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The Epistemological Foundations of the Theory of Objectification

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Abstract: Epistemology deals with the question of how things are known. It is, therefore, not surprising that, implicitly or explicitly, epistemology has been a prolegomenon to educational theories. Yet, the vast array of rationalist, empiricist, and pragmatic epistemologies remain *a*historical. The ahistoricism of these epistemologies is at odds with the main tenets of contemporary sociocultural approaches, particularly those approaches that argue for the cultural situatedness and historical nature of knowledge and knowing. The purpose of this article is to offer a discussion of some elements of an epistemological nature that underpin a cultural-historical theory of teaching and learning—the theory of objectification (Radford, 2008, 2013, 2014a). These epistemological elements (inspired by Hegel's philosophy and some dialectic materialist thinkers) play a crucial role in the theory to conceptualize knowing and learning against the background of cultural-historical modes of cognition and forms of knowability.

1. Introduction

The Oxford English Dictionary (http://www.oed.com/) defines epistemology as "the theory of knowledge and understanding, esp. with regard to its methods, validity, and scope." Epistemology tries indeed to answer the question of *how* things are known. It is clear, therefore, that educational theories cannot go far without resorting to epistemology. In other words, epistemology appears as a prolegomenon to any educational theory.

It is not surprising that epistemology has always been of interest to mathematics educators (see, e.g., Artigue, 1990; 1995; D'Amore, 2004; Glaeser, 1981; Sfard, 1995). Nor is it surprising that the foundational theories in mathematics education—constructivism, and the theory of didactic situations, for instance—resorted to epistemology. Constructivism (Cobb, 1995; Cobb, Yackel, & Wood, 1992) resorted to Piaget's genetic epistemology in the adapted Kantian version offered by von Glasersfeld (1995) and its idea of *viable knowledge*. The theory of didactic situations (Brousseau, 1997) resorted to Piaget's genetic epistemology and also to Bachelard's (1986) epistemology and its idea of *epistemological obstacles*.

Since the question of *how* things are known can be answered in different ways, it is not unexpected that epistemology comes in a variety of kinds: rationalist epistemology, empiricist epistemology, pragmatic epistemology, etc. Yet, as Wartofksy points out, "Historicaly (*sic*), epistemology has been ahistorical" (Wartofsky, 1987, p. 357). What makes epistemology ahistorical is not an inadvertent inattention to history. Rather, it is a shared common essentialism. Wartofsky continues:

> epistemologists have sought to fix the universal and necessary conditions of any knowledge whatever, or to establish the essential nature of the human mind. Thus, whether empiricist or rationalist, realist or phenomenalist, traditional epistemologies have shared a common essentialism. What made such epistemologies different were alternative accounts of what are the fixed, essential modes of the acquisition of knowledge, or what are the universal and unchanging structures of the human mind. (Wartofsky, 1987, p. 357)

Piaget's genetic epistemology is an interesting case in point. In order to understand how we know, Piaget does indeed resort to history (see, e.g., Piaget & Garcia, 1989). However, the mechanisms of knowledge construction that Piaget identifies in his genetic epistemology (i.e., assimilation, adaptation, equilibration) are *universal*; they do not depend on the geographic or the temporal situation. The mechanisms of knowledge construction are both *a*historical and *a*contextual (Radford, Boero, & Vasco, 2000). As a result, history does not play any epistemological constitutive role (other than as a marker of a naturalist phylogenetic evolution of the species). Hence, if Piaget resorts to history, it is only to refute it. The reasons may be found in a kind of essentialism with which Piaget endows his genetic epistemology. Piaget's essentialism is not of a Platonic nature. It does concern the immutability of the objects of knowledge. The immutability concerns rather the manner in which he interprets human action; that is, as schemas that become organized into fixed logical-mathematical structures. He says, "il n'existe pas de donnée expérimentale qui ne suppose, ne fût-ce que pour sa lecture même, une coordination logico-mathématique (de n'importe quel niveau, fût-ce-sensori-moteur) à laquelle cette donnée est nécessairement relative" ["There is no experimental data that suppose, if only for its reading, a logical-mathematical coordination (of any level, even-sensorimotor) to which this data is necessarily relative"] (Piaget, 1973, p. 18).

However, the ahistoricity of epistemology is at odds with the main tenets of contemporary sociocultural approaches, in particular those approaches that argue for the cultural situatedness and historical nature of knowledge and knowing (D'Ambrosio, 2006; D'Amore, Radford, & Bagni, 2006). The question is: Is there a possibility for a non-essentialist kind of epistemology? Is there a possibility for thinking of an account of the way in which we come to know that will really take into account history and culture as epistemic categories?

The answer is yes, and some efforts have been made in the past. A few decades ago, the science epistemologist Marx Wartofsky, in his article *Epistemology Historicized*, offered some ideas about how a historical epistemology would look. Such an epistemology, he suggested, would start "from the premises that the acquisition of knowledge is a fundamental mode of human action" (Wartofsky, 1987, p. 358). But instead of considering human action in a formal or abstract way, as Piaget did, he suggested to understand it as a form of human practice inseparable from other forms of human practice, "inseparable from the historicity of these other modes, that is, from their historical change and development" (Wartofsky, 1987, p. 358). Such an epistemology should be based on the idea that

> the appropriate domain for the study of human cognitive practice is not the abstract and relatively featureless domain of the 'human mind', whether tabula rasa, or packed full of innate ideas or faculties; but rather the concrete, many-

> featured and historical domain of human practices - social, technological, artistic, scientific. (Wartofsky, 1987, p. 358)

Wartofsky (1987) also called attention to the epistemic role of semiotics and artifacts (material objects, symbols, representations) and strongly claimed that since artifacts and symbols have a history, so does cognition: "modes of cognitive practice, perception, thought, ways of seeing and ways of knowing, also have a history" (p. 358). As a result, "modes of cognition change historically in relation to changes in modes of social practice, and in particular, in relation to historical changes in modes of representational practice" (Wartofsky, 1987, p. 358).

