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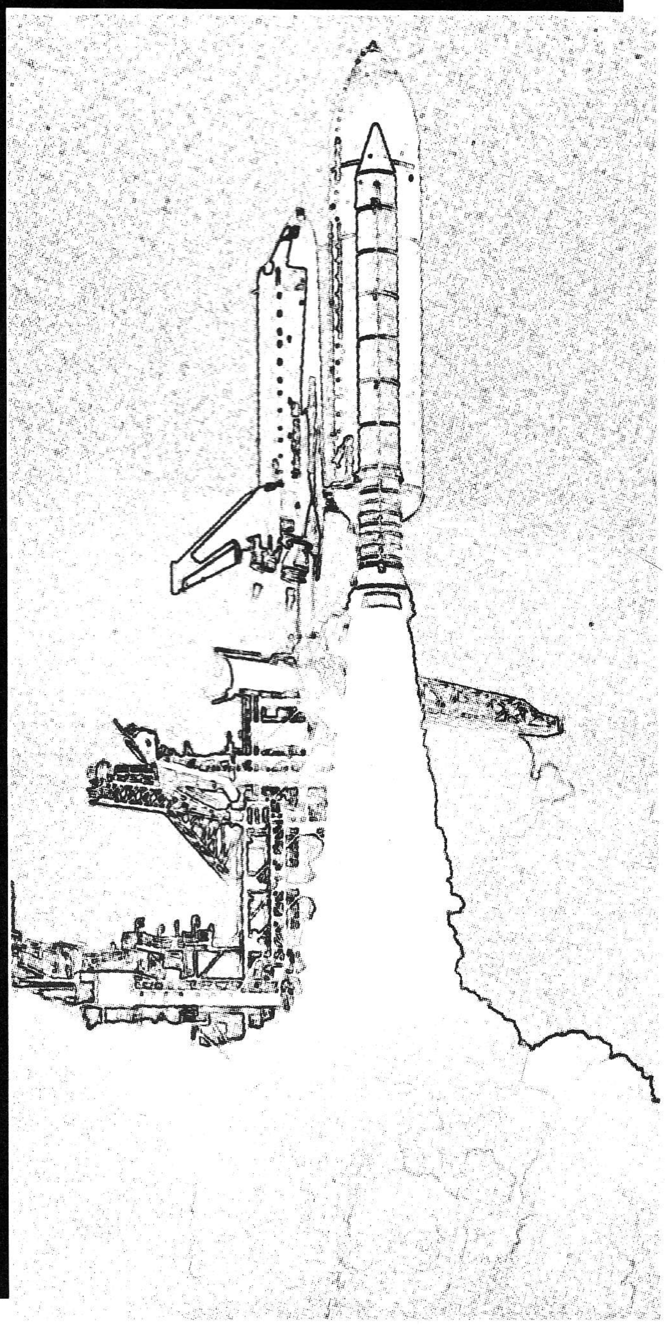

Math and the Shuttle Launch - The Ontario Connection

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▲ Helping Students to Construct and Link Problem-solving Models[1]

Luis Radford
School of Science Education
Laurentian University
Sudbury, Ontario, P3E 2C6

Luis Radford is an associate professor at Laurentian University, Sudbury, Ontario. He teaches in the preservice teacher training program. His main research includes the teaching and learning of algebra and the history of mathematics.

1. Introduction

In the last years, problem-solving has been considered an important element for the teaching of mathematics. For instance, the Provincial Standards in Mathematics have recently emphasized the role of problem-solving as one of the three key components of mathematics learning (Ontario Ministry of Education and Training, 1993, pp. 9-11. See also *Focus on Renewal of Mathematics Education,* OAME, 1993, p.4). The aim of this paper is to present a concrete approach for teaching strategies based in problem-solving methods.

Our starting point is the fact that, often, when students face a new problem they do not have an existing model of resolution. Generally, the construction of a new model which would allow them to solve the problem proves to be a very difficult task. The main idea of our approach is that, in this case, it is preferable to advance towards the process of problem-solving gradually. Instead of constructing a problem-solving procedure for the target problem, it would be better to first solve other simple problems related to it. The problem-solving procedures for the simplified problems will then allow the students to evolve towards a more complex problem-solving method which will, in turn, solve the target problem.

