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Perceiving the General: The Multisemiotic Dimension of Students' Algebraic Activity

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In this article, we deal with students' algebraic generalizations set in the context of elementary geometric-numeric patterns. Drawing from Vygotsky's psychology, Leont'ev's Activity Theory, and Husserl's phenomenology, we focus on the various semiotic resources mobilized by students in their passage from the particular to the general. Two small groups of Grade 9 students are investigated through a four-dimensional analysis: video, audio, transcripts, and written material. The resulting qualitative analysis shows how discourse, gestures, actions, and rhythms orchestrate one another and how, through a complex and subtle coordination of them, the students objectify different aspects of their spatial-temporal mathematical experience. The analysis also suggests connections between the syntax of the students' algebraic formulas and the semiotic means of objectification through which the formulas were forged, thereby shedding some light on the meaning of students' algebraic expression. Some implications for the teaching and learning of mathematics are discussed.

Key words: Algebra; Classroom interaction; Communication; Language and mathematics; Linguistics; Patterns; Relationships in mathematics; Vygotsky

In the course of a regular lesson on algebra, Grade 9 students were dealing with the classical pattern shown in *Figure 1*. They had to continue the sequence of figures up to Figure 5 and then find the number of circles in Figure 10 and in Figure 100.[1] Talking to his two group-mates, Doug[2] says: "So, we just add another thing, like that." Exactly as he utters the word "another" he starts making a *rhythmic sequence* of six parallel gestures. The gestures play a twofold role. First, they highlight the last two circles diagonally disposed at the end of each figure of the sequence. Second,

[1] Throughout the article, names of figures in italics (e.g., *Figure 1*) refer to objects in the article, whereas nonitalic names (e.g., Figure 1) refer to elements of a pattern in the classroom activity given to the students.

[2] The students' names are pseudonyms.

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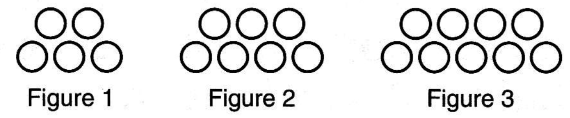

Figure 1. The sequence of figures given to the students in a Grade 9 mathematics lesson.

by being *rhythmically repeated*, they express the idea of something *general*, something that continues *further and further,* in space and in time.

Doug's utterance and body actions can be seen as bearing little (if any) cognitive relevance. This would be the case in theoretical approaches to learning and knowing in which cognitive activity is conceived of as something mental, as something intrinsically subjective, taking place in the head. In this article we articulate a different approach—a semiotic-cultural one—in which cognitive activity is seen in more materialistic terms (see e.g., Geertz, 1973; Goodwin, 2000a; Radford, 1998). More precisely, cognitive activity is considered as an activity embodied in the corporality of actions, in artifacts, and in the historical-cultural systems of signs that the students mobilize to accomplish their goals (Ilyenkov, 1977a, 1977b, 1982; Vygotsky, 1962; Wartofsky, 1979). Concomitant with this view of cognitive activity is our view of mathematics. Instead of considering mathematics to be a set of disembodied truths beyond the vicissitudes of cultures, we see it much like art and poetry, that is to say, as a way of expressing ourselves and of making sense of our world (Radford, 1997; Radford, Bardini, & Sabena, 2005).

This is why, from the theoretical perspective advocated here, Doug's utterances and gestures are considered key elements in the students' unfolding mathematical experience. Not only do these utterances and gestures help the students to focus on certain parts of their perceptual field (e.g., the pair of last circles in each figure), but they also provide the students with a way to express their *sense* of generality. Both that which is attended in perception and the way it is expressed are crucial aspects of processes of mathematical generalization. As Mason, Gramham, Pimm, and Gowar (1985) suggested, "[g]enerality is the lifeblood of mathematics and algebra is the language in which generality is expressed" (p. 8). Now, the question is: What is algebraic language made of?

Certainly, alphanumeric symbolism is part of that language, and indeed an important one. Thus, commenting on the link between generalization and its expression, Kieran (1989) contended that "a necessary component [of algebraic thinking] is the use of algebraic symbolism to reason about and to express that generalization" (p. 165). It is clear that alphanumeric symbolism provides students with a culturally sophisticated way of expressing generalizations and entering the realm of algebra. However, paths to algebra, we want to suggest, are also provided by the various ways that we have to *perceive* and *express* the general. These remarks lead us to argue that, to a certain extent, Doug *is* doing algebra. He is, indeed, producing an algebraic text, although his "algebraic text" is not made up of alphanumeric signs; in fact, his text has a different "texture": It is made up of (spoken) words, rhythm, and gestures.

Doug's algebraic activity may look similar in several respects to the one practiced by neo-Pythagoreans, such as Nichomachus in the 2nd century BC, who, without having modern algebraic symbolism at their disposal, nonetheless investigated complicated relationships among triangular, square, pentagonal, and other numbers (see D'Ooge, 1938), in all likelihood using pointing and other gestures to shape their perceptual process and to express the general. It is unquestionable, though, that the sense of generality achieved through words, gestures, and rhythm is not the same as the one achieved through a formula or a graph. A given semiotic system provides us with specific ways to signify or to say certain things, whereas another semiotic system provides us with other ways of signification. The linguist Émile Benveniste referred to this situation as the principle of *nonredundancy*: "Semiotic systems are not 'synonymous'; we are not able to say 'the same thing' with spoken words that we can with music, as they are systems with different bases" (Benveniste, 1985, p. 235). The same distinction is true of gestures and formulas.

By the same token, Benveniste's nonredundancy principle warns us against the common belief in translatability—the belief that, for example, a formula says *the same thing* as its graph, or that a formula says the same thing as the word problem it "translates."[3] The nonredundancy principle does not mean, however, that what we intend or express in one semiotic system is completely independent from what we express in another one. From an ontogenetic viewpoint, we seize mathematical and other cultural objects in a *progressive manner*, as we involve ourselves in a reflexive sign-and-tool-mediated cognitive praxis.

In this article, we investigate the students' progressive awareness of generality as they reflect on a pattern problem. Our aim is to better understand the way the students attend to the *perceptually given* (e.g., the three first elements of a sequence) and start moving beyond it in their attempt to reach the general. Love (1986) and Mason (1996) have referred to this subtle and crucial process in which the perceptually given is transcended in terms of "seeing" or "noticing" the general in/through the particular.[4] We investigate this process as a phenomenological aspect of the students' cultural sign-mediated mathematical experience. To do so, we present a detailed semiotic analysis of short key passages of the students' activity and focus on those elements (words, gestures, signs, etc.) to which the students make recourse in order to transcend the particular. These microanalyses provide us with a fine-grained anatomy of the genesis of students' generalizations.

