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Edited by

Jim Tattersall
Providence College
Providence, Rhode Island
and
United States Military Academy
West Point, New York

Linking Psychology and Epistemology: Can the History of Mathematics Be a Useful Tool for Teaching Mathematics?[1]

Luis Radford
Université Laurentienne
Sudbury, Ontario

§1 Epistemology and Psychology

By the turning of the 5th century B. C., the materialistic standpoint of the Pythagoreans was challenged by a distinction between what can be grasped by sense-perception and by pure reason. Parmenides, in his known poem *On the Nature*, says that the former leads us to the *opinions*, while the latter leads us to the *truth*. This distinction was a central point for Plato, who insisted that true knowledge (*episteme*) is the knowledge of the Forms and it cannot be achieved through the sense. Plato accommodated the unsensible with the sensible by saying that the sensible phenomena (*aistheta*) participates in the Forms -the *eide*.

Perception and the external world was, however, seen rather suspiciously. In *The Republic* (602 d), he says:

> "A stick will look bent if you put it in the water, straight when you take it out, and deceptive differences of shading can make the same surface seem to the eye concave or convex; and our minds are clearly liable to all sorts of confusions of this kind."

Contemporary Experimental Psychology has been, among other things, interested in explaining the role of perception and the role of our interplay with the external word in the study of cognition and knowledge. As in the past, it has not been done without arousing suspicion in some philosophical circles where the ancient word epistemology has taken the meaning of the philosophical analysis of human knowledge.

Within this context, some philosophers deny any determinant contribution of psychology to epistemology. Corlett (1991, p. 286) quotes Chisholm:

> "Contemporary interest in the nature of knowledge pertains not only to the branch of philosophy called "theory of knowledge" or "epistemology", but also to the fields of information theory, artificial intelligence, and cognitive science. The latter disciplines are not alternatives to the traditional theory of knowledge because they are branches of empirical science and not of philosophy. For the most part, the facts with which they are concerned are not relevant to the traditional philosophical questions. Unfortunately, however, this relevance has been exaggerated by many writers who do not seem to have grasped the traditional problems ..."

Psychologists violating the lands of thinking, millennially reserved for philosophy, are perceived as perpetrating the 'genetic fallacy'. Despite clear resistances, one of the problems that has been discussed in the past years is that of the connections between epistemology and experimental psychology.

It is clear that philosophical and empirical epistemology supposes an idealized subject. Roughly speaking, in the case of the former, the idealized subject must follow some rules (e.g. rules that say how one's perception works; see Chisholm (1966, especially

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chapter 3). From an apriorist point of view, one's behavior (i.e. one's adherence to the rules) does not have to be tested. In the case of the latter, the idealized subject is the result of the comparison of many actual subjects. Often, the idealized subject is a *statistical subject*. It is in this sense that we can speak of 'the' individual and his or her knowledge.

What is at stake in the collaborative projects between epistemology and psychology is whether the study of the philosophical idealized subject's knowledge may shed some relevant light on the psychological actual subject's knowledge and vice-versa. The limits and possibilities of this cooperation is at the core of some current studies –among them Corlett (1991), Goldman (1986), MacKay (1988), and Kitchener (1980).

Corlett suggests a mutual collaboration between epistemology and what he calls Experimental Cognitive Psychology (ECP). Thus, the latter can contribute to the former by providing some experimental data about decision making processes, human attention and memory (*Op. cit*. pp. 288-289). On the other hand, epistemology can contribute to the ECP by providing "philosophical analyses of concepts and arguments regarding hypotheses employed by ECP to explain observed human cognitive processes" (*Op. cit.* p.301). Furthermore, it is suggested that epistemology may help ECP by analyzing the relations between various concepts and difficulties about knowledge.

§2 Psychology of Mathematics and Historic Epistemologic Programs

Considering Corlett's words from the specific point of view of the psychology of mathematics, the problem to which we are led is to specify the *kind* of epistemological analysis that will allow us to understand –from a psychological level– the relations between concepts and difficulties about mathematical knowledge.

Curiously, as P. Kitcher (1984) has pointed out, the history of mathematics, within philosophical epistemological discussions, is usually not considered within philosophical epistemological discussions as a worthwhile source of information to elucidate the nature of mathematical knowledge. In fact, it seems that such a use of the history of mathematics has been practiced in restricted groups of philosophers of mathematics[2]., historians and mathematics education researchers.

By the same token, mathematics education researchers, in their didactic inquiries, tend to have recourse to the history of mathematics rather than to the general philosophical epistemology. For instance, a mathematics education researcher seeking for some non-psychological information about the concept of number would take into account the *historical* development of the concept of number rather than a *philosophical* account (e.g. Husserl's *Philosophie der Arithmetik*)[3].

