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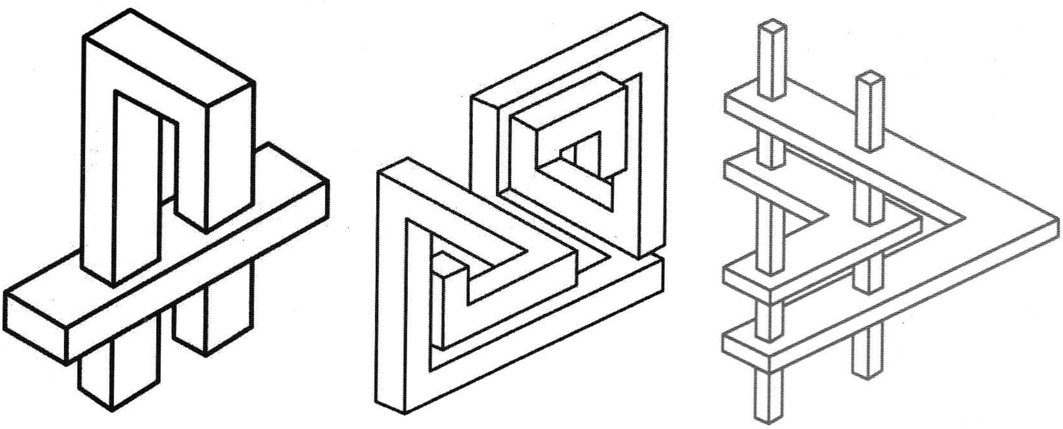

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How to Look at the General Through the Particular: Berkeley and Kant on Symbolizing Mathematical Generality

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Abstract. *In this paper, I briefly examine Berkeley and Kant's ideas about the relationship between the particular and the general and the role that they ascribed to symbols in expressing mathematical generality. I finish with a discussion of some of the differences between the expression of generality in elementary geometry and algebra.*

It was in the early 18th century that Bishop Berkeley argued that it is impossible for the mind to have general abstract ideas. Whatever general idea we have, Berkeley argued, it is always tied to something particular. One of Berkeley's favorite examples was the example of triangles:

> What [is] more easy than for anyone to look a little into his own thoughts, and there try whether he has (…) an idea that shall correspond with the description (…) of the general idea of a triangle, which is, *neither oblique nor rectangle, equilateral, equicrural nor scalenon, but all and none of these at once*? [Berkeley, *A Treatise Concerning the Principles of Human Knowledge,* (PHK), 1710, Introduction, §13].

Berkeley's argument was that whatever triangle we think of, it is necessarily the idea of a *particular* triangle (e.g. an isosceles or rectangle triangle). However, he was not saying that we cannot talk about general notions. For him, the solution of the problem of representing the general rested in the way particulars stand for the general:

> when I demonstrate any proposition concerning triangles, it is to be supposed that I have in view the universal idea of a triangle; which ought not to be understood as if I could frame an idea of a triangle which was neither equilateral, nor scalenon, nor equicrural; but only that the particular triangle I consider, whether of this or that sort it matters not, does equally stand for and represent all rectilinear triangles whatsoever, and is in that sense universal. (Berkeley, PHK, Introduction, §15).

Particulars may function as *signs* of general objects, as he puts it in the following passage:

> suppose a geometrician is demonstrating the method of cutting a line in two equal parts. He draws, for instance, a black line of an inch in length: this, which in itself is a particular line, is nevertheless with regard to its signification general, since, as it is there used, it represents all particular lines whatsoever; so that what is demonstrated of it is demonstrated of all

> lines, or, in other words, of a line in general. And, as that particular line becomes general *by being made a sign*, so the name "line," which taken absolutely is particular, by being a sign is made general. And as the former owes its generality not to its being the sign of an abstract or general line, but of all particular right lines that may possibly exist, so the latter must be thought to derive its generality from the same cause, namely, the various particular lines which it indifferently denotes. (Berkeley, *op. cit.* Introduction, §12; emphasis added)

For Berkeley, hence, not only particulars *can* stand for the general but this is the only way to attain generality. More precisely, for him, generality consists in seeing the particular as something else –seeing it not as particular *qua* particular but seeing the particular as something generic.