A SEMIOTIC-CULTURAL THEORETICAL FRAMEWORK

There is a longstanding tradition that asserts that to notice or to become aware of an object, it is sufficient to have it in front of us. The problem with this tradition

[3] For a discussion of this problem, see Duval (2002) and Radford (2002a).

[4] A detailed review of the literature on the recognition of patterns and generalization is provided by Castro Martínez (1995).

is that it supposes that perception is a kind of passive sensual data receptor that effortlessly receives the object as it is. This sensual-passive view of perception has been challenged by work conducted in several different disciplines (Arzarello, 2004; Carroll, 2001; Miller, 1984; Petit, 2003; Roth & Bowen, 1999; Sonesson, 1994; Wartofsky, 1984). Perception, it is argued, is instead a complex cognitive activity related to the way we experience the objects in our surroundings. For instance, Husserl (1997), the father of phenomenology, wrote, "[p]erception is not some empty 'having' of perceived things, but rather a flowing lived experience" (p. 84). Drawing from Husserl's work, Merleau-Ponty (1945) suggested that what makes an object become an object of perception is our attitude toward it, our way of attending it, or the questions that we are trying to answer (p. 325).

The flow of lived experience (Husserl) and our subjective attitude toward an object (Merleau-Ponty) are related to the *manner* in which we become acquainted with the object of perception, that is, the *activity* that mediates our relationship to the object.[5] This is why

> to explain scientifically the appearance and features of a subjective, sensual image, it is not enough to study the structure and work of sensory organs on the one hand, and the physical nature of the effect an object has on them on the other. It is necessary also to penetrate into the *activity of the subject that mediates his ties with the objective world.* (Leont'ev, 1978, p. 20, emphasis added).

It would be misleading, though, to reduce perception to the subjective realm. If it is true that what we intend in the act of perception is underpinned by our individual attitude toward it, it is also true that prior to the subjective experience, the object of the individual's intention has been endowed with cultural values and theoretical content (Radford, 2006). This is why, as Wartofsky (1979) noted, perceptual activity is not only conditioned by our particular biography or by the species-specific biologically evolved mechanisms of perception but also "by the historically changing 'world' created by human practical and theoretical activity" (pp. 195–196). In other words, the world that appears in front of us is not a new world or one just coming into being. It is a world that, through its diverse situations and contexts, offers (but neither imposes nor *pre*determines) ways of looking at and attending to things.

The short classroom passage mentioned at the beginning of the article may help us here. The way in which the question was asked to the Grade 9 students endowed the figures of the sequence with a certain theoretical content. It offered Doug and his group mates a way to see the circles beyond plain visual stimuli (i.e., not as a mere bunch of circles), but as a part of a theoretical sequence. Generally speaking, in perception, the visual stimuli are *transformed* by a ubiquitous interpretative process where the subjective and the cultural become entangled. It is through this active interpretative process that continuously modifies the perception of the object in front of us that we see what we see and that we go beyond the perceptually seen. Such a process allows us to notice or become aware of general properties that are not visible in the realm of the concrete and the particular. But how is this possible?

[5] We shall come back to the concept of activity in the next section.

How exactly can we see the general in the particular? The passage from the particular to the general is only possible if the particular is not seen as a particular, that is to say, as identical to itself. *The particular must become a sign for something else.*

In our studies on generalization, we have identified three strategies used by the students to transcend the particular. One strategy is to see a particular, p, as *representative* of a vast class of objects $\{p_1, p_2, \ldots\}$. In this case, although the students refer literally to the particular p, what has been said of p applies to the other elements of the corresponding class, for p can be *substituted* for any p_i. The particular p is indeed seen as a *generic object* (Balacheff, 1987). Another strategy is to see p *metaphorically*. In this case, contrary to the generic object, the discussion is not exactly about p. Even if p is the object of discourse, it is so only obliquely. The particular p serves to talk about the general. The students use linguistic expressions such as "Let's say that" or "as if" to mean that they are using a certain particular but not as such (details in Radford, 2000; 2002b). A third strategy is to see p in a *dynamic way*, as the bearer of a specific although general structure of which p is itself an instantiation. This is the case of Doug and his group mates in the example discussed previously.

Of course, as the previous typology suggests, the overcoming of the particular realm is not accomplished through the organ of vision alone. Language becomes an integral part of the world that we see. As Mikhailov (1980) suggested, "[i]t is language that constantly participates in converting the perception and understanding of the external object into self-awareness and self-consciousness" (p. 236). However, as illustrated by Doug's utterance and gestures, we perceive the general through gestures and other signs as well.

The perceptual act of noticing hence unfolds in a process mediated by a multi-systemic semiotic activity in the course of which the object to be seen emerges progressively (Goodwin, 2000b). This process of noticing we have termed a process of *objectification*. In its etymological sense, objectification means to make something (e.g., a certain aspect of a concrete object, like its color, its size, or a property of a mathematical object) apparent. To make something apparent, learners and teachers use signs and artifacts of different sorts (mathematical symbols, graphs, words, gestures, calculators, and so on). We call these artifacts and signs used to objectify knowledge *semiotic means of objectification* (Radford, 2003a).

In previous works, we have discussed the prominent role of gestures and language in students' processes of knowledge objectification. We have provided evidence of the key role of deictic activity, both at the level of gestures, as in pointing, and at the level of language, as when students use terms such as *this* and *that* (Radford, 2000, 2002b, 2003a).[6] In this article, we provide a finer analysis of the passage from the particular to the general in terms of the dialectical relationships between the various semiotic means of objectification mobilized by the students. We are also interested in understanding the impact of the way in which the particular is attended

[6] There is a vast literature on deictics and gestures that has inspired our research, including Goldin-Meadow (2003), Hanks (1992), Kendon, (2004), Kita (2003), Klein (1983), and McNeill (1992, 2000). For gestures in mathematics education, see Roth and Lee (2004) and the papers presented at the 29 PME Research Forum organized by Arzarello and Edwards (2005).

on the constitution of the students' alphanumeric algebraic expressions. It is our contention that insights about the syntax of the students' expressions can be gained by paying attention to the genesis of the students' apprehension of the general. To reach this objective, we focus on the work of two groups of three students each during a Grade 9 mathematics lesson.

METHODOLOGY

Subjects-acting-in Settings

From a methodological viewpoint, small-group work and general discussions are conceived of here as different moments of the *classroom activity*. The term activity—that we borrow from Leont'ev's (1978) theory—has a precise sense: It refers to the *mediated actions* and interconnected sequences of actions (i.e. *operations*) that individuals carry out in the attainment of a *goal*. This sense of activity, better captured by the German term *Tätigkeit* (as something related to the creative transformation and understanding of reality), is different from other more colloquial senses of the term, rendered in German as *Aktivität*, or as merely "doing stuff" (see Davydov, 1999, pp. 45–46; see also Roth and Lee, 2004). That which makes activity (*Tätigkeit*) in Leont'ev's sense different from activity in the colloquial sense is its fundamental epistemological claim according to which, in the course of the activity, individuals relate not only to the world of objects (the *subject-object plane*) but also to other individuals (the *subject-subject plane* or *plane of social interaction*) and acquire, in the joint pursuit of the goal and in the social use of signs and tools, human experience.