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[2]Particularly the generation of philosophers of mathematics influenced (de près ou de loin) by Lakatos' works – in which one of the main questions is to determine the normative criteria underlying decision making processes in front of competitive mathematics research programs (see e.g. Glas, 1993)

[3]For a good sample of examples, see, e. g., the *Proceedings of the History and Epistemology in Mathematics Education*, (Lalonde *et al.* eds., 1995).

However, it is far from easy to have recourse to the history of mathematics in order to collect some data and interpret them in potential ways to be useful for teaching[4]. Indeed, the classical historiography of mathematics gives an account of the most exciting episodes of mathematics throughout time. Usually, this narrative is underlined by an apriorist epistemology of platonistic style. Hence, ancient mathematics is often (implicitly or explicitly) evaluated in terms of Modern Mathematics (see Unguru, 1994, p. 215 ff.). Ancient theories are then seen as potential modern theories; a mathematical idea I_1 is considered better than another idea I_2 if and only if I_1 looks more like the *real or true idea* I_∞ than I_2, where the real or true idea is that of the Modern Mathematics. One of the problems for the psychologist of mathematics and the mathematics educator is that the data provided by the classical apriorist historiographic program cannot significatively contribute to his or her didactic enterprise (see Thomaidis, 1993, p. 71).

Let me give just one example. When Diophantus' *Arithmetica* is seen from this perspective, this monumental treatise appears as a mere compendium of problems solved in a way that "dazzles rather than delights" and Diophantus himself is considered as someone looking only "for correct answers" (Kline, 1972, p. 143). However, when Diophantus' methods are seen from their own historico-epistemological perspective -which requires retracing their links to the Egyptian and Babylonian false position methods and, on the other hand, to the surveyors' geometrical cut-and-paste methods (cf. Høyrup, 1994), Diophantus' methods appear very sophisticated. (A detailed study on this subject can be found in Radford, forthcoming$_1$).

To understand the historical status of a mathematical idea we need not look to its future but to its past. We need to replace the idea under study within its own scientific research program and the cultural conception of mathematics in which such a scientific research program is embedded. The cultural conception of mathematics determines not only its social function but also -in a more abstract level- the conception of mathematical objects themselves.

Additionally to the aforementioned complexity of examining in depth a mathematical idea from a historical point of view for epistemological purposes, there is a just as important point to consider, namely, the specific role that we attribute to history.

What can history reveal to us? What is the kind of truth that history can uncover about a mathematical fact?

In order to stress the relevance of these questions, let us consider here, very briefly, the historical idealism of the nineteenth century. This historical movement assumed a certain coherence *hidden* behind the phenomenal world. One of the most

[4] Of course, a naïve way of linking history and mathematics is to have students solve old problems using modern tools or telling historical anecdotes to the students. A closer look at the question shows that here history appears as a mere strategy to make students approach a curricular subject. Within this approach, the use of history remains on a superficial level. This is why such an approach is *unproblematic* and uninteresting for the psychology of mathematics.

important thinkers of this period was Leopold Ranke, for whom the aim of history was that of grasping this hidden coherence.

Thus, the work of historian was that of uncovering the hidden coherence of the world and exposing it to the sunlight. To achieve this, given that the historical results do not follow the stream of logical deductions, Ranke gave two methods that have to be followed simultaneously:

> "the first is the exact, impartial, critical investigation of events; the second is the connecting of the events explored and the intuitive understanding of them which could not be reached by the first means. To follow only the first path is to miss the essence of *truth* itself..." (Ranke, 1973, p. 7; our emphasis).

The second method was justified by Ranke's ontological position, according to which "there is also [in the nature] the bread of life in the whole and an inner character which speaks through it which can be neither measured nor merely described." (*Op. cit*. p. 7).

At the end of the ninteenth century the hidden coherence of history was questioned. In fact, a new trend in the nature of historical inquiries began to take into account the role played by social and cultural factors, and the role played by our own understanding of reality. Thus, as Iggers said, commenting on some related ideas of Max Weber:

> "Our understanding of reality did not reflect this reality as it really was but answered the questions which the scholar and scientist had asked of it." (Iggers, 1995, p. 134).

