found through the forward heuristics. Forward heuristics lay out the rough draft of the roads on the mathematical map; backward heuristics do the redesigning, the short-cutting, and the road signalling (Vasco 1995,62).

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5.4 Epistemological assumptions framing interpretations of students understanding of mathematics

Luis Radford, Paolo Boero and Carlos Vasco

Two different phenomena need to be linked, in using the history of mathematics to understand better the student processes of learning mathematics and the way in which such an understanding can be used in the design of classroom activities. On the one hand, the learning processes of contemporary students; on the other hand, the historical construction of mathematical knowledge. These phenomena belong to two different theoretical realms: the former to the psychology of mathematics, the latter to an opaque field where epistemology and history (to mention only two disciplines) encounter each other.

The linking of psychological and historico-epistemological phenomena requires a clear epistemological approach. Within the field of mathematics education, different approaches have been used. They differ in their epistemological assumptions and, as a result of this, they provide different explanations of the history of mathematics. They also offer different interpretations of students' understanding of mathematics and suggest different methodological lines of pedagogical action. The aim of this section is to provide an overview of some approaches and their corresponding epistemological frameworks.

5.4.1 The 'epistemological obstacles' perspective

This approach is based on the idea of epistemological obstacles developed by G. Bachelard and later introduced into the didactics of mathematics by G. Brousseau in the 1970s. Brousseau's approach is based on the assumption that knowledge exists and makes sense only because it represents an optimal solution in a system of constraints. For him, historical studies can be inspiring in finding systems of constraints yielding this or that particular mathematical knowledge: these systems of constraints are then called *'situations fondamentales'*. In Brousseau's view, knowledge is not a state of mind; it is a solution to a problem, independent of the solving subject. Within this context, an epistemological obstacle appears as the source of a recurrent non-random mistake that individuals produce when they are trying to solve a problem.

A clear assumption underlying this approach is that an epistemological obstacle is something wholly pertaining to the sphere of the knowledge — a sphere that Brousseau conceives as separated from other spheres. Thus he distinguishes the *epistemological obstacles* from other obstacles, e.g. those related to the students' own cognitive capacities according to their mental development *(ontogenetic obstacles),* those which result from the teaching choices *(didactic obstacles)*

John Fauvel, Jan van Maanen (eds.), *History in mathematics education: the ICMI study,* Dordrecht: Kluwer 2000, pp. 162-167

(Brousseau 1983, 177; Brousseau 1997, 85-7) and those whose origin is related to cultural factors *(cultural obstacles)* (Brousseau 1989; Brousseau 1997, 98-1 14). Of course, the clear-cut division of obstacles into ontogenetic, didactic, cultural and epistemological categories is in itselfan epistemological assumption.

The link between the psychological and the historical phenomena to which we referred previously is ensured by another epistemological assumption: in Brousseau's account, an epistemological obstacle is precisely characterised by its reappearance in both the history of mathematics and in contemporary individuals learning mathematics. He says (translation from Brousseau 1983, 178; Brousseau 1997, 87-8): "The obstacles that are intrinsically epistemological are those that cannot and should not be avoided, precisely because of their constitutive role in the knowledge aimed at. One can recognise them in the history of the concepts themselves."

A third epistemological assumption is to be found in the articulation 'student/milieu'. According to Brousseau, the teacher sets the situation, but the knowledge which will result is due to the student's appropriation of the problem. Thus, the motivation is an exclusive relationship between the problem-situation and the student. In doing this, Brousseau supposes that a kind of isolation between the teacher and the student takes place during the process of solving the given problem.

The interpretation of the student's understanding of mathematics is framed here by the idea that the development of knowledge is a sequence of conceptions and obstacles to overcome (Brousseau 1983, 178). Consequently, the pedagogical action is focused on the elaboration and organization of teaching situations built on carefully chosen problems that will challenge the previous students' conceptions and make it possible to overcome the epistemological obstacles, opening new avenues for richer conceptualisations (for an example, see the way Schneider organised her calculus teaching, §5.3.4).

Sierpinska has stressed that, although the new conceptualisations may be seen as more complex than the previous ones, these do not have to be necessarily related to steps in the development or progress of knowledge: "Epistemological obstacles are not obstacles to the 'right' or 'correct' understanding: they are obstacles to some change in the frame of mind." (Sierpinska 1994, 121).