In the context of the classroom activity to be discussed in this article, *perceiving the general* is the goal of the activity. This goal is clear for the teacher, but, generally speaking, not for the students. Within a didactic teaching-learning project, to make such a goal attainable, the students are required to tackle some mathematical problems. These problems, loaded with historical and cultural conceptual content from the outset, form potential paths toward the goal. Naturally, from the students' point of view, the activity's goal can only emerge as they carry out some actions to deal with and reflect upon the given problems. But, because of Activity Theory's emphasis on the cognitive import of the social plane of interaction, individuals are not considered as isolated cogitators or as mere producers of actions. On the contrary, they appear as *subjects-acting-in settings*, i.e., individuals engaged in the elaboration of an active common relation to historical-cultural reality (Leontyev [Leont'ev], 1981).[7] In tune with our theoretical framework, in what follows, we will

[7] The central ideas of Activity Theory were presented in Leont'ev's dissertation (Léontiev, 1976), followed by two major books (Leontyev, 1978, 1981). The theory has since evolved. For instance, Engeström (1987) elaborated an expanded version of Activity Theory in which the main application is organizational change and learning in the workplace; Cole (1996) attended to the role of the cultural context in ontogenetic development; Radford (2003b) paid attention to the super-symbolic cultural structure that frames situated modes of acting and knowing, and Roth (in press a) has sought to include emotion, motivation, and identity in the theoretical formulation of Activity Theory.

focus on the discursive, sign, and embodied mediated phenomenological processes (the processes of knowledge objectification) underpinning the attainment of activity's goal during the small-group-work phase.

Data Collection and Analysis

The data come from a 5-year longitudinal research program conducted in a French-language school in Ontario, Canada. The data were collected during classroom lessons that were part of the regular school mathematics program. In these lessons, the students spent a substantial period working together in groups of three or four. At some points, the teacher (who interacted continuously with the different groups during the small-group-work phase) conducted a general discussion allowing the students to expose, confront, and discuss their different solutions.

We used three or four video cameras, each filming one small group of students. The videotapes were fully transcribed. In addition to the transcriptions and the videotaped material, we also collect written material such as activity sheets produced during the mathematics lessons and tests. We then carry out discourse, video, and audio analysis using interpretative techniques of qualitative research (see Radford 2000), based on the work of Fairclough (1995), Moerman (1988), and Coulthard (1977).

Guided by our research questions and our theoretical framework, we conducted a *multisemiotic data analysis*. We first identified and selected salient episodes of the videotaped activities. We then refined our analysis with the support of both the transcripts and the students' written material. In particular, we carried out a low motion and a frame-by-frame fine-grained video microanalysis to study the role of gestures and words. Such microanalysis was completed with a voice analysis in terms of word intensity and word temporal distribution using dedicated software. The combination of these four elements (video, audio, transcripts, and activity sheets) provided us with a solid base upon which to deepen our investigation of the several semiotic resources that students mobilize to objectify the targeted mathematical concepts—in this case, the objectification of the general figure of a sequence and its relationship with the produced algebraic formulas. The various semiotic resources (e.g., spoken words, written text, gestures, drawings, symbols, etc.) and their interplay were then interpreted according to our theoretical framework which, in turn, was further refined in the course of the analysis.

The Mathematical Problem

In our classroom-based research, the teacher and our research team design the mathematical lessons, taking into account the curricular content and the students' prior knowledge. The generalization problem that we will discuss here dealt with the study of an elementary geometric sequence (see *Figure 1*). The students were introduced the previous year to the use of letters to build algebraic formulas in the context of patterns. As mentioned previously, the students were required to continue the sequence up to Figure 5 and then to find out the number of circles in Figure 10 and Figure 100. Subsequently they were asked to write a message indicating how to figure out the number of circles in any figure (*figure quelconque*, in the original French), and then to write an algebraic formula for the number of circles in Figure *n*. This problem was intended as a means to revisit the concepts learned the previous year and as a further step in the investigation of the students' processes of generalization.

PERCEIVING THE GENERAL

We discuss two examples concerning the manner in which the students perceived the general. The examples come from two small groups considered representative of their class. We pay special attention to the way the students attended to the particular figures, how they moved on to the general, and the impact of perceiving the general in the particular on the elaboration of their alphanumeric expressions. As we shall see, the students' apprehension of the pattern and the building of generality are underpinned by elements from different semiotic systems (e.g., written signs, words, rhythm, and gestures), which address different aspects of an unfolding sense-making project and whose complex co-ordination ensures a multifarious expression of generality. It will become apparent that the students' sober algebraic alphanumeric formulas still retain traces of the students' gestures, rhythm, and utterances as they attempted to transcend the particular.

Example 1—Embodied and Rhythmic Expressions of the General

As we saw earlier, after having drawn Figure 4 and Figure 5, Doug suggested a particular way to look at the given figures—a way where something general had begun to be noticed, namely, a common feature of the figures. To address the question of finding the number of circles in Figure 10 and Figure 100, the previous common sketchy feature is no longer satisfactory. The students need to refine their perception of the general:

1. *Alice:* No, you just have to always add 1 on the top and 1 on the bottom [inclining her head toward the right when she says "bottom"].
2. *Doug:* Umm. OK. So it's . . . [. . .] How many . . . . How many circles will figure number 10 have?
3. *Alice:* OK. It would be [pointing with her finger to the rows of Figure 2] 11 on the top and then . . . and then . . . 12 on the bottom.

In line 1, Alice suggests that they *see* the figures as divided into two rows. What was previously perceived as a unique object (the couple of two circles at the end of each figure) is now atomized in two separated circles. The shift toward Alice's perception of the figures (*i.e.,* as divided into two rows) is not problematic for Doug, who soon agrees with her point of view (line 2). Line 3 clearly indicates that the early perception of the figures has changed. In fact, the problem related to Figure 10 (line 3) is solved not through the *consecutive* calculation of circles in Figure 6, Figure 7, etc., up to Figure 10, but through a *direct* analysis of Figure 10. It is clear then that the students effectively did transcend the particular. But how did they do it? Let us note that, like Doug, Alice makes recourse to gestures and speech. There is a difference, though. In Doug's case, gestures functioned as a rhythmic ostensible mechanism that conveyed a spatially and temporally embodied sense of generality: The general remained beyond the realm of words and was asserted through rhythm. In the case of Alice, the idea of generality is conveyed by the linguistic adverb *always* (line 1) and the key terms "top" and "bottom." Body, as a semiotic resource, intervenes to *emphasize* the distinction between the two rows through the inclination of her head (as to distinguish the circle added on the bottom from the one added on the top) and, in line 3, as a pointing gesture. Thus, while Doug objectified the idea of generality through a bodily action (a sequence of gestures), Alice does it through a verbalized *schema.*[8] To understand the students' way of transcending the perceptually given, we need to pay closer attention to the terms of which the schema is made.