The post-idealist conception of history gave a new *raison d'être* to the history and led to the rise of new questions and to the implementation of new methodologies. In the case of science, this new orientation is very clear in the 'mature' works of McKeon. His enlightening work on the medieval (Muslim, Greek, Latin, Hebrew) encyclopedias is carried out in terms of structural comparisons that have to be understood in terms of intellectual and cultural factors (McKeon, 1975). Nevertheless, a good number of mathematics historians have not been prompt to follow the new general historiographical trend as their colleagues, the historians of sciences, did. In fact, even though their dominant apriorist program was challenged this century by new philosophical currents (such as intuitionism or nominalism), the idea of considering cultural and social factors as key factors in the understanding of the historical production of mathematics still has not gained much success (despite Tannery's sensitivity to the social role of science; see Guerlac, 1963, p. 808). Lakatos himself made a clear distinction between 'internal history' and 'external history' -and, of course, he considered the latter as a rather accessorial one.

To sum up our discussion, let us stress the fact that psychologists of mathematics tend to have recourse to historico-epistemological analysis rather than to the general philosophical epistemology. Although it seems very reasonable to think of a fruitful interdisciplinary collaboration, this is heavily restricted by differences in their corresponding frameworks. As we said, the dominant apriorist framework in the history of mathematics makes it difficult to use the historical data in order to answer the relevant epistemological questions for teaching. One of these questions may be summarized as follows: *How* does mathematical knowledge grow?

§3 Methodological Aspects of Historico-Epistemologic Research Programs

When Kitcher discusses his realistic evolutionary approach to mathematical knowledge, he does not conceal his surprise that the growth of knowledge is not one of the questions that is usually considered by his philosophy colleagues (Kitcher, 1984, p. 96). The historical growth of mathematical knowledge is, however, at the very center of psychological and epistemological interest in many educational circles.

Likewise, any historiographical program or historico-epistemological research program needs to specify its own aims and methodology. Concerning the latter, we suggest that there are two different methodological problems to face. The 'first methodological problem' of a historico-epistemological research program is to specify how this program may be carried out.

Among other things, we need to specify:

(1) what are the 'interesting' historical data to be collected,

(2) how to collect and to *describe* these data[5].

At another level, there remains what we shall call the 'second methodological problem' –which is specific to psychology and teaching–, namely, the link of the historico-epistemological results to the psychology of mathematics. Two different links, it seems to me, may be considered:

(1´) A link at the level of the conceptual historical processes aiming to shed some light on the cognitive actual subject's processes,

(2´) A link at the level of the construction of teaching sequences –something often called *ingénierie* didactique[6] (of course, (1´) and (2´) are not independent items).

The answer to these questions will depend upon the framework upon which the research program relies. This framework is particularly shaped by:

(α) our own perception of how mathematical knowledge grows;

(β) our philosophical and epistemological hypothesis allowing us to link historical mathematical developments to cognitive ones.

For instance, concerning the 'first methodological problem', a platonistic apriorist program will tend to describe ancient ideas in terms of their likeness to modern ones and the growing of mathematical knowledge will not be considered in terms of intellectual constructs but in terms of *discoveries*. In regard to the 'second methodological problem', such a program will tend to assume (albeit rather implicitly) that the intellectual link between past and present developments is ensured by keeping todays students solving old problems.

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[5] The epistemological description of historical data requires the introduction of non-historical concepts (e.g. *epistemological obstacle* to which we shall return later). It takes place in a metalanguage that aims to answer the question posed by the epistemological inquiries. To better understand the role of the conceptual apparatus allowing one to *describe* epistemologically the historical data, we may contrast the task of the history with the task of the epistemology by taking Unguru's description of the history: "History is primarily, essentially interested in the event *qua* particular event ...History is not (or is primarily not) striving to bunch events together..." (ibid, p. 208). In contrast, the link between events, in regard to the formation and the growth of knowledge, certainly is the task of epistemology. And to do so, new concepts are required.

[6] See, e.g., Artigue, 1988.

As soon as we leave the realm of the restrictive 'internalist' framework of the growth of mathematical knowledge (a realm that is not comprised of simply platonistic approaches), we are led to a new major problem, namely, that of the social dimension of knowledge that may be summarized here, for our purposes, as follows: *How* do social factors participate in the production of mathematical knowledge?
There are some alternatives to the extreme claim that knowledge develops independently of sociological factors of 'internalist' programs. One of them, is the Strong Program in the Sociology of Knowledge, developed by Bloor (1976). The first of its 4 tenets assumes a social causal conception of knowledge and gives to the program an on-going causal orientation that, as Otte (1994, p. 296) says, confines all human facts to human interactions –thus, to *social behaviorism*. Beyond the internalist and social behaviorism standpoints, there remains room enough to develop new approaches. In the field of mathematics education, these approaches tend to oscillate between radical and social constructivism.

