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THE RHETORIC OF GENERALIZATION
A Cultural, Semiotic Approach to Students' Processes of Symbolizing

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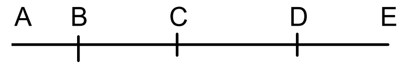
Abstract: *Taking generalization as a cultural semiotic problem, that is, a problem about meaning co-construction occurring in the overlapping territories of writing and speech, this paper attempts to study generalization as a mathematical action unfolding in a classroom discursive Bakhtinian 'text' jointly written by teachers and students in the course of mediated activities. In the case that we shall consider here, what is at stake is the construction and the meaning, in a grade 8 classroom, of a new mathematical object —that of the general term of a sequence or pattern. We shall focus on the problem of how generalization finds expression in processes of sign use (particularly* sign understanding *and* sign production*).*

1. Introduction

This paper is part of an ongoing research program dealing with the students' processes of symbolizing in algebra[1]. By students' processes of symbolizing we mean the different ways in which students come to *understand, use* and *produce* signs. Our work is embedded in a post-Vygotskian semiotic theoretical framework that we elaborated elsewhere (Radford 1998, in print) in which signs are seen as psychological tools, symbolically loaded and intimately linked to the actions that the individuals carry out in their activities. Within this theoretical context, ways of symbolizing (Nemirovsky 1994) are not considered as acultural, pre-given processes. Instead, we consider them as instances of the general modes of signifying resulting of the juncture of sign-mediated activities of the individuals and the *Cultural Semiotic System* (e.g. beliefs, patterns of meaning-making; see Radford 1998) in which activities are subsumed. Mathematical generalizations as well as other mathematical activities are framed by specific and culturally accepted ways of symbolizing. This point can be made clearer if we consider sign use in the historical example of the study of numbers in Antiquity. While mathematicians in the Pythagorean tradition, legitimately used pebbles to investigate some properties of numbers, in Euclid's *Elements* not only the actual pebbles but also any iconic representation of them was completely dismissed and replaced by a referential, non-operational sign-segments/letters language couched in a deductive line of reasoning. The deductive Greek mathematical style was linked, as A. Szabó suggested, to the Eleatan distinction between true knowledge and appearance and the consequent rejection of the sensible world as carrier of knowledge. Moreover, and most important for our discussion, the Eleatan beliefs legitimized new ways of symbolizing which authorized certain rules of sign use and excluded others. How the Euclidean "mathematical generality" could be expressed depended on the Greek conception of the concrete and the abstract, the historical availability of the Phoenician letter-based alphabet adopted by the Greeks (a letter-based written language which was completely different from, for instance, the syllabic Akkadian cuneiform language of the Babylonian scribes) and on the accepted cultural normative dimension in which the use of signs was caught. Let us consider a short example. Proposition 21, Book IX of Euclid's *Elements* reads as follows:

> If as many even numbers as we please be added together, the whole is even.
> For let as many even numbers as we please, AB, BC, CD, DE, be added together; I say that the whole AE is even. For, since each of the numbers AB, BC, CD, DE is even, it has a half part; [VII. Def. 6] so that the whole AE also has a half part.
>
> 
>
>
> But an even number is that which is divisible into two equal parts [id.]; therefore AE is even (Heath 1956, Vol. II, p. 413).

Euclid expresses here "generality" in natural language as a volitive potential action rendered by the comparative formula "as many even numbers as we please". And, within the Euclidean semiotics, the letters allow combinations (in fact assemblages, e.g. AB) which denote segments that stand for non-particular numbers. Interestingly enough, the proof was not recognized (either by Euclid or his later commentators) as lacking generality despite the fact that a drawn segment unavoidably has a particular length as well as the fact that it was actually based on only *four* numbers. As far as we know, the proof was considered completely general by the canons of Greek mathematical thought. As in contemporary classrooms, modes of symbolizing and expressing generality in Antiquity were shaped by the master's and students' beliefs about mathematics, and their mutual understanding and acceptance of legitimizing procedures about mathematical symbolization.