Spatial deictics. We will discuss first the spatial terms "top" and "bottom." The key terms "top" and "bottom" are spatial deictics, that is, terms that indicate something characterized by a distinctly spatial location. They suppose an embodied point of view (an embodied origin). If this origin is changed, top may become bottom, left may become right, and vice versa (Bühler, 1979; Radford, 2002b). By virtue of pointing or indicating something spatially situated, spatial deictics obey the mode of designation of particular objects. In this sense, they share some traits of an indexical gesture. At the same time, insofar as they designate a class of objects rather than a single object, they already point to a generalization. Spatial deictics are pivots located between the particular and the general.

The adverb "always" and the overcoming of time. Because the crux of the generalization of patterns lies in the fact that it predicates something that holds for *all* the elements of a class based on the study of a *few* of them (Sabena, Radford, Bardini, 2005), the spatial and temporal nature of the particular attended figures has to be overcome in the ontogenetic construction of generalization. Adverb terms such as "always" provide the students with a way to come to terms with the problem of time. The action underpinning the schema is thought of as a *potential* action that can be *reiteratively* accomplished.

Deepening perception and the role of silence. The previous remarks suggest a central role for spatial deictics (e.g., "top") and adverb terms (e.g., "always") in the path that the students forge toward mathematical generality. Spatial deictics endow, in a decisive manner, the students' objectified generalizing schema with a geometric meaning. This schema allows the students to go beyond the territory of the organ of vision and offers them a new way of "seeing" or "perceiving" the general. What the students "see" through this new intellectual way of perceiving the general

[8] Here, we are using the term "schema" in a neo-Piagetian sense (see Radford, 2005a).

bears the imprint of the schema's geometrical meaning. The student's written answers to the question about the number of circles in Figure 10 and Figure 100 are in this sense very eloquent (see *Figure 2*).

When asked to write a message describing how to explain to another student what she or he should do in order to find the number of the circles in any figure, Doug relies on the aforementioned geometric meaning:

4. *Doug:* Each. . . . For each figure. . . . You take the number of the figure . . . of the . . . of the. . . . The number of the figure [balancing nervously back and forth on his chair] [. . .] [then, without balancing anymore, he says] let's say that the figure's number is three. You would say one plus three for the top row [moving his pencil in the air from left to right] and two plus three . . . [. . .] No, plus two for the bottom row [pointing with his finger at one of the figures] and plus one [pointing directly to one of the figures] for the top row . . . on . . . of the number . . . of the figure [stressing the words "on" and "of" by pointing his finger toward the table].

Note that Doug does not seem to be at ease with the idea of "any figure." Uncomfortable in this layer of generality, he expresses himself hesitantly, nervously moving on his chair. After the early unsuccessful attempts, Doug abandons this path to generality and, thinking metaphorically, uses Figure 3 as a scaffold. The concreteness of Figure 3 allows him to express the general intended computations. Yet, this crutch is only temporary, and as soon as Doug finishes explaining the computations based on Figure 3, the reference to a particular figure fades away (he even says "no," as if he were making a mistake). The additive actions of adding one plus three and two plus three, which referred to Figure 3, refer now to an *unspecified* figure. Doug talks about "rows" only, without further qualification, even though he points to a concrete figure ("No, plus two for the bottom row and plus one for the top row"). As a matter of fact, Doug avoids *naming* particular figures so as to elude ruining the generalizing process. He still does not find a suitable linguistic expression to talk about the general figure in this layer of generality. Because he still cannot name the general, he refers to it without words. He refers to it with a silent pause. Indeed, Doug's effort to avoid naming particular figures suggests that *the presence of the general is made apparent by its absence at the discursive level.* In other words, by omitting to name the referent, the referent becomes general.

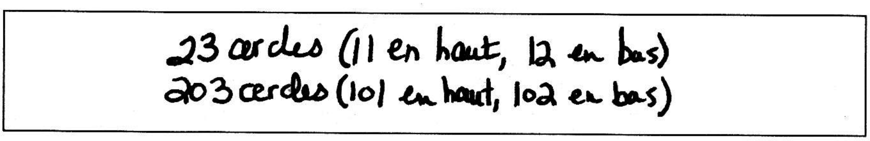

Figure 2. Students' answer indicating the number of circles in Figure 10 and Figure 100.[9]

[9] Translation: "23 circles (11 on the top, 12 on the bottom)" and "203 circles (101 on the top, 102 on the bottom)."

To understand the role of silence in the objectification of mathematical generality, it may be useful to consider the students' mathematical activity as an activity being written in a semiotic text whose signs include words and gestures. Actually, the semiotic text includes more than that: it also includes silences. Like in a piece of music, far from being accessorial, silence plays a crucial role: It is a constitutive part of the text. The same goes, we contend, for the students' mathematical text.

At the end, Doug succeeds in refining the referent through a subtle and important move: the term "any figure" is changed to "the number of the figure." From particular figures such as Figure 3, Doug talks now about "the figure," which is specified by its *number* on the sequence. Doug succeeds here in overcoming what has been termed elsewhere as the "positioning problem" (Radford, 2000, p. 250), that is, the nontrivial problem of referring to a nonspecific figure by the position the figure occupies in the sequence.

The previous students' utterances clearly illustrate the process of knowledge objectification through which they underwent to make the general apparent. The students' written text summarizes this process in a neat way (see *Figure 3*). Similar to the answer provided to the previous questions concerning the number of circles in Figure 10 and Figure 100, the written text carries the geometrical meaning of the operations, intimately related to the students' perception of the figure into two rows. However, although the mention of the rows was previously seen as complementary to the answer, and therefore was placed into brackets, now it becomes part of the answer.