§4 Constructivist Programs

It may not be exaggerated to say that most current research educators and psychologists of mathematics adhere to constructivist frameworks. According to constructivism, the knowledge is not passively received by the cognizing subject nor is the knowledge decoded through apriorist mechanisms. Constructivism claims that knowledge is actively built up by the cognizer through some processes described in terms of a biological metaphor –the processes of equilibration and assimilation.
As almost any theory, constructivism has some different schools: one of them is what is called the *Radical Constructivism*, which states that the aim of the constructed knowledge is to organize the experiential word; furthermore, Radical Constructivism states that the cognizing subject constructs personal representations of the world, but the resulting constructed knowledge does not tell anything about the world. Without denying the physical world, objective knowledge is beyond the scope of the cognizer. Another school is constituted by the *Social Constructivism*, in which the social factors play a central role in the construction of knowledge[7].
We are not going to discuss here the questions that constructivism raises, such as explaining the stability and convergence of our personal representations[8]. What we want to mention is that constructivism has provided a general set of frameworks for the study of psychological phenomena related to the growth of mathematical knowledge and that constructivist conceptions of knowledge may also sketch a path to drive our historico-epistemological inquiries.

However, although historico-epistemological inquiries may be provided with constructivist research paths, these inquiries cannot avoid facing the problem raised by any confrontation between historical mathematical developments and modern cognitive ones (see question (*b*) in section 3).

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[7]A study of Constructivism can be found in Ernest, 1993.

[8]The stability and convergence of our personal representations are two important elements taken into account by other theories to support the idea of the possibility for the individual to get an accessible and true knowledge of the real world (the constructivist point of view is discussed in Glaserfeld, 1990).

The relevance of the question may be better understood if we take into account that mathematical cognitive developments are embedded in modern teaching sequences which are, in turn, embedded in social and cultural contexts which are different from those of the past. This allows that the conditions of the actual psychological geneses of a mathematical concept is ineluctably different from its historical geneses (cf. Artigue, 1990, p. 246). The advancements made during the last years in the sociology of knowledge makes it difficult to accept the idea that ontogeny *must* recapitulate philogeny and leads us to rethink the veritable meaning of any confrontation between past and present conceptual developments. Let us now turn to two of the most prominent historico-epistemological approaches used in educational circles and to examine briefly some assumed hypotheses underlying such a confrontation.

§5 The Reification Approach.

A. Sfard (1995) suggests that there are two kinds of components in mathematics: that of abstract objects and that of computational objects. An individual may see a mathematical concept operationally (as a computational object) or structurally (as an abstract object). She claims that these components are related between themselves. "In a sense", she says, "the abstract objects are just an alternative way of referring to computational processes". Thus, negative numbers refer "to nothing other than the operation of subtracting a number from a smaller one" (p. 37). The new object –in the case of negative numbers–is produced by computations and will later be seen as a structural object. This development passes, according to Sfard's view, through a three-component pattern: interiorization/condensation/reification. In the first step (interiorization) the new object enters the scene; in the second step (condensation) the object undergoes some nontraditional computational processes that require a 'condensation' of it; in the third step, the object achieves a structural status. This new object may enter into a new process "in order to serve as inputs to higher level processes". (p. 138).

The confrontation between historical and psychological conceptual developments seems to be warranted, in Sfard's program, by the fact that some similarities can be noted. She goes a step further when she says:

> "It is probably because of the inherent properties of knowledge itself, because of the nature of the relationships between its different levels, that similar recurrent phenomena can be traced throughout its historical development and its individual reconstruction." (*ibid.* p. 16).

Doing so, Sfard provides us with one of the clearer attempts to explain the similarities between historical and psychological developments. Nevertheless, there are some philosophical and cognitive neo-piagetian assumptions in Sfard's point of view that have to be stressed. She assumes that knowledge can be described in terms of levels connected by some specific relationships. Furthermore, she assumes that there are some *inherent* properties of knowledge that seem to be independent of (or at least capable to be *isolated* from) sociological factors. This last hypothesis –to which we will later return– also appears in the research program based on epistemological obstacles that we will now analyze.

§6 Epistemological Obstacles

Although history of mathematics has been used in teaching contexts for at least a century[9], it was only some twenty years ago that the idea of analyzing the mathematical knowledge from a historical perspective in order to better understand the students' cognitive processes came onto the scene. One of the pioneer works was that of G. Brousseau in the '70s, who transposed into mathematics the notion of epistemological obstacle previously developed by G. Bachelard in his studies on the scientific thought.

Bachelard (1938, p. 13) , in a paragraph that has become famous, says:

> "Quand on cherche les conditions psychologiques des progrès de la science, on arrive bientôt à cette conviction que *c'est en termes d'obstacles qu'il faut poser le problème de la connaissance scientifique*. Et il ne s'agit pas de considérer des obstacles externes comme la complexité et la fugacité des phénomènes, ni d'incriminer la faiblesse des sens et de l'esprit humain: c'est dans l'acte même de connaître, intimement, qu'apparaissent, par une sorte de nécéssité fonctionnelle, des lenteurs et des troubles. C'est là que nous montrerons des causes de stagnation et même de regression, c'est là que nous décèlerons des causes d'inertie que nous appelerons des obstacles épistémologiques."

