MEDITERRANEAN JOURNAL FOR RESEARCH IN MATHEMATICS EDUCATION

AN INTERNATIONAL JOURNAL

Volume 1 No. 2 2002

CYPRUS MATHEMATICAL SOCIETY
ΚΥΠΡΙΑΚΗ ΜΑΘΗΜΑΤΙΚΗ ΕΤΑΙΡΕΙΑ
Nicosia - Cyprus

Ed. in Chief
Athanasios Gagatsis

Mediterranean Journal for
Research in Mathematics Education
Vol. 1, No. 2, 63-72, 2002

GENERALIZING GEOMETRIC-NUMERIC PATTERNS: METAPHORS, INDEXES AND OTHER STUDENTS' SEMIOTIC DEVICES

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ABSTRACT

The goal of this paper is to offer an overview of some of the results of a longitudinal classroom-based research program concerning the emergence of students' algebraic thinking in generalizing geometric-numeric patterns. Adopting an anthropological viewpoint where thinking is seen as a cognitive praxis and drawing from a semiotic-cultural perspective, we pay careful attention to the crucial turning points in the students' initial attempts to grasp the practice of algebra and focus on: (1) the students' first understandings of algebraic generalization and (2) the social means of semiotic objectification used by the students.

INTRODUCTION

As Bednarz, Kieran & Lee pointed out (1996, p. 4), generalization is one of the curricular and teaching options at the forefront of contemporary efforts to introduce algebra in school. Indeed, algebraic generalization gives students an opportunity to start building up their first symbolic expressions by capitalizing on their previous arithmetical experience. Nevertheless, successful access to algebra through generalization requires the students to overcome a certain number of problems. For instance, in a series of studies, MacGregor and Stacey (1992, 1993, 1995) evidenced a number of difficulties that students encounter in recognizing a numerical functional relationship between the variables. They also reported some problems related to finding a suitable algebraic expression for such a relationship. Analyses and results generated by other researchers point in a similar direction. Thus, in a recent paper dealing with pre-symbolic numeric generalization, Sasman et al. observed that, instead of seeking a functional relationship between the variables, students tend to focus on recursive relationship between the terms of the sequence and argued that, in order to better understand the students' cognitive difficulties, more research is still needed (1999, p. 161). Within the discussion of the problems that students face when dealing with the algebraic language, Arzarello *et al.* (1993) stressed the difficulties that students usually have in understanding the sense of the letters and of the symbolic expressions built on those letters. The algebraic language, in fact, obeys a logic of signification that seems very incomprehensible to many novice students.

In 1998, four Grade 8 classes from two schools in Ontario joined a longitudinal classroom-based research program whose goal is to investigate, from a semiotic-cultural perspective, the emergence of students' algebraic thinking. In what follows, I shall provide an overview of this research program titled "Students' processes of symbolizing in algebra" and shall highlight some of the results that we have obtained so far[1].

THE THEORETICAL FRAMEWORK

In order to provide accounts of the emergence of students' algebraic thinking in light of the use of signs and the production of meanings, we adopted an anthropological approach relating thinking to the social practices from which it arises. Among the several general forms in which to theorize such a relation (see e.g. Chaiklin and Lave, 1993) ours elaborates this relation in a way that thinking is conceived as a form of mediated cognitive reflexive praxis (see Furinghetti and Radford, 2002). The mediated character of thinking draws from Vygotsky's work and refers to the role played by artefacts, tools, sign systems and other means to achieve and objectify the cognitive praxis. The reflexive nature of thinking is to be understood in Ilyenkov's sense, that is, as the distinctive component that makes cognition an intellectual reflection of the external world in the forms of the individual's activity (Ilyenkov 1977, p. 252). Thinking as cognitive praxis (praxis cogitans; details in Radford 2003a) emphasizes the fact that what we know and the way we come to know it is framed by ontological stances and by cultural meaning-making processes that shape a certain kind of rationality out of which specific kinds of mathematical questions and problems are posed.