The process through which the students have gone at the different stages of the problem is ultimately reflected in the algebraic formula (see *Figure 4*). More precisely, the answer given by the students still keeps track of its geometrical meaning. For the mathematics expert, brackets in this expression are unnecessary, but seem essential to the students, who do not expand the formula. For the students, the brackets are organizers of the way in which the formula tells us the story of

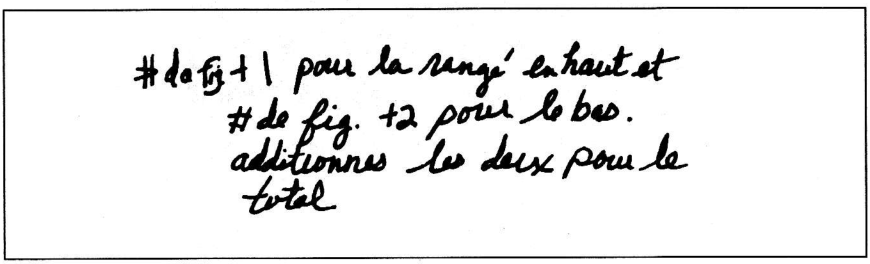
de fig + 1 pour la rangé en haut et
de fig. +2 pour le bas.
additionnes les deux pour le
total

Figure 3. Students' message indicating how to find out the number of circles in any figure.[10]

[10] Translation: "# of fig. +1 for the top row, and # of fig. +2 for the bottom. Add the two to get the total."

$$(n+1)+(n+2)$$

Figure 4. Students' algebraic formula indicating the number of circles in Figure *n*.

their mathematical experience. No further algebraic manipulation is then performed, for the brackets delimitate the computations made on the two rows of the figures.

Example 2—Perceiving with the Eye, Words, and Gestures

Example 2 comes from another group of three students: Jay, Mimi (sitting side by side), and Rita (sitting in front of them). The students begin counting the number of circles in the figures, realizing that it increases by two each time. Then, their attention focuses on the geometrical structure of the figures (see *Figure 5* for some corresponding gestures):

1. *Rita:* You have five here. . . . [pointing to Figure 3 on the sheet].
2. *Mimi:* So, yeah, you have five on top [she points to the sheet, placing her hand in a horizontal position, in the space in which Jay is beginning to draw Figure 4] and six on the . . . [she points again to the sheet, placing her hand a bit lower].
3. *Jay:* Why are you putting . . . ? Oh yeah, yeah, yeah, there will be 11, I think [he starts drawing Figure 4].
4. *Rita:* Yep.
5. *Mimi:* But you must go six on the bottom . . . [Jay has just finished drawing the first row of circles] and five on the top [Jay finishes drawing the second row].

Figure 5. Some gestures occurring in the lines of the dialogue.

Pointing to a specific part of Figure 3, which is given on the sheet, but referring in her speech to Figure 4, Rita makes a link between the two consecutive figures apparent. Through her indexical gesture, she is suggesting a quantitative and qualitative way to apprehend the figures: The former relates to the number of circles of the figure and the latter refers to the way the circles are disposed. This apprehension of the figure is easily adopted by Mimi, and properly described through the spatial deictics "top" and "bottom" (lines 2 and 5). In so doing, the students shift from blunt counting to a *scheme of counting* whose meaning is geometrical, as in Example 1. To perceive this scheme is a crucial step toward the general.

In line 2, Mimi's words are accompanied by two corresponding gestures. Through these gestures, Mimi accomplishes a number of functions: (1) she participates in the drawing process— by entering Jay's "personal space" she offers, in fact, guidance in carrying out the task; (2) she depicts the spatial position of the rows in an iconic way, and (3) she clarifies the reference of the uttered words. In line 5, Mimi does not make any gestures; rather, her *words* are perfectly *synchronized* with Jay's *action*, almost directing him in the action of drawing. In fact, to complete her sentence with the description of the second row, Mimi waits until Jay finishes drawing the first row of circles. We thus see that the general is being perceived through a complex interpersonal synchronization of eye, words, and gestures.

Later, the group work is interrupted by an announcement to the class about a forthcoming social activity. While Mimi and Rita pay attention to the announcement, Jay keeps on working, writing "23" and "203" as the answers for the question about the number of circles in Figure 10 and Figure 100, respectively. When the girls return to the task, they ask Jay for an explanation of his results (see *Figure 6* for some corresponding gestures):

6. *Mimi:* [Talking to Jay] I just want to know how you figured it out.

7. *Jay:* OK. Figure 4 has five on top, right? [with his pencil, he points to the top row of Figure 4, moving his pencil from the left to the right].

8. *Mimi:* Yeah . . .

9. *Jay:* . . . and it has six on the bottom [he points to the bottom row using a similar gesture as in line 7] [...]

10. *Mimi:* [Pointing to the circles while counting] 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11. [Pause] [...] Oh yeah. Figure 10 would have . . .

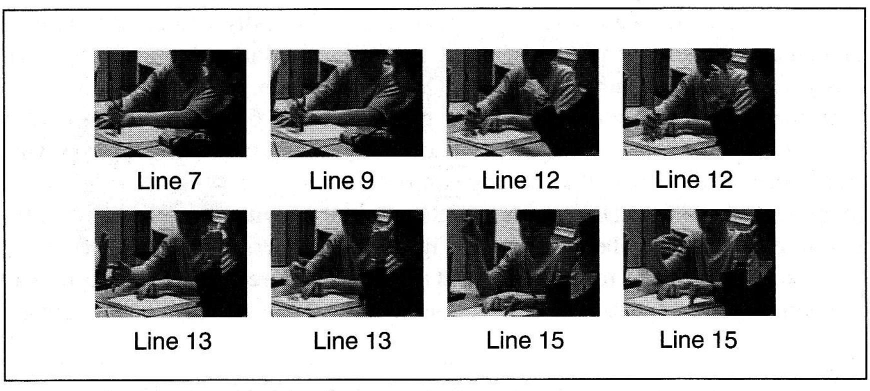

Figure 6. Some gestures occurring in the lines of the dialogue.

11. *Jay:* Ten. There would be like ...
12. *Mimi:* There would be 11 [she is making a quick gesture that points to the air. Jay is placing his hand in a horizontal position] and there would be 10 [she is making the same quick gesture but higher up. Jay is shifting his hand lower down] right?
13. *Jay:* Eleven [similar gesture but more evident, with the whole hand] and 12 [same gesture but lower].
14. *Mimi:* Eleven and 12. So it would make 23, yeah.
15. *Jay:* 100 would have 101 and 102 [same gestures as the previous ones, but in the space in front of his face].
16. *Mimi:* OK. Cool. Got it now. I just wanted to know how you got that.

Developing Mimi's initial idea, elaborated in lines 2 and 5, Jay attains a structural apprehension of the figure that guides him to solve the problem for Figure 10 and Figure 100. Moreover, in his explanation, he uses the same discourse genre as Mimi's: a discourse genre that interweaves word spatial deictics ("top," "bottom") with iconic and indexical gestures.[11] Through iconic and indexical gestures, Jay touches the two horizontal rows into which Figure 4 can be divided. Turning to Figure 10, Mimi (line 12) matches her utterance with two gestures that keep certain specific aspects of Jay's: having one gesture for each row, and their vertical shift. But whereas Jay's gestures point materially to the rows of Figure 4, Mimi's are made in the air (line 11). Indeed, Figure 10 is not in the perceptual field of the students, so new mechanisms of semiotic objectification have to be displayed. This, we suggest, is the role of gestures here. Of course, Mimi could have simply reached the answer using words. The fact that she did not, and that she used gestures, is right to the point that we want to make here: Gestures do not merely carry out intentions or information. They are key elements of the process of knowledge objectification (Radford, 2005b). This point becomes even clearer when the students address the question of Figure 100. The gestures are again made in the air, and this time at a higher elevation from the desk.