2. Helping Students to Construct and Link Problem-Solving Models

In order to give an idea of how our approach can be implemented in the classroom, we shall now present an example.

For our discussion, we will use the following problem[2]:

"Jane Kimble, a grocer, checked her supply of milk and counted 80 containers of milk. Some were 2L cartons and others were 3L bags. Altogether there was a total of 220L of milk. How many of the containers were 3L bags?

(Target problem)

This example belongs to a wide group of problems that we can write, using algebra with two unknowns, in the form of a linear system of equations: $x+y=a$; $bx+cy=d$. However, in the Curriculum Guidelines (Ontario Ministry of Education, 1985), the methods of algebra with two unknowns are introduced only in Grade 10 and, at the General Level, it is done either through graphical procedures (via intersection of straight lines) or through numerical ones (Ontario Ministry of Education, 1985, p. 47). How can we help students in Grades 7, 8 and 9 to construct an algebraic problem-solving model without using two unknowns?

In what follows, we shall suggest a possible didactic path which shows how we can help students in Grade 7, 8 or 9 to construct a sequence of problem-solving models moving from a problem-solving model based on concrete thinking to another one based on symbolic thinking. In order to do so, we shall begin by considering a simplified problem of the target problem.

The simplified problem follows:

"Jane Kimble, a grocer, checked her supply of milk and counted 30 containers of milk. Some were 2L cartons and others were 3L bags. Altogether there was a total of 83L of milk. How many of the containers were 3L bags?"

(Simplified Problem)

2.1 The Trial and Error Method

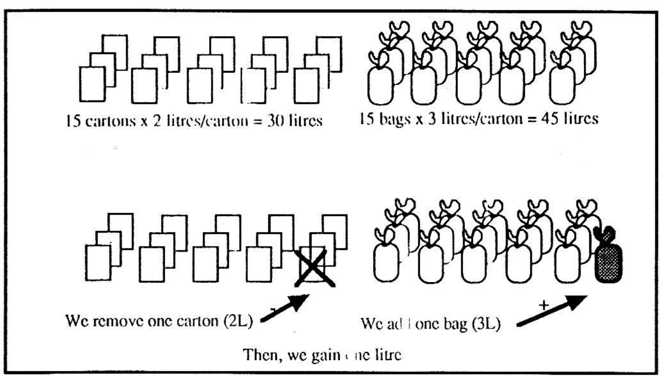
The simplified problem could be solved via trial and error. It is a simple method that has the advantage of requiring knowledge of only simple arithmetical concepts but it has the disadvantage that, to solve other similar problems, it can take a long time to find the answer. (The time will depend on the "size" of the numbers; in general terms, the bigger the numbers the less efficient the method). The didactic problem is then that of convincing students to move on to more complex procedures which can be done if we succeed in convincing them of the "power" of the new procedures. Here is an example of how this might be done.

2.2 Towards a more complex problem-solving model: a manipulative-based method

In past times, when number symbols were not yet invented, the shepherd used to take a little bag containing as many stones as he had sheep. When he returned home in the evening, he compared the stones with his sheep. If there were as many stones as sheep, he knew that he had not lost any of his sheep that day. Once symbols were invented to represent numbers, new procedures for counting and calculating emerged (Pallascio et al. 1993). This historical fact reminds us of the very important role played by manipulatives in the development of concepts. How can we help students move to a more complex problem-solving model using manipulatives?

Let's once again consider the simplified problem stated above. We may begin by supposing that the number of bags is the same as the number of cartons. Thus, we have 15 of each type of milk container. On a table placed in front of the students, we place 15 cartons or plastic boxes containing 2 objects each (two blocks, for instance, each one simulating a litre of milk). Beside this table, there is another table on which we place 15 bags, each containing 3 objects (three blocks, once again simulating litres of milk). At this point we can ask the students to calculate the total number of litres; they will find the answer to be 75 litres.

Instead of replacing the numbers of cartons and the number of bags, as done in the trial-and-error method, we will think in terms of how much we failed in our former assumption. We found 75 litres, but should have had 83 litres. So we missed the correct answer by 8 litres. This means that we must add some bags. Helped by manipulatives, students can see that if they add one bag, they must remove a carton in order to keep the total number of containers equal to 30.