These theoretical considerations led us to investigate the emergence of students' algebraic thinking in its own arena —the classroom, seen as an ethnographic site— and to pay careful attention to the crucial turning points in the students' grasping of the practice of algebra[2]. We shall focus here on two of these crucial, related points: (1) the students' first understandings of algebraic generalization and (2) the social means of semiotic objectification used by the students. That is, those linguistic and non-linguistic means (e.g. pointing, naming or signalling) that allow the students to achieve a fixation of attention thereby making discernible from the undifferentiated horizon of objects —i.e. the *continuum* of all that can be culturally experienced (Eco 1999, p. 52 ff.)— certain aspects of the objects and even to shape and make visible new objects that, as the algebraic concept of the general term of a pattern, are beyond direct perception. While the first point will be developed in Section 3, the last point will be elaborated in Sections 4 to 6.

UNDERSTANDING THE PRACTICE OF GENERALIZATION:
What's your question again?

The social nature of knowledge and thinking stressed in our framework led us to elaborate teaching sequences based on mathematical activities involving different tasks intended to be cooperatively carried out by the students according to a small-group

1. The program is funded by the Social Sciences and Humanities Research Council of Canada.

2. It is true that once a layer of mathematical generalization has been reached, generalization may look natural, endemic, and ubiquitous, as Mason once suggested (1966, p. 66). However, it is equally true that the attainment of a new layer of generalization (e.g. the algebraic one) requires a great deal of new understandings, the mobilization of new signs and the production of new meanings.

working format (in general, the small-groups were comprised of 2 or 3 students. The small-group activities were usually followed by general discussions conducted by the teacher. An emblematic example of generalizing activity is the one concerning the classic toothpick pattern shown in Figure 1. This activity included several tasks, among them the following: (a) to find the number of toothpicks required to make figure number 5 and figure number 25; (b) to explain how to find the number of toothpicks required to make any given figure and (c) to write a mathematical formula to calculate the number of toothpicks required to make figure number 'n'.

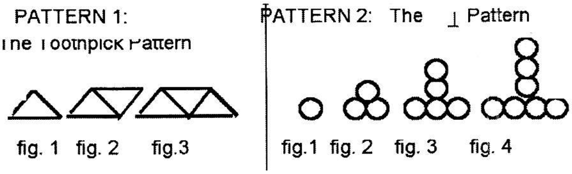

Figure 1. Two patterns given to the students.

One of the turning points in the students' comprehension of algebraic generalization relates to their understanding of what is meant by Figure n. As a noumenal object, Figure n cannot be grasped through perception and hence cannot be pointed to ostensively. It does not exist in the horizon of objects that the students perceive in front of them. It then has to be objectified through diverse semiotic devices (e.g. gestures, words, drawings, signs). Usually, before looking for help from the teacher, the students spend a long time arguing and discussing among themselves. Here is an excerpt from one of the student small-groups (Grade 8 Class No. 1): The first line is a reaction to Dylan who suggested that n is fourteen, because n is the 14th letter in the alphabet.

1. Albert: No it's not!
2. Dylan: Yeah it is!
3. Sylvain: What is n? (*asking the teacher who is coincidentally walking by*) [...]
4. Teacher: "n" is meant to be any number. [...]
5. Dylan: n is what?
6. Teacher: Any number.
7. Dylan: I don't understand.
8. Teacher: You don't understand?
9. Dylan: No. [...]
10. Teacher: (*talking to Sylvain*) Do you understand what n is?
11. Sylvain: Which one? (*pointing to the figures on the sheet*) This, this or this?
12 Teacher: It does not matter which one.

In the analysis of this episode (see Radford, 1999), we suggested that the teacher sought to promote an understanding of "n" as "*n being any number*". The students' reactions show that there is a tremendous difficulty in constructing this specific meaning. In line 11, n is taken as one of the particular figures on the sheet. The teacher then continued:

13. Teacher: (*talking to Albert after a long period in which the students remained silent*) OK. So then, do you have an idea what n is?

