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On the role of representations and artefacts in knowing and learning

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Abstract This article provides a critical commentary on the concepts of representation and digital artefacts in Morgan and Kynigos's article of this Special Issue. To set the context, in the first part, I examine some of the tensions that arose during discussions through the 1980s and 1990s about representation in mathematics education research. Then, I comment on the conceptual differences between Morgan's and Kynigos's approaches. These differences point to different epistemological assumptions that lead to different conceptualizations of artefacts in learning processes. In the last part, I argue that Morgan's and Kynigos's approaches have the merit of moving the discussion about representations to new theoretical horizons. I suggest, however, that a discussion about representations and digital artefacts requires a thematized account of the manner in which the phenomenological artefact- and representation-mediated knowledge produced by students in the classroom relates to the target cultural mathematical knowledge. Such an account, I contend, requires an explicit ontological conception of knowing and knowledge. I conclude the article with an example in which knowledge is considered as codified movement and knowing as the event of its enactment in concrete practice. Within this Hegelian materialist viewpoint, representations are neither predicated in terms of an adequacy between ideas and their representations nor as heuristic devices in meaning making processes. Representations are rather an integral part of the activity of knowledge presentation.

Keywords Representations · Semiotics · Knowledge · Knowing · Learning · Abstract · Concrete · Hegel's dialectical ontology

1 Introduction

Like all species on earth, we live immersed in a material world. However, in contrast to the other species, the material world in which we live bears the imprint of our human existence. The world of nature has been transformed into a species-made niche. And it is in this artificial niche that we come to know, think, act, and feel. It is not surprising then that during the early twentieth century, anthropologists, psychologists, philosophers, and other scholars considered

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humans to be a distinctive *technological* species. Although the use of tools (e.g., for nut cracking) is well-documented in wild chimpanzees and capuchin monkeys (Hirata, Morimura, & Houki, 2009), *Homo sapiens* is the species that has the greatest capacity to create tools and signs to transform nature, to communicate, and to codify knowledge in enduring cultural manners.

Leroi-Gourham suggests that, from an evolutionary viewpoint, there is an entanglement between speech and tools whose roots go back to a bipolar technicity found in many vertebrates. That bipolar technicity culminated in anthropoids possessing two functional pairs: hand/tool and face/language. The millenarian progressive specialization and liberation of the hands—subjected initially to locomotion, gesture, and the grasping of things—evolved towards greater technical possibilities of tool creation and use. In turn, tool creation and use led to a greater degree of freedom of operation of the facial organs and the creation of possibilities for new linguistic and symbolic capabilities. There is, Leroi-Gourham argues, a phylogenetic biological and social correlation in chimpanzees, humanoids, and humans between prehistoric tools and prehistoric language (Leroi-Gourham, 1993; see also Corballis, 2003; Malafouris, 2012).

In dealing with questions of representations and artefacts, the articles in this Special Issue run upfront into a problem that is older than our own species. The articles remind us that, as old as it is, the problem of the epistemic and cognitive role of tools and signs continues to haunt us—perhaps now more than ever by virtue of the unprecedented technological dimensions of contemporary life. And so here we are still trying to make sense of how we think and learn with and through signs and artefacts.

In this article, I provide a critical commentary on the concepts of representation and digital artefacts in Morgan and Kynigos's article of this Special Issue. I start my journey by discussing some of the tensions that arose during mathematics education research of the 1980s and 1990s. Then, I comment on the conceptual differences between Morgan's and Kynigos's approaches. In the last part I argue that Morgan's and Kynigos's approaches have the significant merit of moving the discussion about representations to new theoretical horizons. I suggest, however, that a discussion about representations and digital artefacts requires a thematized account of the manner in which the phenomenological artefact- and representation-mediated knowledge produced by students in the classroom relates to the target cultural mathematical knowledge. Such an account, I contend, requires in turn an explicit ontological conception of knowing and knowledge.

2 Representing knowledge

The problem of knowledge representation has been one of the most investigated and most discussed problems in mathematics education. And it has also been one of the most controversial ones. In a book edited by Claude Janvier (1987)—which came to be considered the book *par excellence* in matters of representation in the 1990s and remains a landmark in its field—Kaput reminds us that all discussion about representation needs to distinguish between two entities: the representing world and the represented world. In the same book, however, von Glasersfeld argues forcefully that we need to remain vigilant about what we mean by the represented world. What is it that representations represent? He goes on to argue that representations cannot be reduced to a problem of communication, nor can they be reduced to a supposed matching between something out there and the representations that allegedly represent it. While the first view reduces knowledge to an improbable straightforward flux of information, the second view, von Glasersfeld contends, rests on the problematic assumption of a correspondence between reality and its representation, where truth "inevitably becomes the perfect match, the flawless representation" (1987, p. 4).

The representational view that, directly or indirectly, inspired mathematics education discussions in the 1980s and 1990s and against which von Glasersfeld cautions us belongs to a longstanding philosophical tradition (for a recent overview, see Lassègue & Visetti, 2002; Havelangue, Lenay, & Stewart, 2003). This tradition affected not only mathematics education but also other disciplines like epistemology, psychology, and linguistics. In fact, it affected all those disciplines in which what is at stake is the problem of knowledge, knowing, and thinking.