In the case of mathematics education—the research domain in which I would like to place this discussion—a historical epistemology should be concerned with the elucidation of the nature of objects of knowledge as cultural-historical entities, particularly with their nature and knowability. Such an epistemology should show how the knowability of mathematical objects is cast within definite evolving historical modes of cognition.

The purpose of this article is to offer a discussion of some elements of an epistemological nature that underpin the theory of objectification (Radford, 2008, 2013, 2014a) and to which the theory has recourse in order to conceptualize teaching and learning, and knowledge and knowing. The kind of historical epistemology to which the theory of objectification resorts draws on Hegel's work and the dialectic materialist school of thought as developed by Marx (1998) as well as some dialectician philosophers and psychologists after him, such as Evald Ilyenkov (1977), Theodor Adorno (1973, 2008), L. S. Vygotsky (1987), and A. N. Leont'ev (1978). The historical epistemology leads to envision knowing and learning against the background of historical modes of cognition and forms of knowability.

Given the legendary contempt that Hegel showed for mathematics (see, e.g., Hegel, 1977, 2009), my enterprise, to say the least, is daunting. Hegel (as well as other famous philosophers, like Heidegger (1977) and Husserl (1970)) was indeed critical of the mathematics of his time. Hegel sensed in a very clear way that mathematics was turning into a technical discipline, which, through its universalist claims and aspirations, sacrifices meaning in the interest of calculations. Hegel, I would say, would have been rather sympathetic to something like a "poetic mathematics," an expressive adventure mediated by an expressive language where subject and object co-inhabit together. In the mathematics of Hegel's time, however, the language of mathematics was already a language without subject. In the mathematics of Hegel's time, the individual had already evaporated from the mathematical discourse. As the German philosopher Theodor Adorno puts it, "The subject is spent and impoverished in its categorial performance; to be able to define and articulate what it confronts . . . the subject must dilute itself to the point of mere universality" (Adorno, 2008, p. 139).

Yet, I will draw on Hegel's dialectics to talk about mathematical objects, knowledge, and knowing. I do think that despite Hegel's well-known idealism and anti-mathematical stance, he provides elements with which to understand knowledge in general and mathematical knowledge in particular.

I shall start by addressing the questions of the nature of mathematical objects and how we think about these objects. It is already a Hegelian insight that thinking and its objects cannot be dealt with separately. To think, indeed, is to think about something. Thinking and this something that is the object of thinking are intertwined and indissoluble.

2. Mathematical objects

There are several widespread approaches to mathematical objects. In this section, I will mention three of them. The first one consists of conceiving of mathematical objects as produced by the mind. This is the approach articulated by Descartes, Leibniz, and other rationalists. In his *New Essays Concerning Human Understanding*, Leibniz says:

> all arithmetic and all geometry are innate, and are in us virtually, so that we can find them there if we consider attentively and set in order what we already have in the mind, without making use of any truth learned through experience or through the tradition of another, as Plato has shown in a dialogue in which he introduces Socrates leading a child to abstract truths by questions alone without giving him any information. We can then make for ourselves these sciences [i.e., arithmetic and geometry] in our study, and even with closed eyes, without learning through sight or even through touch the truths which we need; although it is true that we would not consider the ideas in question if we had never seen or touched anything. (Leibniz, 1949, p. 78)

The mind, therefore, has only to search inside itself to tidy up and order out what is already there to find mathematical objects and what can be said about them.

There is a second approach—chronologically older than the previous one—that goes back to Plato. Plato thought of mathematical objects as *forms*: unchanging entities that populated an ideal world of perfect and intelligible atemporal essences (see, e.g., Caveing, 1996). How do we come to know these unchanging forms? There are two widespread answers.

The first answer comes from Plato. In his *Phaedrus* dialogue, Plato explained the knowability of the objects of knowledge in terms of recollection. Our soul was assumed to have been in touch with the realm of forms, the realm of Truth, when the soul "disregarded the things we now call real and lifted up its head to what is truly real instead" (Plato, 2012, p. 235; 249c). Unfortunately, during our birth in the world, we forgot about Truth and forms. The process of recollection is well illustrated in another dialogue, *Meno*, the one Leibniz was referring to in the previous citation, where a slave is presented as going into a process of reminiscence: he is recollecting knowledge about geometric figures that he already had in a past life.

The second answer is the modern answer: the unchanging forms are *discovered*. In a recent article Côté (2013, p. 375) summarizes the point in the following terms: "the full version of mathematical Platonism means that mathematicians do not invent theorems, but discover them."

Platonism remains very popular among mathematicians, at least if we are to believe Bernays (1935) and more recently Giusti (2000). However, today Platonism does not seems to be articulated in terms of recollection; rather, it seems to be articulated in terms of *discovery*. I know of no mathematician or mathematics educator resorting to Plato's reminiscence theory to explain learning. The theory of reminiscence seems to have fallen out of favour. Yet, we cannot say that Plato has not influenced us. For one thing, Piaget drew on Kant, who, in turn, drew on Plato. Kant drew in particular on Plato's ontology and referred to the forms as *noumena*. In Kant's account (see Kant, 2003), they are prior to, and independent of, human activity. They are *somewhere* already. They exist. To convey that idea of the independence of these objects from human activity, Kant referred to them as *things-in-themselves*.