14. Dylan: Fourteen.
15. Teacher: It may be fourteen ...
16. Albert: (*interrupting*) any number?
17. Teacher: (*continuing the explanation*) ... it may be 18, it may be 25...
18. Dylan: Oh! It can be any number?
19. Albert: (*interrupting*) The number that we decide!
20. Dylan: OK, then, (*taking the sheet*) OK, n can be ... uhh...
21. Albert: Twelve.
22. Dylan: Yeah. [...]
23. Teacher: And if you leave it to say any number? How can we find ... how can we find the number of circles for any term of the sequence (*making a sign with his hands as if going from one term to the next*)?
24. Albert: Figure n? There is no Figure n!
25. Dylan: (*talking to Albert*) He just explained it! n is whatever you want it to be.

The French epistemologist Gaston Bachelard (1986) rightly noticed that all (scientific) knowledge is always an answer to a question. But the question, it has to be added, only makes sense within the scope of its own practice. Here, the students cannot make sense of the mathematical question because we are on the borders of two different mathematical practices (the arithmetical one and the algebraic one). This is why the students tend to interpret the question in arithmetical terms while the teacher directs his efforts to the understanding of the question in its algebraic sense.

Classroom episodes like the previous one, that we encounter again and again in the students' first contact with symbolic algebraic generalization, reminds us of a psychologist who went to Tanganyika to study the natives' capability to grasp and extend a pattern. The test required the subjects to continue a series of coloured pegs placed in holes drilled in a board, e.g. two blue, two red, two blue, two red, etc. After some hopeless efforts, the psychologist was almost going to conclude that the generalizing structuring principle was alien to the Tanganyikan mind when he saw one of the subjects who had failed the test earlier proceed to plant an avenue of trees, two of one kind followed by two of another kind (Bartlett, 1937). Although generalization may be omnipresent and natural, it is so within the confines of its own practice and the conceptual and concrete, sensual activities, signs and meanings that this practice encompasses.

At any rate, the students' attainment of a first rudimentary level of algebraic meaning for Figure n is a crucial step in their insertion into the practice of algebra –a practice that requires a dialectically related and creative effort of students and teachers in order to deal with historically produced meanings and means of action (tools, symbols, artefacts, etc.). As Leont'ev (1978, p. 88 ff.) noticed, meanings lead a double life: meanings as products of society with their history and meanings as 'personal senses', i.e., individualized and personalized meanings by the concrete individuals in their activity. The view of the students' insertion into social mathematical practices that we are conveying relies on the recognizance of the multiple dimensions of individuals' voices –something related to Bakhtin's concept of *heteroglossia* (Bakhtin 1981, 1990)– and runs against a static and monolithic concept of culture. As we shall see in the next sections, the students' elaboration of a primary algebraic understanding and symbolization of Figure n, in fact, unfolds in the tension of personal senses and social meanings out of which a first token of the general Figure n is sketched. However imperfect the sketch may be (as is the case of a group of students from Grade 8 Class No. 3 mentioned below), the primary algebraic understanding of Figure n and its symbolization appear to be meshed with the semiotic objectification through words of such a figure.

Metaphors as an Embodied way of Linking the Particular and the General

As indicated in § 3, usually, before the students actually write a symbolic algebraic formula, we ask them to *explain* how to find the number of toothpicks (or the 'basic' objects of the figures of the given pattern under consideration) required to make any given figure. This task brings the students to a different layer of discourse and generality. The major semiotic and cognitive problem that the students have to overcome is to give shape with words to the noumenal object *Figure n.* Some students take advantage of the geometric configuration of the pattern while others focus on numerical relations between terms. These two strategies stress different views of the relation between the particular and the general (Radford, 2000). In both, however, we find the students having recourse to metaphors. For instance, in an analysis concerning Pattern 2, reported elsewhere (Radford, 2001), to provide an explanation about how to find the number of circles required to make Figure n in Pattern 2, the students referred to Figure 12. However, they were *not* exactly talking about the *particular* Figure 12. Instead, this figure was a means to talk about the still ineffable general Figure n. Figure 12 was suitable for the task because it was not one of those figures that the students materially had in front of them. In this sense, for the students' discursive and objectifying process, Figure 12 shares some elements of abstract objects. Furthermore, this figure was not too big to be handled mentally – like Figure 120 that was first mentioned by the students and then abandoned for Figure 12. In one of her interventions, Anik said:

1. Anik: You could say, uh, the figure ... OK. Say: Let's say that in figure ... um ... Figure 12 (*moving her hands on the desk near the first figures made up of bingo chips, as if she were touching the hypothetical Figure 12*). You'd put 12 chips ...
2. Josh: (*interrupting*) on the bottom ...

Although the metaphorical Figure 12 serves the students as a remarkable semiotic device to achieve a certain level of objectification, Figure 12, nevertheless, is not enough to deal with mathematical generality. The students then displayed a range of subtle discursive resources. A closer look at the dialogue showed that in reasoning about and expressing generality, the students used two key categories of words having two different semiotic functions: a generative action function and a deictic function (see *Figure 2*).

Deictic terms are linguistic units (in our example, 'top', 'bottom') referring to objects in the universe of discourse (in this example, the figures of the pattern) by virtue of the situation where dialogue is carried out. It is the contextual circumstances that determine their referents. As such, deictic terms depend heavily on the context (see Nyckees 1998, p. 242 ff.) and have a particular function in dialogical processes. We term 'generative action function' the linguistic mechanisms expressing an action whose particularity is that of being repeatedly undertaken in thought. In this case, the adverb "always" provides the generative action function with its repetitive character, supplying it with the conceptual dimension required in the generalizing task. The relevance of generative action functions can be acknowledged by noticing that, in our example, generality is objectified as the *potential action* that can be reiteratively accomplished.

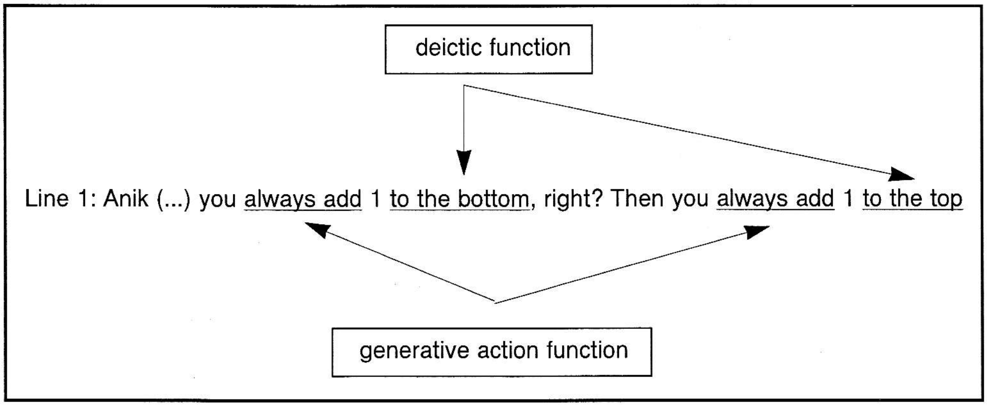

Figure 2. Generative action function and deictic function

Notice that the continuity of the experience of particular figures assures the students that Figure n should have the same *form*. This is crucial in the construction of the general object. MacGregor and Stacey's research has shown that, in *numerical* patterns given in the form of an x/y or two-entry table, students sometimes use a certain rule to investigate the terms located before a specific number and that they change the rule for terms located beyond the number. They ascribed this to the students' insertion of non-mathematical considerations into the problem at hand. One advantage of numeric-geometric pattern models like those in *Figure 1*, may be that it becomes difficult to think that shapes will suddenly change.