In this paper we shall deal with a problem which arises in the algebraic study of patterns, namely, that of generalization. Ordinarily, in such cases, because of curricular requirements (as is the case in the current Ontario curriculum for Junior High-School), generalization is expressed through the semiotics of the algebraic language. Of course, a great deal of experimental research has shown that the algebraic expression of generalization is very difficult for students who are still acquiring the mastering of the algebraic language (see e.g. Rico *et al*. 1996). In accordance with our theoretical framework (Radford 1998, in print), we will attempt to explore generalization as a semiotic problem, that is, a problem about meaning co-construction by teachers and students in the course of mediated activities. We shall focus on the construction and the meaning, in a grade 8 classroom, of a new mathematical object—that of the general term of a sequence or pattern. We are particularly interested in the problem of how generalization finds expression in processes of sign use. Since the general term cannot be ostensively pointed to as one can point to a door or to a desk, the semiotic construction of such a mathematical object acquires a particular didactic interest.

2. The methodology

In our research program we are accompanying for three years some 120 students and 6 teachers in the teaching and learning of algebra. This task includes the teachers', researcher's and assistants' joint elaboration of general and particular goals, the joint elaboration of teaching and learning settings, the video-taping of the lessons, discussions, and feedback. The teaching settings have been elaborated in such a way that the students (who are presently in Grade 8) work together in small groups; then the teacher conducts a general discussion allowing the students to expose, confront and discuss their different achieved solutions. In general terms, we are interested in investigating the students' processes of symbolizing in specific teaching settings about patterns on the one hand, and equations and inequations on the other. In this paper, however, we shall focus solely on the students' and teacher's co-constructive semiotic expressions of the "general term" of a sequence or pattern. The results that we shall present here come from an interpretative, descriptive protocol analysis (Fairclough 1995, Moerman 1988). Because of the length requirements of the article, we shall limit ourselves to the protocol analysis of one of our student groups. The protocol analysis will attempt to disentangle texture forms underlying the process of sign use (particularly *sign understanding* and *sign production*) in terms of the conveyed meaning and of the classroom use of utterances genres (e.g. reading, confronting, requesting, informing). Our question can then be explicitly formulated in the following terms: how do students' and teachers' voices and writings find their way in the construction of the new object (from the student's perspective) of the general term of a pattern or sequence? How do teacher and students deal with the concrete and the abstract in pattern problems? In its most general terms, and taking the term 'rhetoric' as a mode of discourse or text making, the question is: How does the rhetoric of generalization take place in the classroom?

3. Results

The students were asked to work in groups to solve some problems about patterns. In previous activities they investigated some patterns and had to provide answers to questions like *a* and *b* shown below. Questions *c* and *d* required a new kind of symbolic understanding. For the sake of brevity, we will consider here only some excerpts of the episode concerning one 3-student group discussion of questions *c* and *d*. Let us nevertheless mention that, although questions *a* and *b* led to different understandings of how to investigate patterns, the students did not raise problems concerning issues on generalization. The students kept focused on concrete issues raised by those particular questions. The case for questions *c* and *d* was very different.

Observe the following pattern:

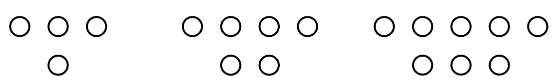


Fig. 1 Fig. 2 Fig. 3

a) How many circles would you have
* in the bottom row of figure number 6?
* in the top row of figure number 6?
* in total in figure number 6?

b) How many circles would you have
* in the bottom row of figure number 11?
* in the top row of figure number 11?
* in total in figure number 11?

c) How many circles would the top row of figure number "n" have?

d) How many circles would figure number "n" have in total? Explain your answer!

Time	*Line*	*dialog / remarks*
1:41	(21)	student 2: *(he writes the answer to the third part of question b while saying)* in total that comes to 24. Wow! This is easy! *(Now he reads question c)* How many circles would the top row of figure number … What? OK. Somebody else!
1:59	(22)	student 1: *(reads the question.)* How many circles… […] What does it mean?
2:07	(26)	student 2: I don't know. *(hitting the sheet with his pencil)*
2:13	(27)	student 1: What's figure n? / *(inaudible)*
2:22	(28)	student 1: (*talking to student 2*) Shut up! I'm going to kill you. … n is what letter in the alphabet?
2:33	(29)	student 2: (*talking to student 1*) Ask the teacher.