We then face the following question: How many cartons must we replace by bags to obtain a total of 83 litres? In adding a bag and removing a carton, we gain 1 litre, but we still require 8 more litres. So we have to add 8 bags and remove 8 cartons from the tables. The students will discover that the solution is 23 bags and 7 cartons.

Changing some data in the simplified problem gives students practice in using the new arithmetic manipulative-based model.

2.3 An Arithmetic Problem-Solving Model

Until now we have used two problem-solving models to solve the simplified problem, namely the trial and error method (where the selected numbers are chosen in a non-systematic way) and an arithmetically organized manipulative-based method.

After using manipulatives and formulating other (similar) problems, the students can be introduced to an arithmetic-abstract problem-solving model, that is, a problem-solving model based only on arithmetic concepts without using manipulatives. In order to motivate this we need to confront students with a problem whose solution, using the current method, is not easy. We can change some data in our grocer's problem in such a way that the quantities are so large that it is tedious to solve the problem using manipulatives. Thus, consider the next problem which is, in fact, our target-problem (see the beginning of section 2):

> *"Jane Kimble, a grocer, checked her supply of milk and counted 80 containers of milk. Some were 2L cartons and others were 3L bags. Altogether there was a total of 220L of milk. How many of the containers were 3L bags?"*

Following the same thought process, the students should be encouraged to face the problem by manipulating ideas instead of manipulating concrete objects.

Suppose that the number of milk bags is the same as the number of milk cartons. We then have 40 of each. Calculate the number of litres. In the cartons, we have 40x2 = 80 litres; in the bags we have 40x3 = 120 litres. We then find that we have 80+120 = 200 litres of milk, but the problem requires a total of 220 litres. We are 20 litres short of the total. So our assumption that we have the same number of bags and cartons is wrong. We can then conclude that we need more bags than cartons. If we add one milk bag, we have to remove one milk carton (in order to keep the number of cartons and bags equal to 80), and we gain one litre of milk. But we need 20 litres, so we have to remove 20 cartons and replace them with 20 bags. So it remains, 40-20 = 20 cartons and 40+20 = 60 bags of milk.

This method of problem solving needs to be internalized by students. The teacher can then propose other problems changing some of the data in the last problem[3]. The teacher and/or students can also propose other problems of the form $x+y = a$; $bx+cy = d$. They can also discuss and construct problems of the type $x \pm y = a$; $bx \pm cy = d$.

The students will realize that the new problem-solving model has the advantage of allowing them to approach a relatively wide family of problems in a direct way; no trial and

[3] *An interesting problem is the following:*

> *"Jane Kimble checks her milk inventory. There are 80 containers of milk: some are 2L cartons and the others are 3L bags. There is a total of 195L of milk. What, then, must be the number of 3L bags?*

Let's suppose, as in the previous problem, that we have 40 milk bags and 40 milk cartons. We find that we have 200 litres instead of 195 litres. Thus, now we have 5 litres more, so we have to replace bags by cartons.

error is required, nor do they require manipulatives.

We have achieved a partial sequence of problem-solving models. We can note, at this point, that one of the biggest differences between the trial and error model and the previous arithmetic-abstract problem-solving model is that, in the latter, we use *hypothetical reasoning* by supposing that the number of bags equals the number of cartons. We then made calculations that allowed us to generate new data that we corrected later. In the trial and error method we simply repeat the same procedure with different quantities until we obtain the correct answer; hypothetical reasoning does not play any role.

Why is hypothetical reasoning important here? Because hypothetical reasoning is the logical base of algebraic thinking in solving word problems, as we will see below. Thus, the abstract arithmetical problem-solving model can help students to construct knowledge that will be useful to them when they learn algebra, and is a vital step in the construction of advanced problem-solving models.

In 2.4 below, we discuss an algebraic problem-solving model that uses only one unknown in solving the target-problem, which is, in fact a *natural generalization* of the arithmetic-abstract problem-solving model.