5.4.2 A socio-cultural perspective

Some Vygotskian perspectives in mathematics education choose, from the outset, a different set of epistemological assumptions. Thus, in Radford's socio-cultural perspective, knowledge is not restricted to the technical character which results when knowledge is seen as essentially related to the actions required to solve problems. Following a socio-historical approach (see eg Mikhailov 1980, Ilyenkov 1977) and a cultural tradition (see eg Wartofsky 1979), knowledge is conceived as a culturally mediated cognitive praxis resulting from the activities in which people engage. Furthermore, the specific content with which knowledge is provided is seen as framed by the rationality of the culture under consideration. It is the mode of that rationality which will delimit the borders of what can be considered as a scientific problem and what shapes the norms of scientific inquiry-for instance, what is an accepted scientific discourse and what is not, what is accepted as evidence and what is not. The mode of the rationality relates directly to the social, historical, material and symbolic characteristics underpinning the activities of the individuals (Radford, submitted). Hence, from a sociocultural epistemological viewpoint, knowledge can only be understood in reference to the rationality from which it arises and the way the activities of the individuals are imbricated in their social, historical, material and symbolic dimensions.

In this line of thought, a problem is never an object on its own, but is always posed, studied and solved within the canons of rationality of the culture to which it belongs (Radford 1997a). For example, the supposed numerical patterned cosmological nature of the universe was an important belief in the culture of the Neoplatonists (as it was in the early Pythagorean schools). Another belief from that early Greek period was that "the paradigmatic relation between the world and numbers is such that what is true of numbers and their properties is also true of the structure and processes of the world" (O'Meara 1989, 18). The problems that they posed, resulting from the aforementioned assumed numerical structure of the world and the investigation of this structure through. non-deductive methods (Radford 1995), were seen as being completely genuine and valid within their rationality and beliefs.

In Radford's socio-cultural approach, the student/milieu relation is sustained by the epistemological assumption according to which knowledge is socially constructed. Instead of seeing such a construction as a diachronic move between the teacher and the student, as is often the case in socio-constructivist accounts, the student is seen as fully submerged in his cultural milieu, acting and thinking through the arsenal of concepts, meanings and tools of the culture. The way in which an individual appropriates the cultural knowledge of his or her culture is often referred to in Vygotskian perspectives as *interiorisation.* Different accounts of interiorisation can be provided. In the socio-cultural approach under consideration, a semiotic, sign-mediated, discursive account sees interiorisation not as a passive process but an active one, in which the individual (through the use of signs and discourse) re-creates concepts and meanings and co-creates new ones (Radford 1998). An experimental historically-based classroom study concerning the re-creation of concepts can be found in Radford and Guérette (1996). A historical case study about the co-creation of new mathematical objects is provided by the invention of the second unknown in algebra by Antonio de Mazzinghi in the 14th century (see Radford 1997b).

In this socio-cultural perspective, the classroom is considered as a micro-space of the general space of culture, and the understanding that a student may have of mathematics is seen as a process of cultural intellectual appropriation of meanings and concepts along the lines of student and teacher activities. Understanding is not seen merely as a unidirectional stage reached by a fortunate student resulting from the sudden awareness of something becoming clear. As Voloshinov (1973, 102) put the matter, "Any true understanding is dialogical in nature", meaning that at the very core of understanding resides a hybrid semiotic matching of different views. Since such a semiotic matching is contextually situated and culturally sustained, there is no question, in this approach, of reading the history of mathematics through recapitulationistic lenses (whether of contents or mechanisms). The history of mathematics is a rather marvellous locus in which to reconstruct and interpret the past, in order to open new possibilities for designing activities for our students. Although cultures are different they are not incommensurable; as explored in Voloshinov's concept of understanding, cultures can learn from each other. Their sources of knowledge (e.g. activities and tools) and their meanings and concepts are historically and panculturally constituted. This is made clear by the fact that most of our current concepts are mutations, adaptations or transformations of past concepts elaborated by previous generations of mathematicians in their own specific contexts.