The concept of epistemological obstacle gives to Brousseau a way to interpret the recurrent mistakes that students make when they learn a specific topic. A recurrent mathematical mistake is not aleatorically produced. There is a logic behind it that is explained in terms of a knowledge that suffices to solve some problems fruitfully but fails to appropriately solve other problems. When applied beyond the limits of its scope or domain of validity, the knowledge produces wrong results.

Given that the logic behind a recurrent mistake may have different causes, Brousseau (1983, p. 177) gives us a classification of sources of obstacles:

(1) an *ontogenetic source* (related to the students' own cognitive capacities, according to their development);

(2) a *didactic source* (related to the teaching choices[10]);

(3) an *epistemologic source* (related to the knowledge itself);

Epistemological obstacles (EO) are entailed by the third source. Within this context, the role of the didactician is, according to Brousseau (1989, p. 42):

(1) to find the recurrent mistakes and to identify the underlying conceptions,

(2) to find the obstacles in the history of mathematics and

(3) to compare the learning obstacles and the historical ones in order to decide whether they are EO.

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[9]Rogers, 1995, p. 108, characterizes Cajori's book *A History of Elementary Mathematics with Hints on Methods of Teaching*, published in 1896, as a text providing an inductivist account of the history of mathematics and hinting at a teaching that mirrors the 'inductivist' progress of this science.

[10]Even though the didactic obstacles of mathematics education researchers are well-known, let me refer to one of the most known examples, that of decimal numbers. Frequently, when decimal numbers are taught in the classroom, they are presented (implicitly or explicitly) as natural numbers having a period. This leads the students to see the number 1.2 as lesser than the number 1.15 (for comparing the decimal parts, the student tends to see 2 and 15 as natural numbers). The way in which we present decimal numbers in the classroom and the language that we use to refer to them makes it possible for students to think of decimals in the wrong way. The resulting mistakes are not of ontogenetic source. They are related to our way of teaching: thay are *didactic obstacles*.

A good number of works in the didactic of mathematics field have been carried out from the perspective of EO. For instance, A. Sierpinska (1985) studied the notion of limit in terms of EO. She takes this same perspective for her study of the notion of function (1992).

In her work of 1985 (pp. 7-8), after mentioning the controversies that the EO raised in didactic circles, she says:

> "Pour notre part, nous retiendrons deux aspects de la notion d'obstacle épistémologique selon G. Bachelard (Bachelard, 1938):
> – l'apparition des obstacles a un caractère inévitable [...]
> – la répétition de leur apparition dans la philogénèse et l'ontogénèse des concepts."

However, in a revision of her program in a later paper (Sierpinska, 1989) she contemplates abandoning the aformentioned second point.

In order to understand some implicit presuppositions of the perspective based on EO, let us make some remarks. First of all, as in the case of Sfard's approach, the scholars who adhere to the EO perspective make a *hypothesis* that there is a conceptual constituent in the knowledge that may be *isolated* from its other constituents (e.g. social constituents). And this hypothesis seems to be placed not at a methodological level but at the level of the conception of knowledge itself. This hypothesis allows one to make the acultural and asocial comparisons mentioned in the 3rd point of Brousseau's program quoted above. Thus, in the didactic studies carried out from the EO perspective, the search of epistemological obstacles is done through the confrontation of todays students' solutions with mathematicians of past cultures: it is supposed that what one is observing and comparing is the 'pure' mathematical knowledge. However, a closer look at the history of ideas suggests that ideas are not accessorily shaped by social concerns (we will discuss an example in the next section).

Our second point concerns the *nécessité fonctionnelle* of obstacles which constitutes the first point addressed by Sierpinska. She insists in its importance some lines later when, referring to Bachelard's words, she says:

> "Ici c'est le mot «nécessité» qui nous semble le plus important. Il souligne le fait que cela n'a pas de sens de chercher à éviter les obstacles; on doit buter contre l'obstacle, en prendre conscience et ensuite le franchir pour progresser dans le développement de son savoir" (p. 8).