Let me turn now to the third account of objects of knowledge—the constructivist account. Although Piaget (1924) drew on Kant, he was quick to remove Kant's aprioristic stance: in Piaget, objects of knowledge are rather the product of the individual's constructions. Mathematics education has largely adopted this sense in order to talk about knowledge.

I would like to highlight that the fundamental metaphor behind the idea of objects of knowledge as something that *you make* or something that *you construct* is that objects of knowledge are somehow similar to the concrete objects of the world. You construct, build, or assemble objects of knowledge, as you construct, build, or assemble chairs. This idea of knowledge as construction is relatively recent. It emerged slowly in the course of the 16th and 17th centuries when manufacturing and the commercial production of things became the main form of human production in Europe. Hanna Arendt summarizes this conception of knowledge as follows: "I 'know' a thing whenever I understand how it has come into being" (Arendt, 1958, p. 585). It is within the general 16th and 17th centuries' outlook of a manufactured world that knowledge is first conceived of as a form of manufacture as well. When Kant writes at the end of the 18th century his famous *Critique of Pure Reason*, he is articulating and expressing, at the theoretical level, the new cultural view of knowledge—the view of the modern period in Western development. In the 20th century, and with Piaget (1970) and von Glasersfeld (1995) in particular, the individualist dimension of the modern view on knowledge was pushed to its last consequences: You and only you construct your own knowledge. For, in this view, knowledge is not something that someone else can construct and pass on to you. Doing, knowing, and learning are conflated. What you *know* is exactly what you have *learned* and *done* by yourself (Radford, 2014b).

As many scholars have pointed out, such a view of knowledge is problematic on several counts. For instance, it leaves little room to account for the important role of others and material culture in the way we come to know, leading to a simplified view of cognition, interaction, intersubjectivity, and the ethical dimension. It removes the crucial role of social institutions and the values and tensions they convey, and it de-historicizes knowledge (see, e.g., Campbell, 2002; Lerman, 1996; Otte, 1998; Roth, 2011; Valero, 2004; Zevenbergen, 1996).

In the following section I explore the idea of objects of knowledge from a neo-Hegelian perspective.

3. Knowledge from a neo-Hegelian perspective

In this section I outline the Hegelian-Marxist dialectical materialist conception of knowledge that is at the heart of the theory of objectification (Radford, 2008). Knowledge, in this theory, is not something that individuals possess, acquire, or construct. The conception of knowledge is rather based on a distinction between two related although different ontological categories: *potentiality* and *actuality*. The potentiality/actuality distinction goes back to Aristotle who used the words *dunamis* and *energia*. Potentiality (*dunamis* in Greek) designates the source of motion, something that is entangled in the material world. Potentiality is synonymous with "capacity" or "ability" or "power." Living things and artifacts—musical instruments, for example—have potentiality; that is, a definite capacity *for doing something*. Aristotle contrasted this potentiality to *actuality*, which is "being-at-work"—something in motion occurring in front of us.

Objects of knowledge (mathematical and other) belong precisely to the category of potentiality, and as such, are *abstract* or *general*; that is, they are conceived of as "undeveloped, lacking in connections with other things, poor in content, formal" (Blunden, 2009, p. 44).

Objects of knowledge are hence not psychological or mental entities. They are *pure possibility*—a "complete totality of possible interpretations—those already known, and those yet to be invented" (Ilyenkov, 2012, p. 150). They are possibility grounding interpretations and actions; for example, possibility of making calculations, or thinking and classifying spatial forms in certain "geometric" manners, or possibilities of imagining new ways of doing things, etc.

Objects of knowledge as possibility are not something eternal and independent of all human experience (like Kant's idea of things-in-themselves or as Plato's concept of forms). Objects of knowledge are social-historical-cultural entities. In fact, they result from, and are produced through, social labour. In more precise terms, objects of knowledge are an evolving culturally and historically codified *synthesis* of doing, thinking, and relating to others and the world.

Let me give you an example. It comes not from humans but from chimpanzees. As we know, some groups of chimpanzees crack nuts. Primatologists have shown the complexity underneath the actions that leads to cracking a nut: the chimp has to make several choices. First, the chimp has to choose the nut; second, the chimp has to choose the first stone where the nut will sit (the anvil stone); third, the chimp has to choose the hammer stone, then choose the precise pressure with which to apply the hammer stone to the nut so that the nut is neither crushed nor left unopened. Figure 1 shows Yo, a member of a chimp community in the southeastern corner of the Republic of Guinea.

Figure 1. Yo cracking a nut while two young chimps watch her attentively (from Matsuzawa, Biro, Humle, Inoue-Nakamura, Tonooka, & Yamakoshi. 2001, p. 570).

Yo and other chimps' sequence of actions for cracking a nut becomes a historically codified *synthesis*, resulting in an object of knowledge—knowing how to crack nuts. We can qualify this object of knowledge as *kinesthetic* in that it involves bodily actions without language and signs. In general, in the case of human objects of knowledge, in addition to artifacts, the actions codified in a cultural synthesis include language and other semiotic systems (diagrams, for instance), providing the resulting object of knowledge with a complexity that may surpass the one found in chimps and other species. The synthesis is often *expressed* in a semiotic system, providing the object of knowledge with a description or definition (a circle *is* …), although the explicit expression of the synthesis is not a condition for the existence of the object of knowledge. How much explicit expression is required will depend on the cultural form of mathematical reasoning that embeds the objects of knowledge. To give a short example, Babylonian scribes were not inclined to provide specific definitions of the objects they dealt with (e.g., circles, squares, rectangles, etc.). They talked about objects without defining them. We find the exact opposite in Greek mathematics. Thus, in Euclid's *Elements*, there is an obsessive need to define the mathematical objects from the outset.