It was Leibniz who in the seventeenth century best articulated the representational conception of knowledge. The foundational assumption is a harmonious agreement between the order of things and the order of ideas. Leibniz articulates it as a *correspondence* between language, thought, and the conceptual structure of reality. In *What is an idea?* (a paper written in 1676), he says: "there must be something in me *which not only leads to the thing but also expresses it*" (Leibniz, 1951, p. 281; italics in the original). The assumption is that the human mind and reality are structured in such a way that there is a link between the human capacity for language and symbolization and the human capacity to understand reality. It is within this context that Leibniz envisioned a language whose signs or characters (and the combinations of them) would result in clear thinking and an efficient manipulation of syllogisms and judgments. This language is what he termed the *universal characteristic*. As Rutherford puts it,

> the universal characteristic would enable us to construct linguistic characters which are transparent representations of intelligible thoughts, something the signs of natural languages typically fail to be, and to reduce logical reasoning to a mechanical procedure relying solely on the substitution of formal characters. (Rutherford, 1995, p. 225)

Considering mathematical signs as a kind of model, Leibniz endeavored to find a method to assign characters or signs to our thoughts so that they would be combined or operated unambiguously—"by a species of calculus" (Leibniz in Couturat, 1961, p. 155)—as in arithmetic and algebraic calculations.[1]

This conception of knowledge has not ceased to affect us (see, e.g., Caveing, 2004; Colyvan, 2001). It is perhaps this emphasis on calculation that Vergnaud, in his enlightening conclusion to Janvier's famous book, finds unconvincing in Kaput's theory of symbol use in mathematics. For Vergnaud, a good theory of symbol use cannot succeed without a good theory of reference, which should include reference to "reality" and our actions in it:

> A good theory of reference is needed and I am struck by the fact that none of the contributors [to Janvier's book] speaks in terms of situations and actions in situations. When the word 'referent' is used, by Kaput for instance, it is mainly in the sense of objects and properties, or in the sense of symbolic systems as organized sets consisting of signs, syntaxes, and semantics. (Vergnaud, 1987, p. 230)

Yet, as good a Kantian as he was, von Glasersfeld shows us that the question of "reality" is precisely what we cannot take for granted. Notwithstanding Leibniz and the enthusiastic computationist paradigm that followed, there is nothing that vouches for a compatible structure between signs and things, or between representations and the represented things.

The Kantian argument on which von Glasersfeld draws is that what we represent are not the things as they truly are, as they are in reality, but as they are given to us in the course of our worldly experiences. In other words, what we humans can know are the objects of our experience, not the objects as such. We do not have access to the conceptual triangle as an

[1] For a more detailed discussion of Leibniz' representational view of knowledge, see Radford (2013a).

ideal entity, but only to particular concrete triangles (e.g., the triangle that we draw on a sheet of paper or that we see on a screen). According to this argument, what is graspable by the human mind is what derives from human sensuous and concrete experience; as a result, the realm of things as they truly are remains out of our reach. We cannot escape the "startling" consequence, Kant concluded, that we do not and cannot have knowledge of things as they really are, i.e., things in themselves (Kant, 2003, p. 23). This is, in a nutshell, Kant's argument about the limits of human reason. This is why, for von Glasersfeld, there would not be valid reasons to accept Vergnaud's empirical argument that "there are parts of the real world that are adequately represented at the signified level as can be proved by the existence of some efficient action in all individuals" (1987, pp. 229–230). Vergnaud would be conflating the phenomenological and the transcendental realms. To avoid such pitfalls, von Glasersfeld suggested that the concept of representation should be revisited so that "the goodness of knowledge" should no longer be "predicated on likeness or representation" (von Glasersfeld, 1987, p. 5).

Yet, even the most cursory view of mathematics practice (be it the practice of professional mathematicians or the one of students in schools) makes patent the omnipresence of representations. As Kaput put it, "*the idea of representation is continuous with mathematics itself*" (1987a, p. 25; italics in the original). Now, if mathematics is "inherently representational" (Kaput, 1987b, p. 159), the question is: What do the signs of mathematics activity represent? Von Glasersfeld's (1987, p. 5) answer is: human concepts that, instead of referring to something transcendental, refer to the individual's endeavors, for knowledge in this neo-Kantian context becomes a question of *bricolage*, that is *viable* knowledge that is produced by individuals as they try to idiosyncratically adapt to the constraints of the outside world.

This pragmatic interpretation of Kant's philosophy that von Glasersfeld elaborates in great detail in other works (e.g., von Glasersfeld, 1995) removes the central role that reason played in Kant's epistemology. Indeed, if there is something distinctive in Kant it is how he resorts to reason as a *regulative* idea of human actions. As Kant conceived it, reason is that intellectual power that yields knowledge with rigorous, necessary, and universal validity. It does not have anything to do with a heuristic process of discovery or viability. Although reason would remain empty without the material and sensuous contents that are derived from experience, for Kant, reason is the *pre-condition* of experience, not its result. Kant defined knowledge "in terms of a validity furnished by a priori elements without which there could be no experience at all" (Smith, 1973, p. 446). In the 1920s, we find young Piaget already moving away from the rationalism that weighs on Kant's epistemology. Piaget said: "Experience and reason are not two terms that we can isolate: Reason regulates experience and experience adapts reason" (Piaget 1924, p. 587). For Piaget, as for von Glasersfeld some years later, an account of human reason has to give up Kantian a priorism. But while Piaget's account—based as we know on a conception of the human mind as governed by logico-mathematical operations—still remained infused with rationalism, von Glasersfeld made every possible attempt to overcome it. To accomplish this, he transformed Kant's principles and ended up offering a *pragmatic* and *subjectivist* view of knowing that, in line with a form of theorizing that goes back to the Enlightenment (Radford, 2012), focuses on the individual's actions and experiences: "the compound of experiential elements that constitutes the concept an individual has associated with a word cannot be anything but a compound of abstractions from that *individual's experience*" (von Glasersfeld, 1987, p. 7; emphasis added).