Semiotic nodes. In their path toward generality, students need to mobilize both language and gestures in a coordinated and efficient way. This coordination takes place in particular segments of the students' mathematical activity where knowledge is objectified. These segments of mathematical activity, characterized by the crucial coordination of various semiotic systems, constitute what we have previously termed *semiotic nodes* (Radford, Demers, Guzmán, & Cerulli, 2003).

In the present semiotic node, which goes from line 6 to line 16, the perfect synchronization of words and gestures allows the students to successfully cope with two intertwined aspects of knowledge objectification. The first one relates to the numerical, discrete, and linear nature of the semiotic activity required to answer the question about the number of circles in Figure 100 or to write an algebraic formula. The second one relates to the spatial and analogical nature of the semiotic activity to apprehend the objects given or represented two dimensionally or three dimen-

[11] Generally, a gesture is considered *iconic* if it resembles the semantic content of speech and *indexical* if it indicates (e.g., points to) objects and events in the concrete world (see McNeill, 1992).

sionally. To cope with the first of these aspects of knowledge objectification, it seems that students emphasize the recourse to language, and to cope with the second, they emphasize the recourse to gestures. These two aspects correspond to what Lemke (2003) termed the two fundamental types of meaning-making: "*meaning-by-kind*" or "typological meaning" (language) and analogical or "topological meaning" (motor gestures or visual figures). Our results suggest that the objectification of knowledge is underpinned by a complex dialectic relationship between these two kinds of meanings.

We shall come back to the previous point in the conclusions. For the time being, let us discuss the way the students tackled the problem of writing a message to explain how to find out the number of circles for any figure (see Figure 7 for corresponding gestures):

17. *Mimi:* Add. Add 3 to the number of the figure! [pointing to the results "23" and "203" on the paper].

18. *Jay:* No! [. . .]

19. *Mimi:* I mean like . . . I mean like . . . . You know what I mean, like, for Figure 1 you will add like...OK [pointing to Figure 1 of the sequence] it would be like 1, 1, plus 3; this [pointing to Figure 2] would be 2, 2, plus 3; this [pointing to Figure 3] would be 3, 3, plus 3.

As underlined by her gesture (line 17), Mimi seems to have observed that the number of circles in Figure 10 and Figure 100 ended with the digit 3 and inferred from it a key to look for a general method. This numerical hint leads her to a new apprehension of the figures, which she soon makes explicit and refines in order to convince Jay. Her explanation is carried out based on Figure 1, Figure 2, and Figure 3 (line 19) and through a number of important indexical gestures on the corresponding drawn figures. In fact, she makes three indexical gestures on each figure to render apparent a specific configuration to her group-mates. In the case of Figure 1, she points successively to the top-left circle, then the bottom-left circle, and finally she sketches a small

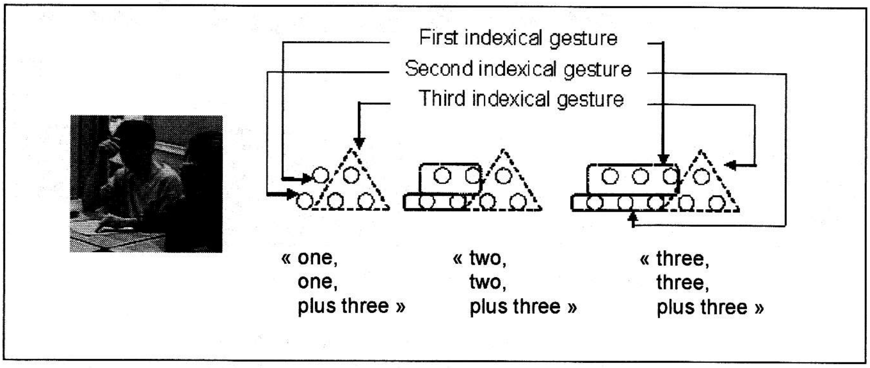

Figure 7. On the left, Mimi making the (first) indexical gesture on Figure 1. On the right, the new apprehension of the figures as a result of the process of knowledge objectification.

triangle surrounding the three left circles on the right (see *Figure 7*). Then, the same words-gesturing scheme is repeated for Figure 2 and Figure 3. Through gestures and words, Mimi objectifies a general structure in a dynamic way that corresponds to the third generalizing strategy mentioned in our theoretical framework.

Rhythm. In addition to gestures and words, a crucial element in the process of objectification is *rhythm*. Rhythm creates the expectation of a forthcoming event (You, 1994) and constitutes, as we shall see, a crucial semiotic device in making apparent the perception of an order that goes beyond the particular figures. As suggested in our theoretical framework, we propose to locate the genesis of generalization at the moment in which the particular stops being seen as such and the attention shifts to the apprehension of something else. As a result, in our account, the genesis of algebraic generalization entails the awareness that something stays *the same* and that something else *changes*. Thus, in order to perceive the general, the students have to make choices. They have to bring to the fore some aspects of the figures (*emphasis*) and leave some other aspects behind (*de-emphasis*).

To get a better idea of the manner in which the students emphasize and de-emphasize the various features of the figures through rhythm, we conducted a *prosodic* analysis of Mimi's key utterance in line 19 ("one plus one plus three" etc.).[12] Our prosodic investigation, carried out using the voice analysis software Praat (Boersma & Weenink, 2006), focused on the temporal distribution of words and word intensity. In the top part of *Figure 8*, the waveform shows a visual distribution of words in time; the curve on the bottom shows the intensity of uttered words (measured in dB).

The waveform allows us to neatly differentiate two kinds of rhythms: *within* and *between* figures. The first type of rhythm, generated through word intensity and pauses between words, helps the students to make apparent a structure *within* each figure. In conjunction with gestures and words, this rhythm organizes the way of

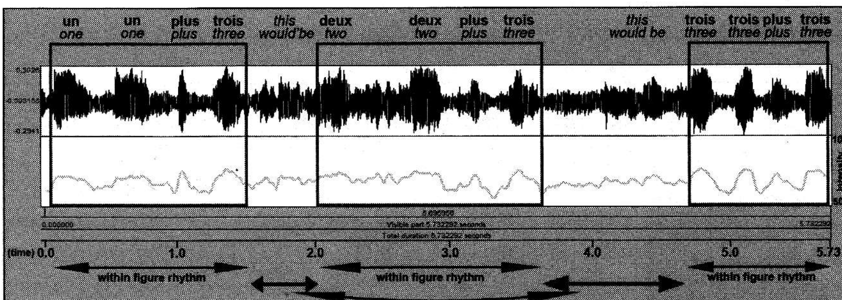

Figure 8. Prosodic analysis of Mimi's utterance.