2.4 An Algebraic Problem-Solving Model

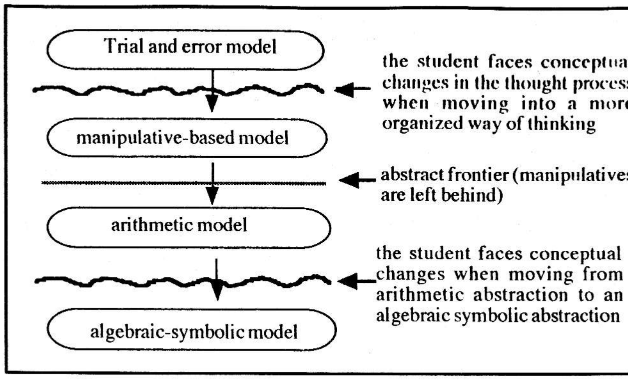
Instead of beginning with a numerical solution (dividing the total number of containers between the cartons and bags, and then deducing *the number* we have to add to the first and remove from the second) we can *suppose* (hypothetical reasoning) that we know this number already. Let x be this number. The *exact* number of cartons is not 40 (half of 80) but 40-x. In the same way, the *exact* number of bags is not 40, but 40+x. Given that a carton holds 2L and a bag holds 3L, we can then express the total number of litres as 2(40-x) + 3(40+x), where this quantity must equal 220 litres. We get the equation:

$2(40-x) + 3(40-x) = 220$

This is one type of equation usually taught in Grade 9.

This algebraic problem-solving model needs to be internalized by students. In order to achieve this, the teacher can propose other similar problems to be solved by the new model, as it was done before in the previous problem-solving model.

The following diagram illustrates the path that we followed in our example. One moves from the trial and error thinking to symbolic thinking. Each level develops problem-solving models that are used in the next level, but in a more complex way.



3. Concluding Remarks for Teaching

As we said before, the main idea of our approach consists in trying to construct a hierarchical progressive sequence of problem-solving models,

$M_1 \rightarrow M_2 \rightarrow \ldots \rightarrow M_n$

that start from a simple model M_1 and that arrives at a more complex model M_n. The choice of the models that precede the final model M_n depends on the M_n model itself and on the knowledge of the students. In our case, we aimed at a model capable of solving the grocer's problem (see beginning of section 2) using the algebraic knowledge presented in the Transition Years. Therefore, we should use algebra with only one unknown. In order to accomplish this, we used three pedagogical principles:

Principle 1: *to simplify the problem.*

Our first principle consists in simplifying the target problem. The simplified problem is then solved through an accessible problem-solving model which will allow, in the next step, to construct a new, more complex model.

Principle 2: *to convince the students.*

Our second principle is that of convincing the students of the necessity to go on to a more complex model of problem-solving. This can be done by presenting a few problems that are difficult or not appropriate to solve by using the present model as we did up above.

Principle 3: *to create a hierarchical link between problem-solving models.*

Our third pedagogical principle states that the next model to be constructed in the hierarchical sequence has to be based on the previous model. We can note, at this point, that the grocer's problem in section 2.3 can be solved by other methods (like almost any problem). One does not need to divide the number of containers in half. One can select any value x between 1 (even between 0) and the number of containers, that is, 80. Although other methods could solve the grocer's problem, the method of dividing the number of containers into two that we used is that which naturally leads to the algebraic problem-solving model at which we aimed. That is the reason why the 'dividing into two method' plays such a central role in our didactic sequence.

The previous principles can be applied to many situations. It remains, however, that the success of our approach in a mathematics class will depend upon the teacher's ability to choose an appropriate simplification of the problem and to obtain a suitable hierarchical sequence of linked models. It will be particularly necessary that the use of concrete models based on manipulatives be coherent with the abstract problem-solving models.

4. Some Historical Remarks

We have seen different methods in solving the same problem. The first one was the trial and error method. The third is a method developed by ancient Babylonian mathematicians, probably during the first Babylonian Dynasty (c. 1900 B.C.). It is called the *false position* method (in fact, in solving the problem, we start with a false solution). The last method is related to the historical emergence of algebra. We can find it in one of the most important books of Greek mathematics, the Diophantus' *Arithmetika* (c. 250). (See Radford, 1992). A detailed account of the historical links between the false position method and the algebraic one presented here can be found in Radford (1993).

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