Because of its emphasis on knowing as personal experience, von Glasersfeld's elaboration of Kant's epistemology leads to a problem that is of extreme importance from an educational perspective—the problem of communication. If individuals construct their own ideas out of their own personal experiences, can they really communicate their ideas to someone else? The answer is not complicated, nor is it surprising. They cannot. In communicating, von

Glasersfeld argued once in an interview, "you interpret what I say according to your own experiences and not according to my experiences" (Pitasi, 2001). The argument, let us note *en passant*, rests on a conception of the individual as a sovereign entity that is the origin and sole source of meaning and intentionality—an individual that somehow manages to live unaffected and insulated from the deeds and meanings of others. However, if communication is not to be pure noise, it nevertheless requires that something common be *shared*, at least to some extent. Now, if one wants to remain coherent with the theoretical neo-Kantian principles, one cannot assume that, in their communicative acts, individuals really share the same things. This would be a gratuitous hypothesis. What can be assumed, by contrast, within the neo-Kantian principles, is that individuals *take* things as shared. Adhering to the constructivist epistemology, Kaput (1987b, p. 176) acknowledged that "direct knowledge of an external world is an epistemic impossibility," yet tried hard to argue, without fully resolving the tensions embedded in such an endeavor, for the possibility of genuinely shared symbols.

As we can see from this short account, Janvier's (1987) seminal book has the great merit of exposing some of the tremendous problems that mathematics educators have been led to ponder when dealing with questions of representation. The problems are located at the junction of ontology and epistemology. There is, on the one hand, the delicate problem of the *nature* of those objects to which the signs we use refer. This is the ontological problem. There is, on the other hand, the no less delicate problem of *how we come to know* through signs. This is the epistemological problem. Nominalism offers some strategies to solve both problems. Its formal variant argues that there is nothing to be represented. Mathematical objects *are* the signs. This was the position advocated by Heine (1872), who claimed that numbers are not represented by signs: they *are* tangible signs (see Frege's (1950) paper for a critique). Another strategy, advocated by some postmodernist thinkers, consists in arguing that there is nothing behind a sign: signs refer to signs in a perpetual movement of referential chains. We live in a world of unlimited semiosis, that is, in an insatiable commerce between signs. For others, these nominalist strategies are but a token of the disenchantment that resulted from the way Kant and other idealists conceived of the problem of the ideal world (Eagleton, 1996). As previously noted, while Kant conceived of reason as a source of ideas and principles, he considered it to have no cognitive reach beyond the realm of sensuous concrete experience. As a result, cognition knows and can know appearances only (Smith, 1973; Strawson, 1966). Hegel criticized Kant for the insurmountable separation that he drew between human reason and the realm of real conceptual things (Hegel, 1977, 1978). He offered a solution in which the conceptual realm is not transcendent in Kant's idealist sense. Transcendence is overcome by a dialectical entanglement between the concrete and the abstract or between the singular and the general. In Hegel's epistemology, the concrete and the abstract are thematized as relationships between potentiality and actuality, and knowledge is no longer predicated in terms of *adequacy* but of *movement*. I shall come to this point later. In the next section, I comment on the manner in which the question of representations appears in the articles in this Special Issue.

3 Digital representations

The articles in this Special Issue revolve around the European ReMath project: Representing Mathematics with Digital Technologies. From the outset, the question of representations is hence invoked. What is distinctive of this project is the confluence of several research teams to tackle a general educational problem, making salient the heterogeneity of the teams' theoretical approaches in the course of a reflexive endeavor to search for some common ground—a shared framework about representations. As Morgan and Kynigos put it,

> Our agenda with respect to representations was part of an important general aim of the project: to attempt to build an integrated theoretical framework that would allow a deeper and more productive conversation between researchers with different theoretical starting points. (Morgan and Kynigos, this Special Issue)

The emphasis on digital representations makes the problem even more interesting—and, of course, more difficult, as the question goes beyond representation and reaches the domain of technology.

My intention is to shed some light, from a semiotic and epistemological perspective, upon the problem of the conceptualization of digital dynamic pedagogic artefacts. To do so, I shall focus on Morgan and Kynigos's article, which brings together two theoretical perspectives involved in this part of the ReMath project and offers an overview of the way the teams dealt with the idea of representation. The theoretical approaches featured in the article are those of the London Institute of Education (IOE), represented by Morgan, and the one of the Educational Technology Laboratory (ETL) of the University of Athens, represented by Kynigos.

The authors start their article by clarifying the respective concepts of representations, alluding, although unproblematically, to the question of the relationship between representations and the mathematical concept that these representations represent. Although both perspectives draw on constructionism, they attend to different aspects of representations and their use.[2] A common point is the idea that representations do not have invariant meanings. Representations are rather thought of as "artefacts" imbedded in processes of meaning production.

Both perspectives emphasize the varied nature of representations, for example, spoken, written, diagrammatic, gestural, or other forms of representation underlying communication. At a certain level of generality—at the level of *what* counts as representation—both approaches seem to be in agreement. But when we move to the level of *how* representations intervene, some differences become noticeable. The IOE perspective puts emphasis on the idea that representations acquire their meaning in their context of use, in interactions occurring within social practices. The IOE perspective further suggests that this meaning is influenced by the resources that participants bring to the interaction. Naturally, this contextual conception of representations shapes the kind of data on which the perspective focuses and the data analysis that is expected: within the IOE perspective, meaning-making processes are examined through the interplay of various representations and multimodal analyses of representations as a whole. The ETL perspective focuses rather on representations as artefacts to which individuals resort to tinker: "representations are not seen simply as objects to which some kind of meaning is attached but as artefacts for tinkering with" (Morgan and Kynigos, this Special Issue). Without denying the contextual and interactionist nature of representations and their meaning, the ETL perspective emphasizes the heuristic potential of representations and introduces the idea of "complexity" of representations. Drawing on Edwards' (1995) work, representations, the ETL perspective reminds us, have *structure* and *functionality*. Structure is linked to the affordances

[2] Constructionism is a neo-Kantian theory of learning based on Piaget's epistemology. As Resnick describes it,

> Constructionism is based on two types of 'construction.' First, it asserts that learning is an active process, in which people actively construct knowledge from their experiences in the world. People don't *get* ideas; they *make* them. (This idea is based on the *constructivist* theories of Jean Piaget.) To this, constructionism adds the idea that people construct new knowledge with particular effectiveness when they are engaged in constructing personally-meaningful products. (Resnick, 1996, p. 2; emphasis in the original)

and possibilities offered by the representations, while functionality refers to the manner in which representations are used by the individual.