In cases where the students' description of the general is less centered on the geometrical shape of the terms of the pattern and is more focused on the numerical relationships between the terms, the embodiment of experience is always manifested even if in a different format. For instance, when dealing with the toothpick pattern, a group of students noticed that the total number of toothpicks in *Figure 1* of the pattern is equal to 1 plus 2, and that the total number of toothpicks in *Figure 2* of the pattern is equal to 2 plus 3, etc. Then a student said: "Yes. Yes. OK. You add the figure plus the next figure ...". Here, generalization is objectified by expressions like 'the figure', 'the next figure' that unveil the spatial-temporal dimension of the general objects dynamically conceived of as 'coming one after the other'. Whether spatial (as in the case of student's utterance shown in *Figure 2* where 'top', 'bottom' are key structural elements or as in the case of a student that referred to the vertical part of the emerging object Figure n as 'the one in the air"), cinematic or a combination of both, metaphors become intellectual tools to make the structure of relevant events apparent thereby creating a new perceptual field. This perceptual field appears hence as sustained by a certain mode of referring or denoting that, in addition to including an arsenal of deictic and other contextual and temporal situated terms, has the particularity that the knowing subject becomes (implicitly or explicitly) interwoven with the object of knowledge. In *Figure 2*, we see the explicit insertion of the individual through the pronoun "You". Subject and object also become tied together in the aforementioned utterance related to the toothpick pattern: "Yes. Yes. OK. *You* add the figure plus the next figure ...". As a result, the individual perceives the general *perspectively*. And, as I argued elsewhere (Radford, 2002; 2003b), this has tremendous implications when the students build their first symbolic expressions. Indeed, in the first place, algebraic generalizations involve objects that do not have spatial-temporal characteristics. And, secondly, in algebraic generalizations, the individual does not have access to a perspectival view of the objects. To provide adequate symbolic expressions, the students then have to grasp the *'rhetoric of algebra'*, that is, a novel mode of text production beyond the embodiment of situated experience, having its own mode of denotation that requires, in particular, a de-timing (Arzarello, 2000) and a de-spatialisation of the objects and of the symbolic narratives in which they are contained.

How then will the students proceed to the des-embodiment of their spatial-temporal embodied situated experience? How are they to produce the voiceless symbolic algebraic expressions? I want to exemplify the students' difficulties in the disembodiment of experience through what I termed the 'positioning problem'.

THE 'POSITIONING PROBLEM' AND THE LOSS OF THE PERPECTIVAL VIEW

When the students had to build a formula for the number of toothpicks in Figure n, they could not have recourse to the contextual-bound deictic terms –either *descriptional* like 'top', 'bottom', 'horizontal', 'vertical' or *spatial* like 'next'– as well as to the indexical demonstratives like 'this' and 'that' that proved to be central to express generality in the previous non fully symbolic layer of discourse. In the realm of algebraic symbolism, they no longer enjoyed the perspectival view with which natural language empowered them. How then do they proceed to symbolize, e.g., the verbally expressed pattern "the figure plus the next figure"? What the students actually did was to change strategy and instead of symbolizing "the figure plus the next figure" they symbolized the pattern "the figure plus the figure plus one". This, in turn, led them to produce the symbolic expression (n+n)+1. As the analysis of the dialogue suggests (Radford, 2003b), the students' trouble with the first observed pattern resides in the symbolization of 'the next figure'. Such a symbolization is related to a general semiotic problem – the 'positioning problem'. What I mean by the 'positioning problem' is this difficulty that the students, very often, face when referring to the elements of the pattern not through spatial deictic terms but in terms of their unspecified ranks. At its core, resides the untranslatability of deictics into algebraic symbols. This untranslatability is what Anik is experiencing when, referring to Pattern 1, she says: " OK. You can say ... you make ... OK you add the figure ... oh my God, how do you say it [in algebraic symbols]... The figure plus the next figure?".

At this point, one of the options for novice students is to try to bypass the 'positioning problem'. Which is exactly what Anik and her group did by changing strategy. As discussed in (Radford, 2003b), the teacher made substantial efforts to bring the students to symbolize the first pattern. And when the students finally produced the symbolic expression (n+1)+n (which translated the expression 'the next figure plus the figure') they did not recognize it as denoting the same mathematical object as the expression (n+n)+1. As we shall see in the next section, this Fregean problem is related to the meaning of signs with which novice students endow their very first algebraic expressions.