Kant tackled the problem of the relationship between the general and the particular in different terms. His starting point was the distinction that, according to him, prevails between mathematics and other forms of intellectual endeavor, such as philosophy. For Kant, the generality of mathematical concepts resides in the fact that mathematics, as opposed to philosophy, proceeds by *construction* of its objects. This construction makes recourse to what Kant called "intuition" – i.e. to our capacity of being sensuously affected *in concreto* by the objects of the world. This intuitivity is not, however, of the order of the general: for Kant it simply means something particular. Talking about figures of geometry, Kant explains the idea of construction and the generalizing role of the particular in the following way:

> The single figure which we draw is empirical, and yet it serves to express the concept, without impairing its universality. For in this empirical intuition we consider only the act whereby we construct the concept, and abstract from the many determinations (for instance, the magnitude of the sides and of the angles), which are quite indifferent, as not altering the concept 'triangle'. (Kant, *Critique of Pure Reason* (CPR), (1781, 1787; A714/B742)

What ensures the link between the empirical, particular triangle that the geometer uses in his or her proof and the general triangle is the *schema*, i.e. the universal conditions (or rules) of its construction, "Thus we think a triangle as an object, in that we are conscious of the combination of three straight lines according to a rule by which such an intuition can always be represented." (CPR, A105).
Kant's answer to the riddle of the general is hence different from Berkeley's. Although Kant agrees with Berkeley that "No image could ever be adequate to the concept of a triangle in general" and that an image "would never attain that universality of the concept which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled" (CPR, A140-41/B180), he does not see the geometer reasoning on a generic object but on a rule (schema). It is the

schema that ensures the universality of the concept that the intuition expresses on the manifold of representations.

Let me now turn to the expression of generality in algebra. What does Kant have to say about the schemas of algebra? Algebra cannot work on the basis of the ostensive nature of geometric intuitions. What are hence these algebraic intuitions? Kant says:

> mathematics does not only construct magnitudes (*quanta*) as in geometry; it also constructs magnitude as such (*quantitas*), as in algebra. In this it abstracts completely from the properties of the object that is to be thought in terms of such a concept of magnitude. It then chooses a certain notation for all constructions of magnitude as such (numbers), that is, for addition, subtraction, extraction of roots, etc. (Kant, CPR, A717/B745)

What Kant is telling us is that we can represent an unknown triangle through application of its schema and draw a three-sided figure, and that, in algebra, we can represent the actions of adding, subtracting, etc., but how can we represent an unknown number or a variable? It would be unusual and odd to represent it by, say, the sign "3". The algebraic sign, Kant remarks, has to be "adopted" and hence to be "conventional". It has to be a *symbol*, in Peirce's sense. Kant continues:

> Once it has adopted a notation for the general concept of magnitudes so far as their different relations are concerned, it exhibits in intuition, in accordance with certain universal rules, all the various operations through which the magnitudes are produced and modified. When, for instance, one magnitude is to be divided by another, their symbols are placed together, in accordance with the sign for division, and similarly in the other processes (Kant, *ibid*)

For Kant, hence, the difference is this: the constructions of geometry are of an ostensive nature while those of algebra are *symbolic*. Obviously, Kant's account of algebraic schemas oversimplifies the semiotic problem of the meaning of signs in algebra. The schemas of algebra go beyond the rules of arithmetic. Algebra is not merely an arithmetic disguised with letters. Algebra introduces new general objects such as unknowns, variables, and parameters that do not have an exact equivalent arithmetic counterpart. Otherwise, the practice of algebra would still be the practice of arithmetic with a different code. Because of its ostensive nature, geometry rests to a large extent on perceptual imagination. The case of algebra is different, even if, historically speaking, algebra borrowed –at least in one of its traditions– its syntax and meaning from geometry (Høyrup, 2002). Symbolic algebra *à la* Vieta requires a different form of endowing signs with meaning. As Berkeley noticed, in algebra "to proceed right it is not requisite that in every step each letter suggest to your thoughts that particular quantity it was appointed to stand for." (*op. cit.* Introduction, §19). Even more: the possibility of formal or syntactic manipulations of signs rests on not thinking of the particular semantic aspects of the quantity that the sign represents. In many cases, as Berkeley remarked, letters cannot suggest a concrete image correlated to something in the world, as the square root of a negative number, and yet the calculations can go on.

Ontogenetically speaking, what could the Kantian schemas of variable look like and what would the role of symbols be? The construction of X-Y value tables may be one of the schemas. But as classroom research suggests (Bardini, Radford, Sabena, 2005), the ensuing numerical variational experience is insufficient. The emergence of new symbolic relations between variables seems to be required in order to attain new complex forms of seeing the general in the particular and to achieve the decontextualisation that the concept of variable requires.

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