How can we explain the difference in emphasis between the IOE and the ETL approaches? To understand these differences, the authors suggest that we have to inquire into the theoretical conceptions conveyed by the perspectives. They mention three: (1) the conception of cognition and its concomitant concept of learning, (2) the role attributed to representations in the learning process, and (3) the concept of meaning. I comment on these ideas in the next subsections.

3.1 Learning and cognition

From the IOE perspective, learning happens in social interaction and can only be predicated at a social level: "the IOE's perspective only allows one to speak of learning in terms of changes in patterns of interaction, without any move to take such changes as evidence of changes in individual cognition" (Morgan and Kynigos, this Special Issue). From the ETL perspective, learning happens also in social interaction, but it can be genuinely predicated on individuals through the manner in which they generate meanings and their meanings evolve through social interaction.

Why is it that the IOE perspective considers changes in pattern interaction as *evidence* of learning but not as evidence of changes in individual cognition? Is it because sound inferences cannot be drawn from the social plane of interaction to the cognitive plane of the individual? Or is it because learning is considered to be *intrinsically* social and hence inferences from the social to the individual planes would be vacuous? In the first case, the problem would be of a *methodological* nature. It would make sense to talk about learning from an individual perspective, but the theory would not have the analytic conceptual and methodological tools to investigate it, given that the focus of attention is in the *interaction* and the manner in which individuals employ multimodal representations in *interactional* settings. In the second case, the problem would be of an *ontological* nature. The concept of learning would be assumed to be *social* so that the question about individual cognition and learning would not be really meaningful. Cognition would be interactional through and through. The article is not clear in this regard. However, there are reasons to believe that the problem for the IOE perspective is rather ontological: later in the article we are told that "The IOE approach conceives of learning as changes in patterns of interaction rather than as intra-personal changes" (Morgan and Kynigos, this Special Issue). The ETL approach, by contrast, influenced in a greater manner by constructionism, adopts a person-centered perspective and investigates how the person under consideration produces meanings in interaction with others persons and digital artefacts. The ETL approach does not seem to find problematic the transitions from the social to the individual and vice versa: "The ETL perspective discusses the extent to which a representation is conducive to the generation of meanings." The bi-directional transitions between the social and the individual, however, are not simple to account for. It requires a theoretical description of interaction and how it relates to learning and cognition (Radford, 2011). How should we interpret communication and interaction within the ETL perspective? If we go back to the neo-Kantian principle that we discussed earlier, we realize how von Glasersfeld had to be careful in describing cognition as an individual phenomenon and its implications on communication and interaction. We may not agree with the manner in which he transformed or adapted Kant's principles; yet, his account is perfectly sound and coherent. For the ETL perspective, is social interaction a simple space for individuals to catalyze and refine their personal meanings? Or is it something more complex, where (for instance) the social and the individual become entangled to a point where an effort to disentangle them is no longer possible? Unfortunately, the article is not very clear about where the ETL perspective places itself in this respect. However, the analyses of the students' actions with a digital microworld (MoPiX) give sometimes the impression that the ETL perspective leans towards something approximating the second case. Yet, at other times, it overtly ascribes to the cognitive predications of first-person learning of constructionism (see, e.g., the end of the last section of the article).

Be it as it may, the fundamental difference concerning the concept of cognition and learning in the IOE and ETL approaches brings in, as we can expect, interesting consequences concerning the role attributed to representations.

3.2 The role attributed to representations

Indeed, even if both approaches attend to a similar range of representations (e.g., the mathematical formal symbolism, graphs, etc.), they differ in their understanding of *why* and *how* individuals resort to them. The theoretical principles of a theory or perspective serve indeed as an orienting vector not only to produce and select data but also to *interpret* them with the end of answering research questions (Radford, 2008a). Thus, the microworld MoPix, designed by the IOE team, was intended to provide a multi-semiotic environment "with rich potential for making meanings drawing on multiple resources" (Morgan and Kynigos, this Special Issue) that would prove conducive to exploring how students build multimodal mathematical meanings in interaction. "The focus of the IOE team's research is on exploring the kinds of meanings facilitated by the multiple resources available to students" (this Special Issue). Behind such a view is a social semiotic conception of thinking and learning. By contrast, since the ETL perspective assumes that learning happens as people tinker with representations, the ETL team focused on the heuristic view of representations, the meanings that students ascribed to them, and the relationship between these meanings. This is why the ETL team displays an unambiguous interest in the symbolic formalism of MoPiX and the access it provides to the "deep structure" of the microworld. The IOE perspective, by contrast, rests on a more pragmatic stance; the symbolic formalism is *one* of the various semiotic systems in play, including the students' everyday linguistic expressions. We see hence that while the IOE approach relates the reasons and manners in which students deal with representations (the "why" and the "how") to a form of social experience of mathematical knowledge mediated by a plethora of multimodal representations, the ETL approach relates them to an inquisitive endeavor that remains constrained and afforded by the structure and function of the microworld.