[12] *Prosody* refers to all those features to which speakers resort in order to mark the ideas conveyed in conversation in a distinctive way. Typical prosodic elements include intonation, prominence (as indicated by the duration of words), and perceived pitch. Some works on prosody include Bolinger (1983), Goodwin, Goodwin, and Yaeger-Dror (2002), Roth (in press a, in press b), and Selting (1994).

counting. The other type of rhythm appears as a result of generated "transitions" between the counting processes carried out by Mimi when she goes from one figure to the next. These "transitions" are generated at two different levels. At the *lexical level*, Mimi uses the expression "this would be," whose semantic value indicates the hypothetical nature of the emerging counting schema. At the *temporal level*, this same expression allows Mimi to accomplish a separation between the counted figures.

Table 1, derived from Praat prosodic analysis of Mimi's utterance, provides us with a precise idea of the *within* and *between* figures rhythm. The data in row 3 indicate that $a_{33} < a_{32}$; $a_{38} < a_{37}$, $a_{3\ 13} < a_{3\ 12}$; thus the time elapsed between the additive preposition "plus" and the uttered number prior to it is *consistently shorter* than the elapsed time between the two uttered numbers before "plus." It is also interesting to note that, in the case of Figures 1 and 2, the elapsed time between "plus" and the following word is shorter than the time between "plus" and the uttered number before it (i.e., $a_{34} < a_{33}$, $a_{39} < a_{38}$). Hence, the preposition "plus" does not merely play the role of an arithmetic operation. It plays a key prosodic role in emphasizing and de-emphasizing aspects of the figures.[13]

The prosodic analysis helps us understand the students' mechanisms of emphasizing and de-emphasizing features of the figures and sheds light on the sophisticated ways in which rhythm is used as a semiotic device in the students' phenomenological apprehension of the general. This is why it may be worthwhile to think of algebraic generalization as a process similar to the creation of a sculpture or a painting. Some elements are brought to the fore; others are left in the back. Both are important, for it is through their *contrast* that one notices what has to be noticed (Radford, 2002b). Rhythm accentuates this contrast in the students' semiotic activity. It heightens the relief of the constant and the variable in the act of generalization.

Returning to Example 2, the students have now paved the way for tackling the problem of expressing the general in the alphanumeric semiotic system of algebra. However, the path towards the symbolic expression of generality passes through an intense use of examples:

20. *Mimi:* The number of the figure like . . . we'll say that the figure is 10 [gestures with an open hand as to indicate a row on the desk], you'll have 10 dots [similar gesture on the desk] plus 3 [sort of grouping gestures a bit more to the right and to the bottom, on the desk] right? [Pause] No. . . .

21. *Jay:* No.

22. *Mimi:* You double the number of the figure.

23. *Jay:* Ten plus 10 [pointing to the sheet]

---

13 Even if the temporal distributions of words for the first two speech segments ($0.157 \leq t \leq 1.348$; $2.161 \leq t \leq 3.463$) are quite similar to that of the third speech segment ($4.793 \leq t \leq 5.633$), as suggested by the waveform, the data indicates that the duration of the latter (0.840 s) is shorter than the duration of the former (i.e., 1.191 and 1.302; see row 4). Because the students did not need to go beyond Figure 3 to objectify the counting schema, one of the reasons may be that the generalization was achieved during the investigation of the first two figures, the third figure hence playing the role of verification. This particular status of Figure 3 is also suggested by the fact that $a_{4\ 10} > a_{4\ 5}$ and may explain why the intensity of the words uttered here is generally higher than the intensity displayed in talking about the first two figures (see row 1).

Table 1
Intensity and Time Data of Mimi's Utterance

	One	One	Plus	Three	This would be	Two	Two	Plus	Three	This would be	Three	Three	Plus	Three
1. Intensity (dB)	76.58	77.52	80.04	81.93		78.72	78.61	77.44	80.66		81.73	81.24	77.94	80.38
2. Time(s)	0.157	0.665	1.025	1.348		2.161	2.798	3.158	3.463		4.793	5.116	5.347	5.633
3. Time(s) between consecutive words		0.508	0.36	0.323			0.637	0.36	0.305			0.323	0.231	0.286
4. Total time(s)		1.191					1.302					0.840		

Note. Rows 1 and 2 show the intensity and time position of words (both measured at the middle of the duration of the word). Row 3 gives the elapsed time between consecutive words. Row 4 gives the total time of the speech segments.

24. *Mimi:* [Interrupting] So it will be 20 dots plus 3 [pointing to the number 23 on the sheet]. You double the number of the figure and you add 3, right? So Figure 25 will be 50 . . . 3. Right? That's what it is.

The students' dialogue suggests the presence of an intense dialectics between the concrete examples and the incipient general. In this dialectics, the concrete accomplishes a twofold function: First, it endows the students with the means to "craft" the general; second, it provides the students with a way to validate their general statements (see also Arzarello, 2000 on ascending and descending control).

Through words, gestures, pauses, and rhythm, the students organize and reorganize the concrete and move up to new layers of generality. Drawing from this structuring function of the concrete examples, the students end up contracting the generic term "the number of the figure" to a more concise expression—"the figure"—that will finally appear represented by the letter *n* in the algebraic formula:

25. *Jay:* You double the number of the figure, then you add 3 [. . .]
26. *Mimi:* So it's always the number of the figure times 2, 75 times 2, then you add 3.
27. *Jay:* Figure times 2 plus 3.

During this dialogue, Jay (under the close supervision of Mimi) is writing the message shown in *Figure 9*, as required by the task. The text is a mixture of mathematical symbols and terms in natural language. Undoubtedly, the comma is the most interesting element: It translates, in a written form, the spatial and temporal characteristic of one crucial distinctive event objectified in the course of the students' mathematical experience, namely, the distinction between the constant and the changing elements in the figures, as the students perceived them.