The Indexical Nature of Students' First Algebraic Signs

In (Radford, 2000) several students' algebraic generalizing strategies were examined. This analysis was carried out in terms of the various meanings with which signs were endowed by the students and the semiotic role that students ascribed to signs as a way to convey relations between the particular and the general. One of the reported results was the identification of the nature of the signs that the students tend to use in the elaboration of their first algebraic formulas: it turned out that these signs appear genetically related to the arithmetical concrete actions and to the objectification of these actions in speech. More specifically, novice students often use algebraic symbols as *marks* or *abbreviations* of key words belonging to a *discursive* non-symbolic semiotic layer. Thus, the students' symbolic expression n x 2+2 mirrors the utterance "The term times two plus two" previously produced during the students' discursive activity (Radford, 1999, p. 95). Following Peirce's terminology, I suggested that the students' first algebraic signs were *indexical* in nature, inasmuch as they stand for their objects in such a way that, like pointers, they appear as *indicating* the place of the objects to which they refer[3].

Although (degenerate) indexes offer students a gate through which to walk into the realm of algebra, they bear important semiotic limitations. We saw, for instance, in one of the classroom sessions, how impossible it was for a group of students to successfully add "n+n". The difficulty resides in the fact that indexical signs cannot be added. As long as they are still pointing to their objects, one cannot collect them and merge them into a single new symbolic expression. The occurrence of indexical signs in novice students unfolds in the realm of a sequentially framed experience in which the signs remain contextually anchored. As a result, *the algebraic expression is seen as a mnemonic device reflecting the actual course of the flow of calculations.* Algebraic expressions thus become inseparable from the underlying (mental or concrete) actions. For the students, (n+n)+1 and (n+1)+n are two different expressions because they signify two different actions.

To become a more fluent practician of algebra hence requires being able to endow indexical meanings with non-indexical ones. And this demands the effacement of the individual from the actions that s/he performs. The difficulty of the effacement of the individuals in the analysis of the action that they produce was noticed by Piaget (1955) many years ago. The children's "*décentration*" of their actions was indeed seen by him as something of great importance in the development of logical thinking. One of the reasons for the ontogenetic persistence of the action as a link between subject and object may be that, as Vygotsky (1997) suggested, actions appear as a formidable source of meaning in the emergence of the child's semiotic activity.

Synthesis and Concluding Remarks

3. Peirce distinguished between two different kinds of indexes: *genuine* indexes and *degenerate* indexes. A genuine index involves an existential relation with its object, whereas a degenerate index involves a referential relation (see Peirce, 1955, p. 108). Clearly, the students' use of signs in their emergent understanding of algebraic activity corresponds to the second kind.

In this paper, we discussed two related crucial turning points in the students' reflective insertion into the phylogenetically constituted practice of algebra. Adopting an anthropological approach and seeing thinking as a kind of *cognitive praxis*, the first point that we discussed dealt with how students understand algebraic generalizations. The second point addressed the question of the social means of semiotic objectification that students use to shape and make apparent the noumenal and unperceivable object Figure n of a geometric-numeric pattern. While the first point sheds some light on the kind of tensions occurring during the expansion of the students' personal senses to attain some of the social meanings required in algebra, the second point evidences some of the means of semiotic objectification used by the students. Although space constraints did not allow us to go any further, the sketched results, nevertheless, make it possible to reformulate the students' acquisition of the first rudiments of the algebraic language as required in algebraic generalizations in terms of the disembodiment of meaning and experience. Although we have succeeded in unveiling some of the students' processes of semiotic objectification (e.g. those recurring to metaphors or deictic and generative functions of language) and have gained an insight into the understanding of the students' difficulties in achieving algebraic generalizations, the engineering of finer teaching activities leading to the classroom processes of disembodiment of concrete, spatiotemporally situated meaning and the mechanisms of its re-embodiment as required in algebraic symbolic language still requires further research.

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