Despite these important theoretical differences in the principles of the theories, there are some similarities—for example, the common constructionist background, the interest in knowledge representation, and digital artefacts. The interplay between similarities and differences leads the perspectives to attend sometimes to similar passages in the students' activity, while interpreting it differently: "For both teams, the connectivity between symbolic and animated graphic modes is identifiable as significant to the solution process and of interest theoretically" (Morgan and Kynigos, this Special Issue). The reasons, as the author stress, are not the same: From the IOE perspective, the connectivity between these two semiotic modes offers a potential for meaning construction. From the ETL perspective, the connectivity is interpreted as evidence of meaningful knowledge connections. Yet, meaning and its production are not conceptualized in the same way.

3.3 Meaning

At the end of the article, in a retrospective mood, the authors mention differences surrounding their concept of meaning:

> while the constructionist perspective is concerned with meaning as (an ultimately individual) cognitive phenomenon, for social semiotics meaning is conceptualised as

> the establishment of shared orientations through communication in interaction between individuals—meaning is located in the interaction rather than acquired by the individual. This difference reflects a fundamental difference on the object of the two theoretical perspectives: constructionism may be characterised as a theory about cognition and its development while social semiotics is a theory about signs and the ways they function. (Morgan and Kynigos, this Special Issue)

As we can see, in the first sense, meaning is considered to be something individually produced, even if it is produced in the course of interaction with others. Meaning appears as a psychological construct. The individual is posited as the producer and origin of meaning and intentionality.

In the second sense, meaning is located not in the individual but in interaction. It does not come from within. Meaning sprouts from within the collective. Meaning is not psychological: it is truly social.

We reach here another profound distinction that explains fundamental differences in the approaches under consideration. In the perspectives here discussed, the digital microworld MoPiX seems to be considered as something that "helps" or "assists" the cognitive or social production of meanings. In the case of the ETL, the digital artefact offers a sophisticated aid with the help of which the students can tinker in more complex ways and make meaningful connections. In the case of the IOE perspective, the digital artefact provides the students with a multi-semiotic space of social experiences. As a result, the theoretical conception of the artefact is not exactly the same.

Despite these differences, both approaches put a strong emphasis on the idea that the students' use of the digital artefact MoPiX is somehow expected to lead to the mathematical meanings of the target knowledge:

> The objects of the MoPiX microworld were designed to behave in mathematically coherent ways, providing an environment that, by exploring and building models within the microworld, was intended to allow students to construct orientations to concepts such as velocity and acceleration consistent with conventional mathematical and physical principles. (Morgan and Kynigos, this Special Issue)

In the case of the IOE team, the authors tell us, there is an influential institutional demand to reach this target knowledge. However, in both approaches, it is not completely clear how the generation of students' meanings could end up coinciding, approximating, or relating to the mathematical meanings that are the target of the mathematical activity. That which moves meaning towards its target (i.e., its *teleology*) still needs to be accounted for (Radford, 2006). In the case of the IOE approach, it is not clear how the collective multimodal patterns may end up related to the target knowledge. In the case of the ETL approach, it is not clear how the evolution of the students' meanings would culminate in the expected mathematical meanings. In both cases, the role of the teacher is not made clear.

The problem of the link between the knowledge and meanings that arise in the course of the students' activity and the cultural target mathematical knowledge is certainly one of the most difficult to tackle and is present in all teaching and learning theories. In the case of educational theories that focus on technology, the problem reappears in terms of how the contextual and local meanings of students' activity generated with and through the artefacts relate to the target meanings. My contention is that a theoretical account of this problem rests on assumptions about knowledge and knowing. I dwell upon this problem in the upcoming sections. I will start with a discussion of the cultural–historical dimension of artefacts, which paves the way to a broader discussion of knowledge and learning.

4 The cultural–historical dimension of artefacts

Educational digital artefacts are not naïve objects. Quite the contrary, they are rather complex. Their complexity, however, cannot be reduced to the technical problems that their manufacture entails. They are also complex in another sense: educational digital artefacts are bearers of a *historical intelligence* (Lektorsky, 1995; Pea, 1993) structured in definite societal ways. This structuration makes things appear and behave in specific cultural manners—in the case of MoPiX, to make things "behave in mathematically coherent ways" (Morgan and Kynigos, this Special Issue).

Unavoidably, indeed, artefacts (digital or not) bring in historical meanings that are more than something that provides room for heuristic or sensuous multimodal experimentation. The historical intelligence embodied in artefacts and actualized in social activity is far from neutral; it *affects* the students' meanings in distinctive and profound ways, by suggesting definite forms of action and reflection, and potential lines of social and cognitive development. What this means is that artifacts are ideological, in Althusser's (2012) sense. That is, they are part of reproductive practices that ensure the sustainability of knowledge in society. Let me refer here to language—generally considered the human cultural artefact *par excellence*. As Goodwin notes, language is often considered immaterial or part of the subjective mental activity of speakers. What is missed, Goodwin (2010, p. 105) argues, is that language constitutes material-like public structures "sedimented in the world." These sedimented public structures help individuals to think, to express themselves, and to act. Even our deepest and personal thoughts (political, scientific, esthetic, poetic, etc.) can only be rendered actual through conceptual categories that are the sedimentations of human intelligence and previous human activity. Now, the sedimented cultural–historical conceptual categories serve not as mere cloths with the aid of which we dress up our feelings and ideas. They constitute the foundations of our thought. In other terms, as original and idiosyncratic as they can be, our thoughts are ineluctably shaped by the conceptual categories that language conveys. This is why language is a conspiracy against what could be a direct experience of the world (Baxandall, 1971). When in his Poem IV Pablo Neruda writes: "The morning is full of storm/ in the heart of summer// The clouds travel like white handkerchiefs of good-bye,/ the wind, travelling, waving them in its hands" (Neruda, 1976, p. 9),[3] he is expressing something very personal and novel. Yet, as new as it might be, the poem is made possible by the *entanglement* of the poet's life and the metaphorical and affective historical societal structuring categories that language makes available to him (e.g., summer, clouds, to wave, movement, departing). It is for that reason that poetry is not the mere expression of emotions but the constitution of emotions through the expressivity of language. If ideas or feelings do not go through this process of semiotic expression, they would remain abstract, formless, simple, and vague impressions—purely physical sensations. It is this phenomenon that Vygotsky (1987, p. 243) describes when he quotes the poet Mandelshtam saying: "I forgot the word that I wanted to say,/ And thought, unembodied, returns to the hall of shadows."