At the end of the mathematics activity, what became noticed through a complex coordination of hands in the space (line 20); rhythm and pauses in speech (line 19);

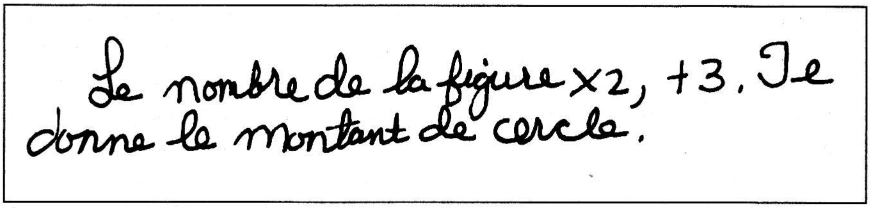

Figure 9. The students' text.[14]

and nouns, deictics, and adverbs (e.g., lines 23 and 26) reaches now an extremely concise formulation. Indeed, soon after writing the above text, Jay provides the formula shown in *Figure 10.* The "space" to be occupied by each one of the five symbols that now constitutes the formula (i.e., "n," "×," "2," "+," and "3") was progressively prepared by the students' previous joint mathematical experience. In other words, from an ontogenetic viewpoint, the students' alphanumeric algebraic syntax is a contracted expression of a vivid mathematical experience previously objectified by an impressive coordination of diverse semiotic means of objectification. Thus, the symbolic letter *n* is the "semiotic contraction" (Radford, 2002b) of the "number of the figure" that has been so often quoted before, either directly or by means of examples, and the whole formula is the crystallization of a semiotic process endowed with its situated history. It is a history in which each sign acquired a distinctive meaning and which may explain why the students do not simplify the formula into the more standard expression $2n + 3$. It seems that the formula still contains the remnants of the *narrative* side of algebra (Radford, 2002a), where signs play the role of narrating a story and where the formula has not yet reached the autonomy of a detached symbolic artifact. The students' formula is still an index pointing to the students' vivid mathematical experience.

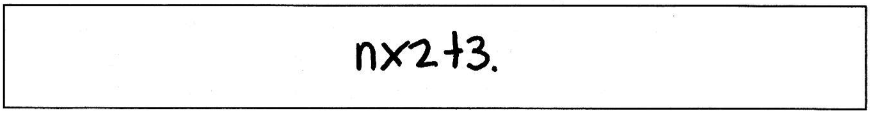

Figure 10. Jay's formula.

SYNTHESIS AND CONCLUSION

In this article we dealt with the genesis of algebraic generalization of patterns. Drawing from previous research on the topic (e.g., Kieran, 1989; Lee, 1996; Lee & Wheeler, 1989; MacGregor & Stacey, 1995; Mason, 1996; Mason et al. 1985), we elaborated the idea of generalization as a shift of attention that leads one to *see*

14 "The number of the figure × 2, + 3. It gives you the amount of circles."

the general *in* and *through* the particular. Our theoretical choice led us to suggest that, from an ontogenetic viewpoint, a process of generalization can be considered as a phenomenological process, more precisely as a process of *objectification*, that is, a process of making something *apparent*. Since conceptual objects or conceptual relations cannot be seen as we may see a chair, objectification relates here to the students' possibility to become conscious or aware of something. In terms of Leont'ev's Activity Theory, this awareness is the result of an unfolding common activity out of which a common relation to cultural reality is forged. This is why this awareness cannot be seen as the mere result of a passive reception of sensual data. As Piaget remarked, the dawn of consciousness "is not merely a sort of interior illumination" (Piaget, 1976, p. vii). On the contrary, it is a subtle, active, creative, imaginative, and interpretative social process of gradually becoming aware of something through the use of words, gestures, mathematical symbols, graphs, artifacts, etc.—in short, through *semiotic means of objectification.* Within this context, in order to retrace the students' grasping of generality, we focused on the phenomenological import of the diverse semiotic means of objectification (language included) to which the students made recourse in transcending the particular. Our microgenetic analyses underlined the role of some of the semiotic systems that the students use to accomplish this. Words equip them with embodied cultural conceptual categories (such as "top," "bottom") to carry out the objectification of the general. The kinesthetic motion of the hand in gesturing provides the students with a kind of visual and sensory-motor representation of the general figure and helps the students to *imagine* it (e.g., metaphorically, generatively, or dynamically). Rhythm makes apparent the perception of an order that goes beyond the particular figures and ensures the synchronization of the topological and typological aspects of knowledge objectification emphasized by words and gestures.

From an educational viewpoint, what can be gained by formulating the problem of generalization in this way? To answer this question, let us first state one of the principles of our semiotic-cultural approach: We believe that the students' mathematical thinking cannot be fully captured by only paying attention to what the students write (e.g., their formulas). As our results suggest, in order to think mathematically, the students use, in fundamental ways, other semiotic systems than those confined to the written sphere. Effort should hence be made to understand these other semiotic systems underpinning the students' mathematical experience.

In saying this, we do not want to minimize the role of the written in cognition. Rather, our plea is intended to call attention to the variety of semiotic resources and the complex coordination that the students make of them—a *semiotic symphony* played in the halls of semiotic nodes, so to speak. It seems to us that the cognitive value of these other semiotic systems still remains marginalized or simply ignored in our contemporary mathematics curricula and textbooks. Mathematics still remains understood as something made around the paper and the pencil. The awareness that doing mathematics is much more than just writing it may lead us—teachers and mathematics educators—to recognize the value of those other semiotic systems in the day-to-day teaching of mathematics and in students' development of mathematical thinking.

Naturally, we are not suggesting abandoning the alphanumeric algebraic semiotic system. Generality, we contend, can be variously expressed. The previous remark leads us to the question of the cognitive difficulties that students encounter when asked to express the general through mathematical formulas. Our microgenetic analyses shed some light on this problem. As implied by our analysis, the students' algebraic symbolism conveys a meaning deeply rooted in their mathematical activity as mediated by the diverse semiotic means of objectification. For instance, the spatial deictics "top" and "bottom" impressed their mark in the syntax of the formula "$(n + 1) + (n + 2)$." Rhythm, in an even subtler manner, also impressed its mark in the message produced by the second group of students, where it appeared under the form of a comma (see *Figure 9*).

Some of the cognitive difficulties that students have to overcome in their learning of the algebraic alphanumeric symbolism are related to the formidable semiotic contraction imposed by the fact that what they have to express now has to be done with a very few signs (three or four, as in the final formulas). The semiotic contraction entails a *compaction* of meaning, which may be noticed when we compare Doug's initial sensory-motor sense of expressing generality to the formula "$(n + 1) + (n + 2)$." There is nothing that would insinuate *a priori* a natural link between Doug's gestures and the formula. Through bodily actions, the students find a way to emphasize and de-emphasize events; natural language offers the students a rich array of deictics, adverbs, verbs, and so on; even the recourse to silence in an utterance helps the students to make something noticeable. The few letters and other characters that constitute the standard set of signs of contemporary school algebra do not have the same expressivity and signifying scope. Their mode of signification is completely different. As Benveniste (1985) reminds us, semiotic systems are not synonymous. Each one of them offers a different form of expressivity and objectification. Reaching the algebraic alphanumeric formula is therefore not the mere business of translation, for total translation is simply impossible. It is rather a matter of becoming aware of how abstract relations are said and expressed through different signs. Hence, it may be advantageous to see the students' path toward the production of the alphanumeric formula as a progressive refinement of a general idea which, by being objectified in and through diverse semiotic means of objectification, leads to a deeper awareness of general mathematical relationships. And it may be equally advantageous to understand that mathematical generality can be variously expressed and that some students will find it more difficult to express it in some ways than in others.

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