5.4.3 The 'voices and echoes' perspective

Let us now turn to the epistemological assumptions underlying Boero's 'voices and echoes' perspective (see §5.3.1). His point of departure is the fact that some verbal and non-verbal expressions (especially those produced by scientists of the past) represent in a dense way important leaps in the evolution of mathematics and science. Each of these expressions conveys a content, an organisation of the discourse and the cultural horizon of the historical leap. Referring to Bachtin (1968) and Wertsch (1991), Boero & al (1997) called these expressions *voices.* Performing suitable tasks proposed by the teacher, the student may try to make connections between the voice and his/her own interpretations, conceptions, experiences and *personal senses* (Leont'ev 1978), and produce an *echo,* a link with the voice made explicit through a discourse. What the authors have called the *Voices and echoes game* (VEG) is a particular educational situation aimed at activating students to produce echoes through specific tasks: *"How might X have interpreted the fact that Y?";* or *"Through what experiences might Z have supported his hypothesis?";* or: "*What analogies and differences can you find between what your classmate said and what you read about W?* ".

The epistemological assumptions underlying the VEG, partly presented in Boero & al (1998), concern both the nature of 'theoretical knowledge' (the content to be mediated through the VEG), and the cognitive and educational justifications of the VEG. As regards the nature of theoretical knowledge, in mathematics and elsewhere, some characteristics were highlighted drawing on the seminal work of Vygotsky about scientific concepts (see Vygotsky 1990, chapter 6). In particular, theoretical knowledge is systematic and coherent; validation of many statements depends on logico-linguistic developments related to basic assumptions (axioms in mathematics, principles in physics, etc.).

In relationship to the problem of transmitting mathematical theoretical knowledge in school, the preceding description was refined by taking into account Wittgenstein's philosophy of language as well as recent developments in the field of mathematics education by Sfard. The following aspects of theoretical knowledge in mathematics were considered as crucial, concerning both the *processes of theory production* (especially as regards the role of language) and the *peculiarities of the produced theories:*

- theoretical knowledge is organised according to explicit *methodological requirements* (like coherence, systematicity, etc.), which offer important (although not exhaustive) guidelines for constructing and evaluating theories;
- definitions and proofs are key steps in the progressive extensions of a theory. They are produced through *thinking strategies* (general, like proving by contradiction; or particular, like 'epsilon-delta reasoning' in mathematical analysis) which exploit the potentialities of language and belong to cultural tradition;
- the *speech genre* of the language used to build up and communicate theoretical knowledge has specific language keys for a theory or a set of coordinated theories — for instance, the theory of limits and the theory of integration, in mathematical analysis. The speech genre belongs to a cultural tradition;
- as a coherent and systematic organisation of experience, theoretical knowledge vehiculates specific *'manners of viewing'* the objects of a theory (in the field of mathematical modelling, we may consider deterministic or probabilistic modelling; in the field of geometry, the synthetic or analytic points of view; etc.).

In Boero *et al.* (1998), the authors claim that the approach to theoretical knowledge in a given mathematics domain must take these elements into account, with the aim of mediating them in suitable ways. Concerning the problem of 'mediation', the assumption is made that, depending on its very nature, *each of the listed peculiarities is beyond the reach of a purely constructivistic approach.*

The authors' working hypothesis is that the VEG can function as a learning environment where the elements listed above can be mediated through suitable tasks, needing 'active imitation' in the student's 'zone of proximal development'. The first teaching experiments, reported in Boero *et al* 1997, Boero *et al.* 1998, Garuti 1997, Lladó & Boero 1997, Tizzani & Boero 1997, were intended to provide experimental evidence for this hypothesis.

The three perspectives mentioned in this section have shown a variety of ways of conceiving the production of knowledge. Each of them relies on different epistemological assumptions. It is evident from this that different epistemological assumptions lead to different interpretations of the history of mathematics, as well as different ways of linking historical conceptual developments to the conceptual developments of contemporary students.

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5.5 Conclusions: guidelines and suggestions for future research

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The various issues addressed in this chapter, and the related teaching experiments and didactical analyses briefly described, show clearly that while 'naive recapitulationism' has persisted in many forms, the relation between ontogenesis and phylogenesis is now recognised to be much more complex than was originally believed. The relations between history of mathematics and learning and teaching of mathematics can be extremely varied. Some teaching experiments may use historical texts as essential material for the class, while on the other hand some didactical analyses may integrate historical data in the teaching strategy, and epistemological reflections about it, in such a way that history is not visible in the actual teaching or learning experience.