The same argument can be applied to MoPix and other digital and non-digital artefacts. Like language, they provide from the outset an array of cultural–historical conceptual categories (e.g., velocity, acceleration) and a conceptually structured space through which the students experience the world and, always in novel ways, form their thoughts. Velocity, in its modern conception, appears as a rate of heterogeneous variables (space and time) that was unthinkable in Greek antiquity. Aristotle, for instance, thought and reasoned in ways where he

[3] "Es la mañana llena de tempestad/ en el corazón del verano. // Como pañuelos blancos de adiós viajan las nubes, / el viento las sacude con sus viajeras manos." (Neruda, 1976, p. 8)

had to compare homogeneous things between themselves only, without possibility for instance of thinking of velocity as something quantifiable. He used rather cultural comparative superlatives like faster, slower. In *Physics* he says: “the quicker of two things traverses a greater magnitude in an equal time, an equal magnitude in less time, and a greater magnitude in less time” (Aristotle, 1984, p. 393 [Physics, VI, 232b20-232b15]). The modern concept of velocity that MoPiX conveys mobilizes already the idea of velocity as a *number* that measures *heterogeneous* elements. It offers to the students a specific conceptually structured space in which to think. As far as I can see in Morgan and Kynigos’s analyses, the effect of the artefacts’ historical intelligence on the students’ semiotic and cognitive activity is missing. The absence of the historical dimension of artefacts in teaching and learning contemporary educational studies seems indeed to be the norm. Maybe such an absence is a symptom of the manner in which epistemological and educational research has traditionally understood knowledge and knowing.

Let me expand on these ideas through the discussion of a short and simple example that brings together knowledge, knowing, and the role of artefacts therein. The example I would like to refer to is nut cracking in wild chimpanzees.

5 Knowledge, knowing, and artefacts

Nut cracking in chimpanzees is not an obvious process. As primatologists note, it comprises the following steps:

1. The chimp picks up a nut;
2. Puts it on a particular surface: an anvil stone,
3. Holds another stone (the hammer stone),
4. Hits the nut on the anvil stone with the hammer stone, and
5. Eats the kernel of the cracked nut (see Fig. 1).

Studies in the wild suggest that it takes 3 to 7 years for the chimp infants to learn the process. Infants do not necessarily start by using a hammer stone and the anvil. The proper attention to the objects, their choice (size, hardness, etc.), and subsequently the spatial and temporal coordination of the three of them (nuts, anvil, and hammer) is a long process. Often,

Fig. 1 Yoyo cracking a nut while the young chimps watch her attentively (from Matsuzawa et al., 2001, p. 570)

young chimps of about 0.5 years manipulate only one object (either a nut or a stone). They may choose a nut and step on it. As chimps grow older, they may resort to the three objects, but not in the correct sequence of nut cracking behavior, resulting in failed attempts. A key aspect of the process is the appearance of suitable cracking skills—for example "the action of hitting as a means to apply sufficient pressure to a nut shell to break it" (Hirata et al., 2009, p. 98).

Nut cracking is learned as a social process that primatologists term "education by master-apprenticeship" (Matsuzawa et al., 2001). What the term means is that wild chimps do not teach it in explicit ways to their offspring. Summarizing 13 years of field research, Matsuzawa et al. (2001, p. 571) write: "We never encountered episodes of active teaching … We never observed chimpanzee mothers perform molding (grasping the hands of infants for guidance), or giving appropriate stone tools or good-quality nuts to be cracked to their infants." The young chimps, who usually remain with their mother until the age of 4 to 5 years, observe attentively how the mother cracks nuts and then try to do it by themselves, even without apparently understanding the goal of the process at first.[4]

Not all chimpanzee groups crack nuts, and those groups where nut cracking occurs do not all crack the same variety of nuts. Primatologists believe that nut cracking developed somewhere in West Africa and was subsequently conveyed socially from one generation to the next. The nut cracking practice eventually spread out among neighboring groups as a result of chimps' immigration (Hirata et al., 2009, p. 88; Matsuzawa et al., 2001, pp. 569–70). At any rate, the accomplishment of nut cracking appears as a social practice where technology plays a fundamental role.

I would like to suggest that "knowledge," in this case knowledge of how to crack nuts, appears as a conceptual or "ideal form" that cannot be equated to *this* or *that* particular sequence of coordinated actions with *these* or *those* stones. The ideal form is a social and cultural codification of these actions beyond each one of its concrete instances or realizations. It is nut cracking as an ideal form that lends the generality to each one of its specific solutions. This is what Deleuze means when he asserts that "problems are Ideas themselves" (1994, p. 163).