I will come back to this point later. For the time being, let me say something about the idea of *synthesis* that is central to the definition of objects of knowledge that I have just suggested. The synthesis to which I am referring is not the kind of synthesis Kant (2003) put forward in the *Critique of pure reason*; that is, the synthesis of a legislative reason that merges and subsumes an array of cases into a (pre-) given concept. The materialist concept of synthesis that I am suggesting here conceives of synthesis as *codified labour*. More precisely, it refers to a nascent relationship of various and different actions out of which these actions become recognized as *different* and the *same*. In the chimps' example, a synthesis of actions that have been carried out by *different* chimps with *different* stones and *different* nuts in *different* moments become recognized as *same-yet-different*. The synthesis that leads to the object of knowledge makes this object general in the sense that it does not relate to this or that situation (these stones and nuts, Yo, or another particular chimp): It is a *synthesis* of different singularities and as such it is *not* an abstraction, but a synthesis that contains the divergence and contradictions of the singularities that it attempts to hold together. It is a *synthesis of non-identity*, which confers the object of knowledge with its *internal contradictions*. Instead of being a flaw or imperfection, the non-identitarian synthesis confers the object of knowledge with an irreconcilable nature vis-à-vis the synthesized items. The resulting and unavoidably internal contradictions are precisely what afford the further development of the object of knowledge. As bearer of contradictions, the object of knowledge indeed opens up room for further actions and new interpretations and creations. As such, the object of knowledge always points to what itself is not.

To come back to the chimpanzee example, some chimps crack nuts of certain kinds, but not others. The nut-cracking object of knowledge may eventually evolve if the chimps start including other nuts or hitting nuts with objects other than stones (which has happened in some chimp cultures where tree branches started being considered). Initially, Yo's community cracked coula nuts, but not panda nuts. After some time, they started cracking panda nuts as well. A new synthesis occurred. The new synthesis *sublated* the previous one, giving rise to a development of the object of knowledge.

The same may be said of mathematical objects of knowledge. They are cultural and historical synthesis, of dealing with, for instance, certain kind of situations or problems that mathematicians solve through a sequence of well identified steps. The situations become expressed as, for example, "linear equations." Figure 2 shows a linear equation and a sequence of well identified steps to solve it by a group of 11–12-year-old Grade 6 students.

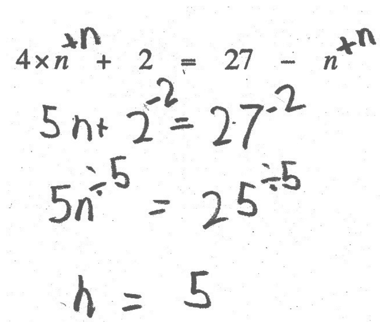

Figure 2. A group of Grade 6 students' sequence of steps to solve a linear equation.

Let me note, however, that linear equation knowledge is not the sequence of signs we see on the paper. Linear equation knowledge is a synthesis, a codified way of dealing with problems or situations like the one shown in Figure 2. Linear equation knowledge is pure cultural possibility—possibility of thinking about indeterminate and known numbers in certain historically constituted analytical ways. In the example referred to in Figure 2, linear equation knowledge has been *realized* or *actualized* in a singular instance, the solving of the equation $4xn + 2 = 27 - n$. This actualization of the linear equation knowledge is what, following Hegel's dialectics, may be termed a *singular*. So, from potentiality, linear equation knowledge has been put in motion and passed from something general (something undeveloped and poor in content) to something concrete and actual; something noticeable and tangible, in short, a singular. This singular *is not* the symbols themselves shown in Figure 2, but the embodied, symbolic, and discursive actions and thoughts required in solving the specific equation $4xn + 2 = 27 - n$. In the singular, mathematical knowledge *appears* as both concrete *and* abstract. It cannot be concrete only; nor can it be abstract only. It is both *simultaneously*.

But what is it that makes possible the movement from potentiality to actuality, or from the general to the singular? The answer is *activity* or *labour* (Radford, 2014a). Indeed, knowledge as general, as synthesis, is not an object of thought and interpretation. It lacks *determinations*. The lack of *determinations* renders knowledge impossible to be sensed, perceived, and reflected upon. It can only become an object of thought and interpretation *through specific problem-posing and problem-solving activities*.

What this means is that objects of knowledge cannot be accessed directly. They are not immediate objects. They are *mediated*. They are *mediated by activity*[1]. So, when we talk about knowledge and its objects, we have three elements: the objects of knowledge themselves, their actualization in the concrete world (singulars), and the activity that mediates them (see Figure 3).

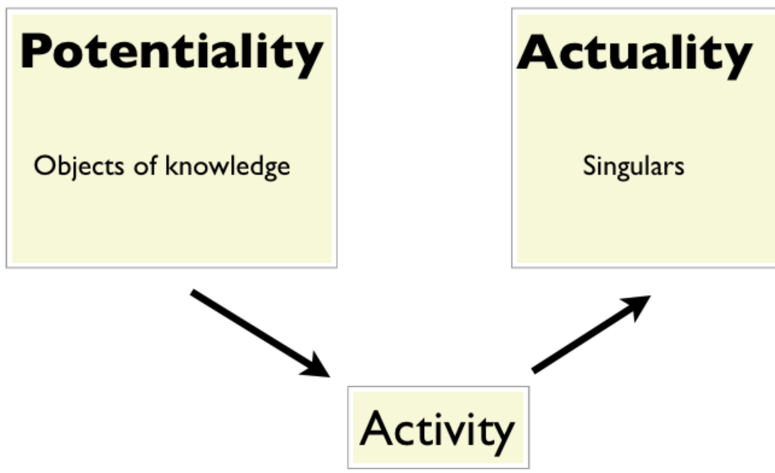

Figure 3. Singulars as the actualization of objects of knowledge through a mediating activity.