In the EO perspective, the knowledge, in its deepest essence, once it has been dispossessed of all of its other constituents and put in its most marvelous nakedness, will, when walking, strike, *inavoidably,* its obstacle. If we see the difficulties that the western medieval mathematicians had in facing the negative numbers, and if we see the difficulties encountered by our students today, we are led to think that, effectively, positive numbers constitute an obstacle for the emergence of negative numbers. If we retrace, however, the negative numbers to chinese mathematicians, we see that they overcame the difficulty of handling negative numbers through a very clever representation of negative numbers by coloured rods and did not seem to face the same problems (or at least not with the same intensity) as the ones that their italian medieval colleagues did. Even a simpler example is that of the concept of zero. The concept of number (conceived as a certain amount of units set together, according to Euclide –*Elements* VII, def. 2) could be an obstacle for the emergence of the concept of zero for a Greek mathematician. Yet it does not seem to have been the case for a Mayan priest. What seems to be an obstacle may cease to be one when seen from a pancultural perspective.

Our third remark deals with a certain perception of mathematical knowledge. According to the EO perspective, the EO are seen as causes of stagnation and even of regression of knowledge (see Bachelard's quotation above); they produce slowness and troubles in the development of knowledge. As in the case of the natural evolution, this teleological perception of knowledge suggests, it seems to me, a being fighting against itself in order to keep itself always evolving. Thus, a period of stagnation in the history of a certain concept is seen as a period in which the knowledge is *failing* to go further. If a certain conception of number achieves our goals and suffices to solve the problems with which we are confronted on an everyday basis, why must it be changed? If a certain conception of mathematics fulfils the needs we have very well, why must we change it? A clear example of this last point is the neo-pythagorician mathematical activity developed in some Hellenic cities at the end of the Antiquity. Mathematicians engaged in this direction, such as Theon of Smirne and Nichomachus of Geresa, did not change their approach to figurative numbers based on patterns by a euclidean one (as their more or less contemporary colleague Diophantus of Alexandria did) neither because they did not know it nor because they could not practice it but because a euclidean deductive perspective did not fit their scientific goals[11].

Our fourth remark concerns the view that the EO perspective imposes on past mathematics. Effectively, an EO appears as a wrong or a non-achieved conceptualization of a certain domain of mathematics. Its overstepping is supposedly accomplished when a new enlarged conceptualization appears. However, often, at a psychological level, the expected enlarged conceptualization is seen as the one that the history of mathematics shows us. Thus, history becomes in some way *normative* for the study of psychological phenomena. One implicit assumption seems to be that history could not turn out differently. However, a closer look at history suggests that in many controversial points of the development of mathematical ideas, there were some alternative programs engaged in competition. This shows the co-existence of different cognitive tendencies at the same point in the history of mathematics and warns us that there is no *a priori* reason to think that our students will spontaneously follow the path showed by winning mathematical programs. Furthermore, the criteria used to evaluate competitive mathematical research programs were not "mathematical" criteria only, but there were also social criteria (Glas, 1993; Radford, 1995).

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[11] Janvier *et al.* (1989, p. 64) seem to have perceived this uncomfortable position to which we are led when we consider, within the perspective of EO, the knowledge as a being trying continually to go further. For they say, after having discussed the idea of obstacle as a knowledge that is set up against another knowlege already in place, "L'expression franchir un obstacle épistémologique nous apparaît donc inappropriée, car elle évoque *une course à obstacle* dans laquelle le mouvement est contrecarré par ce qui se trouve devant, ce qui constitue une analogie incorrecte, l'obstacle épistémologique se trouvant derrière. Vaincre l'obstacle épistémologique semble plus juste et la démarche sous-jacente combine rationnalité et affectivité." (our emphasis)

Another remark, addressed by M. Artigue (1990, p. 260), is that the concept of EO is mainly based in its property of producer of errors. This makes the field of study of EO very restrictive, given that a very large number of mathematical developments were not, historically, motivated by errors encountered in mathematical procedures or solutions. To articulate this point, Artigue gives the example of *quantièmes* -unit fractions- to express fractions in ancient Egypt –an example developed by Brousseau himself. She says: "La conception des quantièmes ne produit pas de résultats faux. Si elle est abandonnée, c'est parce qu'elle se révèle inadaptée à certains problèmes, à certaines opérations." Artigue (*ibid*, p. 260)

Certainly, there are other criteria to further develop the mathematical knowledge than the criteria of avoiding errors. The History of mathematics is full of such examples. For instance, the emergence, in the late Middle-Ages, of the concept of second unknown in algebra was not motivated by errors using algebra with a single unknown but to make the calculations easier (Radford, 1994; Bednarz *et al.* 1995). The emergence of litteral symbolic language followed this criterium of efficiency also[12].