Knowledge as an ideal form (here, knowledge of how to crack nuts) does not have anything to do with Platonic forms. Rather than considering the nut-cracking Seringbara community of chimps that inhabits the mountain forests of Mt. Nimba in the Republic of Guinea as resorting to Platonic forms or to Kantian things-in-themselves, I wish to argue that they would be resorting to culturally and historically constituted embodied processes of reflection and action. The nut cracking "ideal form" is to be understood as a *general prototypic* way of doing things. Rather than sitting in an eternal realm of ideas, this ideal form is encoded in cultural memory as a pattern or sequence of actions. But this ideal form cannot exist if it is not carried out in practice. Its mode of existence is indeed its practical appearance in the concrete world. And vice versa: every process of nut cracking is possible insofar as it appears as the *manifestation* or the *incarnation* of the ideal form. The general and the singular hence come together. They come together in the concrete spatial–temporal and unique *event* where nut cracking occurs (Radford, 2013a).

As language affects individuals in the case discussed previously, so the ideal form affects the young chimps. It affects them in that it offers a way to act in the world, to perceive what is around, and to perceive it with meaning. In Fig. 1, we see an adult chimpanzee named Yo cracking Coula nuts. With her right hand, Yo places the nut over an anvil and, in a coordinate manner, she holds the stone hammer with her left hand, while the young chimps to her left and

[4] For instance, they play with the stones; see http://www.youtube.com/watch?v=bpRu1Zg-128

right watch her attentively. The young chimps do not yet master the relatively sophisticated motor and conceptual skills that are required to accomplish the cracking of the nut. These skills are, for them, pure potentiality. They will become part of the young chimps' repertoire of action and reflection after a long period of intense practice and observation—only after a lengthy process of objectification through which they come to recognize (i.e., to grasp sensuously and conceptually) the ideal form of nut cracking, thereby opening new possibilities for change and innovation. We can now ask the question: What is the epistemic role of the chimps' technology? What do the stones *represent*?

To answer this question, we need to come back to the definition of knowledge that was suggested above. As it was discussed at the beginning of this article, the longstanding tradition of "representational" frameworks works under the assumption of a demarcation between the representation and the object of the representation (often rendered as a demarcation between signifier and signified). Within this context, knowledge, as von Glasersfeld noted, is predicated in terms of *representational adequacy*. If we conceive of knowledge not as something to be represented, but rather as *movement*, as I suggested previously—more precisely as a culturally codified sequence of actions that are continuously instantiated in social practice—then the stones do not represent something in the classical representational sense. What they do, rather, is to become an integral part of the manner in which the patterns of sequence of action are instantiated in the concrete world. The stones neither "represent" something nor mediate the chimps' activity. *They are part of the activity that we may term "knowing"*—that is, of the enactment and actualization of the culturally codified forms of movement, in this case the complex motor and conceptual sequence of actions that lead to the successful cracking of nuts. In the terminology that we have developed elsewhere (Radford, 2002), the stones operate as *semiotic means of objectification*, that is, as means of signification in the disclosure of the ideal through its appearance in the individuals' concrete world of labor, actions, and deeds.

I would like to go further and argue that what I have said about nut cracking and its technology applies to knowledge in general. It applies to the mathematical or physical concepts of velocity and acceleration and of the MoPiX microworld—acknowledging, of course, that in this case the cultural codification of sequence of actions rests on a successive historical embedment of sequences of actions that gives to the modern mathematical concepts their general and abstract character. Digital technologies add layers of complexity to the practice of mathematics and certainly deserve a detailed investigation in teaching and learning. Digital technologies do not merely expand our possibilities. They transform the possibilities of what is knowable and the manner in which something can be known. As Lakatos argued, the telescope was not only an instrument that allowed Galileo to

> 'observe' mountains on the moon and spots on the sun… his 'observations' were not 'observational' in the sense of being observed by the—unaided—senses… It was not Galileo's—pure, untheoretical—observations that confronted Aristotelian theory but rather Galileo's 'observations' in the light of his optical theory. (Lakatos, 1970, p. 173)

Technologies become part of the practice of knowledge, not of its representation. Knowledge, in fact, within the theoretical Hegelian view that I am advocating here is not representable. Nor is it something to be "attained." Because knowledge is *movement*, knowledge is rather something that we instantiate through sensuous and material critical reflection in activity (Radford, 2012, 2013b). From this perspective, the practical activity with the stones or with the MoPiX microworld is not a replica of something immanent, nor its substitute: it is the concrete mode of existence of the culturally codified form of movement that constitutes knowledge. From this perspective, knowledge implies individuals who simultaneously come to know in sensuous, material, and affective ways by presenting themselves to others and to the world in the presentation or appearance of knowledge.

6 Synthesis and concluding remarks

This article was intended as a commentary on the ideas of representation and digital artefacts conveyed in Morgan and Kynigos's article in this Educational Studies in Mathematics Special Issue. The article shows the potential of the methodology of the ReMath project, namely the creation of a space of confluence of several research teams to tackle a general educational problem, namely a reflexive endeavor to search for a shared framework about representations. The ensuing dialogue allowed the research teams to realize the importance of certain key theoretical concepts that underpin conceptions surrounding the idea of artefacts, in particular key concepts such as cognition and learning, the role attributed to representations in the learning process, and the concept of meaning. In the first part of the article, I summarized some of the intense discussions that took place during the 1980s and 1990s in mathematics education research on representations. The summary stresses the tension that resulted from calling into question the assumptions surrounding the traditional view of representations and its concomitant division between represented and representing worlds. We are profoundly indebted to von Glasersfeld for the intellectual lucidity with which he tackled the problem of representation. He showed us the inherent problematic basis of naïve realism.

Certainly, the ReMath project suggests that important progress has been made since the landmark discussions embodied in Janvier's book and other seminal works of the time (e.g., Goldin & Janvier, 1998). Morgan's and Kynigos's approaches have the merit of moving the discussion about representations to new theoretical horizons. I think that Vergnaud's criticism concerning the absence of situations and actions in situations no longer applies. Constructionism and social semiotics put indeed at the heart of their analysis individuals and their deeds in the processes of knowledge representation. The question of whether Vergnaud would be satisfied with the theoretical conception of "situation" in the IOE and ETL perspective, however, is a different matter.