By moving from potentiality into the actual realm of the sensuous, the sensible, and the perceptible (something called in dialectical materialism the *ascent from the abstract to the concrete*), objects of knowledge appear instantiated in singulars.

This movement or ascent from the abstract to the concrete should not be interpreted as a repetition in the mechanical sense of technology. To do so would amount to a complete misunderstanding of the dialectical materialist view of objects of knowledge as *empty forms of difference*, or, as Deleuze (1968, p. 8) says, as "invariable forms of variation." It would also amount to conflating two different layers of ontology: potentiality and actuality. This is what Figure 3 tries to clarify.

4. Concepts

[1] In opposition to other approaches, in the dialectic materialist epistemological approach I am outlining here, the access to the objects of knowledge is not ensured by signs, but by the individuals' activity. The activity does involve signs, but it is not the signs that reveal the object of knowledge; it is rather the activity.

From the individual's viewpoint, what emerges from the actualization or concretion of objects of knowledge through their singulars is the appearance of the object of knowledge in the individual's consciousness. In Hegel's terms, it is a *concept*. In this sense, the concept is a *production*, which means, etymologically speaking, the "bringing forth" or "coming into being" of something. It is in this sense that we refer to classroom mathematical knowledge here: something—an object of knowledge—that comes into being in a singular through classroom activity. The concept is the appearance of the object of knowledge in the student's consciousness through the singular, as afforded by the mediating activity.

Given the crucial mediating nature of the activity in concept formation, we should emphasize here the importance of the classroom activity. If the classroom activity is not socially and mathematically interesting, the ensuing concept and conceptualization will not be very strong. There is hence a pedagogical need to offer activities to the students that involve both the possibility of strong interactional participation and deep mathematical reflections (Radford, 2014b).

Naturally, the historically codified way of cracking nuts or dealing with linear equations is not something that chimps in the first case, and students in the second case, grasp directly. It is at this point that we need to consider the concept of *learning*.

In the following section I describe this concept from the viewpoint of the dialectic materialist theory of objectification.

5. Learning

For the young chimpanzees that happen to live in a nut-cracking chimpanzee culture, like Yo's culture, the nut-cracking object of knowledge is pure possibility. The young chimps have to *learn* how to do it. As summarized previously (Radford, 2013), studies in the wild suggest that it takes from 3 to 7 years for the infant chimp to learn the nut-cracking process. Infants do not necessarily start by using a hammer stone and the anvil. The proper attention to the objects, their choice (size, hardness, etc.), and subsequently the spatial and temporal coordination of the three of them (nuts, anvil, and hammer), is a long process. Often, young chimps of about 0.5 years manipulate only one object (either a nut or a stone). They may choose a nut and step on it. As chimps grow older, they may resort to the three objects, but not in the correct sequence of nut-cracking behaviour, resulting in failed attempts. A key aspect of the process is the appearance of suitable cracking skills—for example, "the action of hitting as a means to apply sufficient pressure to a nut shell to break it" (Hirata, Morimura, & Houki, 2009, p. 98).

How do chimps learn? They learn by participating in the activity that realizes the object of knowledge. They learn by what Matsuzawa *et al*. (2001) call the "master-apprenticeship" method: They observe (see Figure 1), then they try.

We can formulate the question of learning as follows. As *general*, objects of knowledge (i.e., culturally codified ways of doing and thinking) are not graspable or noticeable. In order for an object of knowledge to become an object of thought and consciousness, it has to be set in motion. It has to acquire cultural determinations; that is, it has to acquire content and connections in a process of contrast with other things, thereby becoming more and more concrete. And the only manner by which concepts can acquire cultural determinations is through specific *activities* (in our previous chimpanzee example, through "master-apprenticeship" activities). Learning emerges from the sensuous and conceptual awareness that results from the realization of the object of knowledge (e.g., cracking nuts, solving linear equations) in its concrete realization or individualization.

Let us notice that we always grasp objects of knowledge through the singular that instantiates it; that is, through its individual realization. This is the paradox of learning: in learning we deal with singularities (we solve specific equations, like $4xn + 2 = 27 - n$ or $3xn + 5 = n + 9$, etc.). Yet, what we are after is none of those or any other specific equation. We are after culturally constituted *ways of doing and thinking* that can only be grasped obliquely, in an indirect manner, through our participation in the activity that makes this way of thinking present in the singular.

I can now formulate the concept of objectification through which we thematize learning. Objectification is this social co-transformative, sensuous sense-making process through which the students gradually become critically acquainted with historically constituted cultural meanings and forms of thinking and action. Those systems of thinking (algebraic thinking or statistical thinking, for instance) are there, as potentiality for the novice students. When the students cross the threshold of the school for the first time, the objects of knowledge are pure open potentiality. It is through processes of objectification embedded in the activity that mediates the potential and the actual, and through their realization in concepts, that these systems of thinking will become objects of consciousness and thought.

To simplify our terminology, let us refer to the knowledge of a culture as the ensemble of cultural ways of doing and thinking. Because of the continuous transformation of these ways of doing and thinking, which "change historically in relation to changes in modes of social practice" (Wartofsky, 1987, p. 358) and the social production of the individual's existence, the knowledge of a culture is a flexible and dynamic *system*. This system, along with the ensuing ensemble of realizable mediating activities through which knowledge can be actualized, set the parameters of historical modes of cognition and forms of knowability. These modes of cognition and forms of knowability frame, in turn, the scope of concepts that can be produced at a specific time in a specific culture. These concepts also constitute a dynamic system, which, to distinguish from Knowledge, we term Knowing. We then get the diagram shown in Figure 4.