§7 Synthesis and Concluding Remarks

In a work written some fifty years ago, Crombie suggested that a style of thinking is determined by commitments to conceptions about nature and to conceptions of science. The first deals with "general schemes of existence and its knownability to man", the second deals with the "organization of scientific inquiry, arguments and explanations" that are specific to each style. These two conceptions together determine, he said, "the kind of arguments, evidence and explanation that will give satisfaction, because the supposedly discoverable has been discovered in conformity with the acceptable criteria." (reprinted in Crombie 1995, p. 232).

When the history of mathematics is seen through Crombie's lens, it becomes clear that our look to the past cannot be taken without extreme precautions. An historical mathematical fact cannot be observed without taking into account its 'ecological' intellectual system (which comprises Crombie's two commitments). Otherwise, we may make the error of 'presentism' as so-called by historians –i.e. to see the past according to our modern norms, criteria and values. This makes the work of the didactician more difficult than expected when he or she turns his or her attention to the past in order to collect some data to better understand the modern students' psychological learning processes.

Beyond the inevitable specific difficulties of what we called the 'first methodological problem' –difficulties to which any conscious historian is ineluctably confronted (see section 3)– there is another question that does not concern the history and which is at the core of the 'second methodological problem': is it really possible to relate historical and psychological developments?

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12 Probably moved by similar ideas, Sierpinska, in the revised version of her research program (1989, p. 142), attenuates her first standpoint and introduces the idea that in the epistemological analyses we might study not only the EO (which gives, according to her, a 'pessimistic' point of view) but the "moteurs d'évolution" as well.

This question was framed, in section 1, by an emerging trend that encourages a cooperative work between epistemology and psychology. This made us examine some methodological and philosophical aspects of such a cooperation –although here we considered just a direction of the cooperation, namely, how historical epistemology may help psychology. In this line of thought, we examined some assumptions underlying two of the clearer approaches used nowadays by didacticians –Sfard's reification process and the approach based on epistemological obstacles. One point common to the two was the assumption that knowledge may be isolated from its 'ecological' system and may be analyzed in terms of its conceptual kernel[13]. There is, however, another issue. According to this issue, the knowledge is rooted in its 'ecological' system. Likewise, in some non separable topological spaces, knowledge is not separable; it is bound forever to its own social and cultural context through the chains of the cultural and social conceptions of the world. Mathematical knowledge grows within socially accepted thought structures which give sense to the mathematical problems with which we come up.

Of course, the idea of seeing mathematical knowledge as a social construct is not a new one. Above, we referred to Bloor's *Strong Program* which, in turn, was preceeded by other works (e.g. K. Mannheim's *Ideology and Utopia*, published in 1929 and translated into english in 1936). In one of the most recent reflections on the socio-historical accounts of mathematics, M. Otte (1994, p. 309) says: "The development of knowledge does not take place within the framework of natural evolution but within the frameworks of sociocultural development" and "Knowledge is necessarily social knowledge."

In order to provide an idea of how mathematical knowledge may be rooted in its own social context and cannot be extracted from it, let us briefly discuss an example -the example promised in the previous section in our first remark on EO.

It is well-known that the emergence of mathematical proof formed part of a social intellectual trend consisting of a style of argumentation that flourished *ca.* the 5th century B.C. in some Greek philosophical circles. It was probably encouraged by the Eleatic discussions about the Negation of Being. Within this context, one of the problems was to explain the existence of things in terms of a few principles (e.g. the fire, the air). Mathematical objects were seen as forming part of the unsensible world and attainable only by thought. The junction of a fight against senses (as Plato's words remind us at the beginning of this article) and the deductive style of the current philosophical arguments based on a few principles made it possible to transpose the same idea to the realm of mathematical objects. Even if I am oversimplifying for the sake of brevity, we can realize that the emergence of mathematical proof was profoundly rooted in its sociocultural context. Mathematical proof was compatible with the Greek commitment to their conceptions of nature and science.

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13 Of course, this does not mean that the scholars of the EO perspective do not recognize that cultural factors play a role in the development of knowledge (e.g. Brousseau, in his paper of 1989, added to his taxonomy of 1983 a new category of obstacleobstacles: the *cultural* obstacles).

Let us now consider a modern internet surfer student for whom a deductive reductivist style of argumentation is far from being his or her natural social way of communication in scientific and non-scientific matters. Confronted with the beginnings of mathematical proof, she will experience many difficulties in understanding this strange way of linking propositions to each other to reach a conclusion that was probably already visible on the figure that the teacher drew. In contrast with ancient Greeks, our modern student cannot perceive that what is at stake is the ancient Greek sociocultural conception of things (as *eide*) and the refusal to think in terms of concrete things (see e.g. Plato, *The Republic*, 510d). Both sociocultural commitments do not have room in modern sociocultural conceptions of the world[14].