At any rate, in the course of my commentary I suggested that both approaches (the IOE and the ETL) fall short of thematizing the manner in which we account, at a theoretical level, for the link between, on one hand, the knowledge and meanings that arise in the course of the students' activity with artefacts, and, on the other hand, the cultural target mathematical knowledge. As the discussion presented in Sections 4 and 5 intimate, such a theoretical account rests unavoidably on the concepts of knowledge and knowing that we adopt. Sections 4 and 5 were an attempt to provide an example that draws from ideas of the theory of objectification (Radford, 2008b, 2012, 2013b; Roth & Radford, 2011) and its cultural–historical Hegelian conceptions of knowledge and knowing.

Although it is difficult for me to see how history appears organically in the ETL perspective, I think that history does appear in the IOE approach, but not in its cultural dimension; it appears in its social dimension only, in the manner in which society imprints its differential marks on the individual—e.g., in the manner in which individuals use everyday language in the classroom, indexing their social experience. While I agree with the IOE approach that meanings sprout in interaction, we have to insist, I think, that meanings do not sprout from social interaction *only*. Meaning is historical. Thus, the mathematical meanings that appeared in the classroom passages mentioned in the article (e.g., those pertaining to the link between symbolic and graphic representations or those about velocity and acceleration) do have a cultural history. Their history goes back to Aristotle, Oresme, Descartes, Wallis, and other philosophers and mathematicians (see Radford, 2008c). It is precisely their cultural historical status that makes these meanings recognizable, something without which they could not become objects of knowledge in a curriculum. The question is: How could we conciliate the idea that meaning is historically constituted—and hence already there, prior to the students' interaction—with the idea that, at the same time, it sprouts in interaction? How can we assert that meaning is simultaneously new and old? These questions lead us back to the fundamental ontological problem of representations that we discussed in Section 2 of this article. The problem reappears here in terms of the referent and meaning of representations. One can still think of reference and meaning as pertaining to something a priori, transcendental, as posited by Platonist ontologies. But, as we saw in the second part of this article, there would not be a way to overcome the line demarcating reference and meaning as phenomenological products and reference and meaning as transcendental entities. Within this context, the best that we can do is to adopt a naïve hope that phenomenological and transcendental meanings will somehow miraculously correspond to each other—a move against which Kant has already cogently argued. Or we can think of mathematical meanings as historically and culturally constituted. Instead of lying in a Platonic world of forms, meanings would be located in the culture and would preexist the students' classroom activity, much as the chimps' nut cracking knowledge of the Guinean forests is located in their culture and enacted in their social practices.

But how are we to account for the relationship between these cultural–historical meanings and the meanings that the students produce in interaction? The IOE perspective is clear in this respect: meanings are not "acquired." How then do the phenomenological meanings sprouted in collective interaction "get in touch" with the historical–cultural mathematical ones? Again, the argument of a "natural" or self-directed evolution from the phenomenological meanings towards the cultural one seems difficult to sustain. Such an argument would assume a problematic stance towards mathematics: it would suppose that mathematical thinking goes necessarily into the same developmental direction. Tones of data from anthropology and ethnomathematics show rather the opposite (Radford, 2008d). Mathematical thinking cannot be conceived of as something endowed with its own logical developmental agenda (thus, problem-solving-based Babylonian mathematics is in several profound aspects essentially different from the theoretical apodictic Greek mathematics codified in Euclid's *Elements*). The history deposited in artefacts, I suggested, offers lines of development that are, generally speaking, in tune with cultural knowledge (e.g., the quantifiable concept of speed and acceleration of the MoPix microworld). But this is not enough. The teleology of meaning, as far as teaching and learning are concerned, lies in the teaching–learning activity. In other words, it is not the artefacts than impress meaning with its teleology. Meaning's teleology is driven by the activity of which artefacts are a part. Now, there are several ways in which to conceptualize teaching–learning activities. To give but one example, teaching–learning activity can be conceptualized as a game played between two opponents (the teacher and the student), as in the Didactical Theory of Situations (Brousseau, 1997), or as a joint communal endeavor of the teacher and the students in the pursuit of a goal (Radford & Roth, 2011). In the latter, learning occurs as the target ideal form is brought into existence by the teacher and the students, in the event of the ideal form's concrete appearance in the phenomenological world. Here, the old and the new, the particular and the general are brought together in a movement where one is the condition of existence of the other. To come back to Edward's distinction between structure and function, as a result of its design, classroom teaching–learning activity (which establishes the manner knowledge is called into action) offers an a priori range of epistemic possibilities; it offers the structural dimension and background out of which knowledge can be enacted. Classroom teaching–learning activity *also* offers the functional dimension: It opens a space for the enactment of possibilities, where the potential is transformed into something actual through the students' concrete actions. This dual nature of classroom teaching–learning activity makes possible the incarnation of the ideal form in the particular process of the activity as event—something always unpredictable, as they depend on how teachers and students will engage in the activity.

The ReMath project also investigated another dimension that I can only touch on briefly, namely the role of context in mathematics education theories. The question of context brings in a dimension of theories that is often omitted in educational theorizing. It reminds us that theories are not only the formulation of research questions framed by theoretical principles and scrutinized through methodological procedures. The principles of a theory in fact embed theoretical conceptualizations of the effective determination of relationships between individuals who live in a concrete, specific, and historical national context. The question of context reminds us that research occurs in the terrain of practice—social, cultural, economical, political practice—and not in a theoretical self-contained realm. Our theories bring ineluctably cultural outlooks and political and ethical problems, questions of power and knowledge distribution that work in subtle ways that we still need to explore.

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