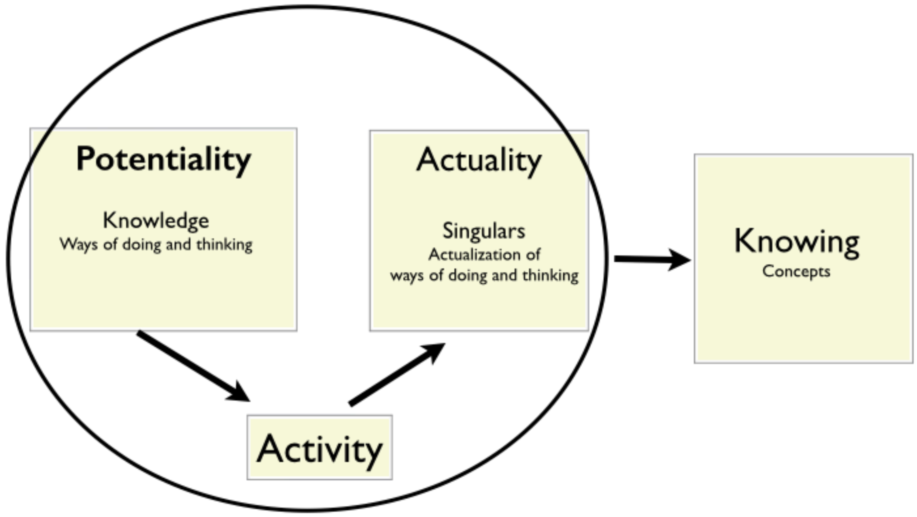

Figure 4. Knowing as what is grasped by the individuals in the realization or actualization of knowledge through activity.

Let me come back once more to the algebra example. Algebra includes several *themes* (generalization, linear equations, polynomial equations, abstract algebra, etc.). A specific culturally codified way of thinking and doing is associated with each one of these *themes*. For example, a linear equation way of thinking and doing comprises ways of posing, reasoning, solving, and dealing with situations susceptible to being expressed in what is meant by linear equations in a certain period and in a certain culture (linear equations may refer to equations involving specific semiotic systems, some type of coefficients only, e.g., positive integers, etc.). Through their cultural codified synthesis, these ways of thinking and doing, which have emerged out of practical cultural activities, may be different from one culture to another (we can think, for instance, of the Ancient Chinese linear equations or those that we find in the Ancient Greek tradition). These ways of thinking and doing (Knowledge in our terminology) have been refined in the course of long cultural processes, which often involve societal contradictions (as it will become clearer in the next section). They become potentiality. Figure 4 suggests that the activity provides them with particular determinations, depending on the nature of the activity. Thus, the knowledge that is actualized in solving the equation shown in Figure 2 involves forms of thinking that include positive and negative coefficients, but not fractional or irrational coefficients. What is therefore actualized is something specific. This is the singular: the actual appearance of the general. Now, the concept is what the students grasp of this singular. In other words, the concept is constituted from what has actually become the object of consciousness for the students in the course of their joint labour with the teachers—the sensuous and actual way of thinking and doing *as encountered* and *cognized* by the students. Concepts, of course, are not isolated entities; they also constitute systems—which we have termed Knowing.

6. The politics of knowledge

The idea of knowledge as an ensemble of ways of thinking and doing as developed in the previous section—that is, as culturally codified syntheses of people's actions— allows us to grasp their cultural and historical nature. We have already contrasted the Euclidean insistence in defining things with the Babylonian mathematical thinking where definitions are not required. Euclid worked within the aristocratic Athenian tradition that confers to language an epistemic value that we do not find in the practical approach of the bureaucratic structure of Mesopotamian cities. From the dialectic viewpoint outlined here, ways of thinking and doing are evolving entities rooted in, and informing, practical activity. They arise as the synthesizing effect of activity and, in turn, affect activity (and the individuals who participate in the activity). They are cause and effect, although not in a causal manner. Rather, they are simultaneously cause and effect in a dialectical sense.

To understand a specific way of thinking and doing, we have to turn to the culture in which they operate and to ask about the activities in which people engage. But a real understanding of a specific way of thinking and doing requires, because of the developmental nature of it, looking at it *historically*. As Ilyenkov (1982, p. 212) put it, “A concrete understanding of reality cannot be attained without a historical approach to it.” The developmental nature of knowledge is based on the internal contradictions they bear within themselves—contradictions that appear and reappear in the confrontation of objects of knowledge in concrete activities, where they give rise to concepts. These internal contradictions are not logical flaws, nor are they merely epistemological contradictions between opposing or competing purified entities. They are replete of social and societal contradictions. In fact, the internal contradictions of objects of knowledge reflect the societal contradictions from where they emerge. There are questions of social, cultural, and political legitimacy that are brought to the fore that favour some ways of doing to the detriment of others (see, e.g., Shapin, 1995).

Let me finish with an example. The Italian mathematician Rafael Bombelli wrote a famous treatise in the 16th century—*L’algebra*. Commenting on Bombelli’s goal, Jayawardene notes:

> Whereas the works of his [Bombelli] predecessors contained many problems of applied arithmetic (some of them solved by means of the methods of algebra), Bombelli's Algebra contained none. His were all abstract problems. In fact, in the introduction to Book III he said that he had deviated from the practice of the majority of contemporary authors of arithmetics who stated their problems in the "guise of human actions": "sotto velame di attioni, e negotij humani ... (come di vendite, compere, restitutioni per- mute; cambij; interessi; deffalcationi, leghe di monete, di metalli; pesi; compagnie, e con perdita, e guadagno, giochi, e simili altre infinite attioni, e operationi humane)." He said that these men wrote with a different purpose-they were practical rather than scientific-and that he, on the other hand, had the intention of teaching the higher arithmetic (or algebra) in the manner of the ancients. (Jayawardene, 1973, p. 511)

Bombelli’s decision has to do with the social competition among cities for work and prestige in 16th century Italy. He has recourse to the Renaissance praised value on the ancient Greeks. And after having included practical problems in his manuscript “in the guise of human actions,” he drew on Diophantus’s work and removed the practical problems. *L’algebra* was made more “scientific” and was directed not the merchants or the abacists but to the aristocratic audience of scientific thinkers of his time (for the social context, see, e.g., Bernardino, 1999; Biagioli, 1993; Hadden, 1994).