Yet there is another example that deserves to be discussed. In order to do so, we need to remember that the number one was not considered as a number by Greek mathematicians. Not merely because it was not a multitude of units or monads but overall because of the deepest philosophical meaning of the "monad" linked to the beginning of things –one of the most fundamental problems of Greek thinking[15]. This special status of the monad brought some problems into the realm of calculations. Thus, Euclid himself was sometimes led to prove the same theorem twice because of the distinction between the "veritable numbers" and the "monad" (see, e.g. *Elements* VII-9 and VII-15). If we isolate the 'pure' knowledge, we could see VII-15 as an unnecessary repetition of VII-9. The problem is that if we proceed to an isolation act, we denaturalize the phenomenon and we are no longer observing it. The Greek knowledge about the concept of "one" has a social component that cannot be deleted from the concept. The mathematical concept is embedded in a philosophical one. To ask Euclide to delete it means to ask him to renounce commitments to the Greek conception of the world.
However, commitmemts to conceptions of mathematics are not merely concomitant to philosophical points of view. What we are trying to stress is that commitments to conceptions of mathematics are, first of all, *social* commitments. Indeed, to understand our point, we need to look at our two examples through their sociocultural contexts and remember that mathematics was not just a phiosophical speculation but was profoundly linked to a way of living in the world. The history of the first pythagoreans -who were spread out in communities in Greece and Italy- were engaged in ethical and political matters. Their conception of life influenced all antiquital thought (e.g. the Academy, the Neo-Pythagoriceans, Middle Platonist). They established rules of conduct by which to live and to be good. Iamblichus, referring to the education of a youth, says:

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[14]where –one might be tempted to say– there exists all that can be seen: it is unplausible that *virtual reality* games were known for their modern success in ancient Greece!

[15]The association of the monad with the beginnings of things may be retraced in many Greek texts. For instance, in the *Theologomena Arithmeticae* -a book belonging to the pythagorean tradition and often attributed to Iamblichus, we read: "The monad is the non-spatial source of number ... Everything has been organized by the monad, because it contains everything potentially" Iamblichus, ed. Waterfield 1988.

"Pythagoras then introduced him to the rudiments of arithmetic and geometry, illustrating them objectively on an abacus, paying him three oboli as a fee for the learning of each figure. This was continued for a long time, the youth being incited to the study of geometry by the desire for honor, with diligence, and in the best order" (Iamblichus' *The Life of Pythagoras*, in: Guthrie, ed. 1987, p. 62).

Note, however, that one may accept that our two examples effectively show how mathematical knowledge is rooted in its own sociocultural context, but at the same time, one may argue that our examples are two among the best examples of *cultural obstacles*. Let us suppose that this is the case. Schematically, when we compare two cultures, C_1 and C_2 ,in regards to their performance in front of a certain problem p, we have the following issues ("+" means success and "-" failure).

		C_1 +	C_1 –
C_2	+	a	b
	–	c	d

The case *a* corresponds to a success of both cultures in front of the same problem. If we designate by C_1 the ancient Greek culture of Euclid's times, and if we designate by C_2 the modern culture of our internet surfer student, then, the case c corresponds to our first example. According to our assumption, the differences "+" and "–", occurring in cases b and c, are caused by cultural obstacles. Why shouldn't the issue "–" and "–", represented by case d, be caused by cultural obstacles also? Why are they usually attributed to *epistemological* obstacles?

The problem, I think, is that it is not possible to distinguish between epistemological and cultural obstacles. Knowledge is ineradicable from its sociocultural context. Any similarity between past conceptualizations and today's cognitive ones cannot constitute any proof of the existence of a 'pure' mathematical knowledge. Any similarity merely proves that two different conceptualizations (underlined by different sociocultural contexts) may lead to similar responses in very specific points.

To end our discussion let us return to our main concern: is it possible to envisage a cooperative work between an historical epistemology and the psychology of mathematics?

The previous remarks about the mathematical knowledge do not mean that a collaborative work between historical epistemology and psychology is impossible. Historico-epistemological analyses are useful for teaching. Historico-epistemological analyses may provide us with interesting information about the development of mathematical knowledge within a culture and across different cultures. As a result, we will have at our disposal some elements that can help us to better understand the mathematical content of our modern curricula (Artigue, 1990, gives the example of rigor in mathematics; the case of mathematical proof that we sketched before may be another example). The way in which an ancient idea was forged may help us to find new ideas that may be compatible with modern curricula in the context of a project of *ingénierie didactique* –keeping in mind, of course, that an ancient problem or an ancient mathematical situation will never again be the same. It seems that Heracleitus was right when, standing on the bank of the river and looking at the flow of the water, said that it is not possible to step twice into the same river.

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