While some knowledge of history of mathematics may help in understanding or perhaps even anticipating some of our students' misunderstandings, a careful didactical analysis using history of mathematics is necessary in order to try to overcome students' difficulties. History may be a guide for designing teaching experiments but it is only one of many approaches, more or less essential, more or less visible, of the whole didactical setting. Therefore, one of the necessary conclusions of this chapter would be that any use of history in the teaching of mathematics needs an accompanying didactical reflection.

This way of putting things creates an asymmetry between history and didactics which may not reflect their actual relationship. Indeed any attempt to put in relation the history of mathematics and the teaching or learning of mathematics necessarily induces an epistemological questioning both of individual cognitive development and of the interpretations of the historical development of mathematics. What happened in the past and what may be likely to happen in the classroom are obviously different phenomena because they are based in very different cultural, sociological, psychological and didactical environments and because contemporary didactical contexts and historical periods conform to very different constraints.

Beyond these differences, the act of teaching is legitimated by the belief that what is taught in the classroom bears some similarity with professional mathematics. However, the knowledge to be taught *(savoir enseigné)* is a transformation of the knowledge of 'professional' mathematicians *(savoir savant)* even if it uses the same vocabulary, notions, and so on, and it is rare that historical processes are taken into account explicitly while writing curricula. Historians of mathematics may object that this is a nonsense. On the other hand, it would also be a nonsense to try to impose a reconstruction of history in the teaching process, in a very strict

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recapitulationist paradigm. As Chevallard says (translated from Chevallard 1991, 48):

> Another direction for research consists in being aware that the planned didactical construction of knowledge is a specific project within the teaching process, bearing an *a priori* heterogeneity with the scientific practices of knowledge, and not immediately reducible to the corresponding socio-historical geneses of knowledge.

Nevertheless, teaching is still organised in such a way that there is a social demand that the knowledge to be taught must appear as close as possible to the official knowledge of mathematicians. In this sense, an epistemological reflection on the development of ideas in the history of mathematics can enrich didactical analysis by providing essential clues which may specify the nature of the knowledge to be taught, and explore different ways of access to that knowledge. Nevertheless what appears to have happened in history does not cover all the possibilities.

Figure 5.5: Nicolaus Copernicus, in front of the Polish Academy of Sciences in Warsaw, seen through the interpretative lens first of Polish history, then of the Danish sculptor Bertel Thorwaldsen, then of a British photographer in the 1990s. Now an inspiration to Polish students, in the 19th and 20th centuries many who had only vague understanding of his achievements were nevertheless agitated about whether Copernicus was Polish or German. The sphere and the compasses have long been symbols to represent a mathematician to the gaze of passers by.

We cannot reconstruct the past with any certainty. Not only are we missing essential data (for example, lost texts, ephemera, unpublished material or oral exchanges) but also a historical fact or event is never pristine. A fact or event is always seen through interpretative lenses and hence will only be partial and subjective. We face essentially similar difficulties when analysing didactical events.

To this extent history and mathematical pedagogy share common theoretical issues with regard to the necessity for epistemological reflection. We need not only to look through history in order to try to improve the teaching of mathematics but also to elaborate common ('echoing') ways of exploring historical and didactical situations. This could be a very challenging issue for future research which could be approached from different viewpoints. It could be a new way of raising the issue of cultural influences in the development of mathematics.

We have said above that what happened in history does not cover all the possible ways of access to one specific element of knowledge. Yet, when setting up a teaching programme, one should try to analyse as many ways of access to the knowledge as possible. This is an important part of any didactical analysis where the use of history can be informative. However, this work is usually confined within the limits of an official curriculum. Indeed, traditions in curricula are sometimes so strong that our views, even as researchers in mathematics education, on the organisation of knowledge are limited because of the strong cultural influences that unconsciously guide our thoughts about the different possible organisations of a curriculum. Because history is temporally and culturally distant from the mathematics taught in our usual curricula, it may provide us with some unusual ways of access to knowledge that could be of considerable didactical value. Of course, this can be possible only if one does not look at history through the lens of 'modern mathematics'. In this sense, another line of development for future research would be a reflection on certain parts of the curriculum in relation to an epistemological reflection on its historical developments.

It may be added that, among the areas for further research, it seems important that mathematics educators and teachers should become more closely involved in co-operative efforts to develop and implement lessons and modules using the history of mathematics as we have shown here. In a similar manner, collaborative work between historians of mathematics and mathematics educators can contribute to better elucidation of the problem of the link between the epistemological and psychological aspects of the conceptual development of mathematical thinking.

Reference for §5.5

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