In Bombelli's *L'algebra* we find a neat synthesis of contrasting and contradictory views that are synthesized along the lines of the societal conflicts that ended up in the creation of one of the most elaborate symbolic systems to deal with Renaissance algebra, although abandoned later for other symbolic systems. Bombelli's work shows the sublation of commercial algebra and its development into a scientific version, while showing at the same time that development of objects of knowledge are, as cultural synthesis, cultural and political.

What are the implications of the outlined cultural-historical, dialectic materialist approach to knowledge and knowing? I would like to mention two. First, by considering objects of knowledge as syntheses of people's labour—syntheses that present themselves as potential sources of new interpretations and actions—we move away from situationist, distributionist, individualist, and interactionist accounts of knowledge formation that are at odds to account for the historicity of knowledge and their cultural nature. Second, the dialectic materialist approach emphasizes *the role of activity* in producing knowledge. In doing so, we can reconceptualize knowledge not as the subjective deeds of individuals, but as something that emerges from, and attempts to respond to, problems of a societal nature. Objects of knowledge are bearers of contradictions. These contradictions are not the result of imperfections. They reflect the variety of the individuals' perspectives, interests, and needs (practical, but also aesthetic, ethical, and others) that we find in a culture. They reflect also the manner in which power is distributed in a culture. As a result of these contradictions, they remain open to be expanded, transformed, or refuted in practice.

My account of knowledge and knowing suffers, though, from a lack of attention to the individuals who are in the process of knowing. In fact, this is the inadequacy of the formulation of the problem of *subject* and *object* in traditional epistemology. The subject appears as already given, and invariable. The problem is posed as if the knower is already there, fully constituted or constituted through his or her own deeds. What is inadequate in this way of posing the problem of the relationship between subject and object is that it misses the fact that there is a dialectical relationship between knowing and becoming. We are knowing because we are becoming. And we are becoming because we are knowing (Radford, in press). In the same way that cultures offer ways of thinking and doing, they offer ways of becoming. It is my hope that we will soon start exploring this dialectic between knowing and becoming in more systematic ways.

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References:

Adorno, T. W. (1973). *Negative dialectics*. New York: The Seabury Press.

Adorno, T. (2008). *Lectures on negative dialectics*. Cambridge: Polity.

Arendt, H. (1958). The modern concept of history. *The Review of Politics*, *20*(4), 570-590.

Artigue, M. (1990). Épistémologie et didactique [Epistemology and didactics]. *Recherches En Didactique Des Mathématiques*, *10*(2 3), 241-286.

Artigue, M. (1995). The role of epistemology in the analysis of teaching/learning relationships in mathematics education. In Y. M. Pothier (Ed.), *Proceedings of the 1995 annual meeting of the canadian mathematics education study group* (pp. 7-21). University of Western Ontario.

Bachelard, G. (1986). *La formation de l'esprit scientifique [The formation of the scientific mind]*. Paris: Vrin.

Bernardino, B. (1999). *Le vite de' matematici* [Lives of mathematicians]. Milano: Franco Angeli.

Bernays, P. (1935). Sur le platonisme dans les mathématiques [On Platonism in mathematics]. *L'Enseignement Mathématique*, *34*, 52-69.

Biagioli, M. (1993). *Galileo, courtier*. Chicago: Chicago University Press.

Blunden, G. (2009). Foreword. In *Hegel's logic* (pp. 7-101). (W. Wallace, Trans.). Pacifica, CA: MIA. (Original work published 1830)

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.

Campbell, S. (2002). Constructivism and the limits of reason: Revisiting the Kantian problematic. *Studies in Philosophy and Education*, *21*, 421-445.

Caveing, M. (1996). Platon et les mathématiques [Plato and mathematics]. In E. Barbin & M. Caveing (Eds.), *Les philosophies et les mathématiques* (pp. 7-25). Paris: Ellipses.

Cobb, P. (1994). Where is the mind? Constructivist and sociocultural perspectives on mathematical development. *Educational Researcher*, *23*(7), 13-23.

Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view in mathematics education. *Journal for Research in Mathematics Education*, *23*(1), 2-33.

Côté, G. (2013). Mathematical Platonism and the nature of infinity. *Open Journal of Philosophy* , *3*(3), 372-375.

D'Ambrosio, U. (2006). *Ethnomathematics*. Rotterdam: Sense Publishers.

D'Amore, B. (2004). Il ruolo dell'epistemologica nella formazione degli insegnanti di matematica nella scuola secondaria [The role of epistemology in high school teacher education]. *La matematica e la sua didattica*, *4*, 4-30.

D'Amore, B., Radford, L., & Bagni, G. (2006). Ostacoli epistemologici e prospettiva socio-culturale [Epistemological obstacles and the sociocultural perspective]. *L'insegnamento Della Matematica E Delle Scienze Integrate*, *29B*(1), 12-39.

Deleuze, G. (1968). *Différence et répétition [Difference and repetition]*. Paris: Presses Universitaires de France.s

Giusti, E. (2000). *La naissance des objets mathématiques*. Paris: Ellipses.

Glaeser, G. (1981). Épistémologie des nombres relatifs [The epistemology of whole numbres]. *Recherches En Didactique Des Mathématiques*, *2*(3), 303–346.

Glasersfeld von, E. (1995). *Radical constructivism: A way of knowing and learning*. London: The Falmer Press.

Hadden, R. W. (1994). *On the shoulders of merchants*. NY: State University of New York Press.

Hegel, G. W. F. (1977). *Hegel's phenomenology of spirit*. Oxford: Oxford University Press (First edition, 1807).

Hegel, G. (2009). *Hegel's logic*. (W. Wallace, Trans.). Pacifica, CA: MIA. (Original work published 1830)

Heidegger, M. (1977). *The question concerning technology and other essays*. New York: Harper Torchbooks.

Hirata, S., Morimura, N., & Houki, C. (2009). How to crack nuts: Acquisition process in captive chimpanzees (pan troglodytes) observing a model. *Animal Cognition*, *12*, 87-101. Retrieved from Google Scholar.

Husserl, E. (1970). *The crisis of the European science*. Evanston: Northwestern University Press.

Ilyenkov, E. V. (1977). *Dialectical logic*. Moscow: Progress Publishers.

Ilyenkov, E. V. (1982). *The dialectic of the abstract and the concrete in marx's capital*. Moscow: Progress Publishers.

Ilyenkov, E. (2012). Dialectics of the ideal. *Historical Materialism*, *20*(2), 149-193.

Jayawardene, S. (1973). The influence of practical arithmetics on the algebra of Rafael Mombelli. *Isis; An International Review Devoted to the History of Science and Its Cultural Influences*, *64*(4), 510-523.

Kant, I. (2003). *Critique of pure reason*. (N. K. Smith, Trans.) New York: St. Marin's Press. (Original work published 1787)

Leibniz, G. W. (1949). *New essays concerning human understanding*. La Salle, Ill: The open Court. (Original work published 1705)

Leont'ev, A. N. (1978). *Activity, consciousness, and personality*. Englewood Cliffs, NJ: Prentice-Hall.

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm? *Journal for Research in Mathematics Education*, *27*(2), 133-150.

Marx, K. (1998). *The German ideology, including theses on Feuerbach and Introduction to the critique of political economy*. New York: Prometheus Books.

Matsuzawa, T., Biro, D., Humle, T., Inoue-Nakamura, N., Tonooka, R., & Yamakoshi, G. (2001). Emergence of culture in wild chimpanzees: Education by master-apprenticeship. In T. Matsuzawa (Ed.), *Primate origins of human cognition and behavior* (pp. 557-574). Tokyo: Springer.

Otte, M. (1998). Limits of constructivism : Kant, Piaget and Peirce. *Science & Education*, *7*, 425-450.

Piaget, J. (1924). L'expérience humaine et la causalité physique [Human Experience and physical causality]. *Journal de Psychologie Normale et Pathologique*, *21*, 586-607.

Piaget, J. (1970). *Genetic epistemology*. New York: W. W. Norton.

Piaget, J. (1973). *Introduction à l'épistémologie génétique [Introduction to genetic epistemology]*. Paris: Presses Universitaires de France.

Piaget, J., & Garcia, R. (1989). *Psychogenesis and the history of science*. New York: Columbia University Press.

Plato. (2012). *A Plato reader: Eight essential dialogues* (C. Reeve, Ed.). Indianapolis, IN: Hackett Publishing Company.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In L. Radford, G. Schubring, & F. Seeger (Eds.), *Semiotics in mathematics education: Epistemology, history, classroom, and culture* (pp. 215-234). Rotterdam: Sense Publishers.

Radford, L. (2013). Three key concepts of the theory of objectification: Knowledge, knowing, and learning. *Journal of Research in Mathematics Education*, *2*(1), 7-44.

Radford, L. (2014a). De la teoría de la objetivación [On the theory of objectification]. *Revista Latinoamericana De Etnomatemática*, *7*(2), 132-150.

Radford, L. (2014b). On teachers *and* students: An ethical cultural-historical perspective. In P. Liljedahl, C. Nicol, S. Oesterle, & D. Allan (Eds.), *Proceedings of the joint meeting of PME 38 and PME-NA 36* (Vol. 1, pp. 1-20). Vancouver: PME.

Radford, L. (2015). The phenomenological, epistemological, and semiotic components of generalization. *PNA* (in press).

Radford, L., Boero, P., & Vasco, C. (2000). Epistemological assumptions framing interpretations of students understanding of mathematics. In J. Fauvel & J. V. Maanen (Eds.), *History in mathematics education. The ICMI study* (pp. 612-167). Dordrecht Boston London: Kluwer.

Roth, W. M. (2011). *Passibility: At the limits of the constructivist metaphor* (Vol. 3). New York: Springer.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior*, *14*, 15-39.

Shapin, S. (1995). *A social history of truth*. Chicago: University of Chicago Press.

Valero, M. (2004). Postmodernism as an attitude of critique to dominant mathematics education research. In P. Walshaw (Ed.), *Mathematics education within the postmodern* (pp. 35-54). Greenwich, CT: Information Age Publishing.

Vygotsky, L. S. (1987). *Collected works* (vol. 1). R. W. Rieber and A. S. Carton (Eds.). New York: Plenum.

Wartofsky, M. (1987). Epistemology historicized. In A. Shimony & N. Debra (Eds.), *Naturalistic epistemology* (pp. 357-374). Dordrecht: Reidel Publishing Company.

Zevenbergen, R. (1996). Constructivism as a liberal bourgeois discourse. *Educational Studies in Mathematics*, *31*, 95-113.