In this passage the students are trying to make sense of the expression "figure number n" contained in question *c*. As we noticed elsewhere (Radford 1996), the general term of a geometric or arithmetic pattern cannot be explicitly expressed within the semiotic system (SS) of the objects of the pattern itself. Even to pose the question, it is necessary to go "out" of the first SS (which will include, in the case of elementary school arithmetic, the basic "well-formed" expressions using the ten digits and certain signs like those required for the elementary numerical operations, equality and so on) and to rely on another richer SS (e.g. a meta-language). In the case of our text, we had recourse to the algebraic language to talk about the general term. As the engineering of the problem given to the students suggests, the idea of generalization that we decide to use resides in an experiential dynamics attempting to go beyond the concrete terms of arithmetic. As expected, the transcript indicates, however, that the students' understanding of the question remained circumscribed to their arithmetical experience. We reach here a nodal point in the development of the classroom activity whose unraveling will require the elaboration of new meanings. While student 2 bluntly abandons the quest for meaning (an action accompanied by exasperation as line 26 suggests), student 1 started a cardinal-arithmetic plan: to display the letters of the alphabet and to figure out what position n occupies in that order:

2:54 (32) <u>student 1</u>: How many circles would the top row of figure 14 have? n is fourteen.
3:00 (33) <u>student 2</u>: No it's not!
3:01 (34) <u>student 1</u>: Yeah it is!
3:02 (35) <u>student 3</u>: What is n? *(asking the teacher who coincidentally is walking by)*
3:04 (36) <u>student 2</u>: *(talking to the teacher)* What is n? We do not know.
3:08 (37) <u>teacher</u>: *(turning the page and reading the question aloud)* How many circles would the top row of figure number n have?
3:13 (38) <u>student 3</u>: What is n?
3:15 (39) <u>student 1</u>: n is fourteen because n is the fourteenth letter of the alphabet. Right?
3:20 (40) <u>student 2</u>: *(counting aloud the letters that student 1 wrote on the table previously)* one, two three, four, five, six, seven, eight, nine, ten, eleven, twelve, thirteen, fourteen.

Student 2 does not agree with student 1's interpretation about what n is and his arithmetical rephrasing of the question ("confronting" utterance in line 33). The arrival of the teacher serves as a potential way to overcome the conflict. The teacher, nevertheless, offers as an answer a re-reading of the question only. Interestingly (and probably because of the teacher's laconic answer) student 2 seems to change his mind and to agree in investigating student 1's idea; thus he starts counting the letters of the alphabet. Noticing that the students are taking an unexpected path that may take them away from the intended algebraic meaning, the teacher consents to explain a bit further:

3:28 (41) <u>teacher</u>: "n" is meant to be any number.
3:32 (42) <u>student 2</u>: OK.
3:33 (43) <u>student 1</u>: n is what?
3:35 (44) <u>teacher</u>: Any number *(in the meantime student 3 comes back to question a.)*
3:39 (45) <u>student 1</u>: I don't understand.
3:41 (46) <u>teacher</u>: You don't understand?
3:42 (47) <u>student 1</u>: No. […]
3:44 (49) <u>teacher</u>: *(talking to student 3)* Do you understand what n is?
3:45 (50) <u>student 3</u>: Which one? *(pointing to the figures on the sheet)* this, this or this?
3:46 (51) <u>teacher</u>: It does not matter which one.

The rhetoric of generalization has now taken a different turn. The teacher launches the understanding of "n" as "*n being any number*". The students' reactions show that there is a tremendous difficulty in constructing this specific meaning. Student 3's answer (line 50) suggests that this difficulty is linked to a very specific semiotic problem that we will term as the "multiple representation problem". Since sign-numbers, in the semiotics of arithmetic, generally refer to a single object (i.e. although in some contextual instances the number five may be represented as e.g. 6-1, nobody will agree that, in the base 10 arithmetic, two different 'basic' representations like "7" and "5" represent the same number) it appears very hard to conceive "n" (which by the way as sign has a similar 'basic' iconicity as "7" or "5") as representing more than one number (see line 50). This is corroborated by the following lines:

4:10 (53) teacher: *(talking to student 2 after a long period in which the students remained silent)* OK. There, do you have an idea what is n?

4:12 (54) student 1: Fourteen.

4:13 (55) teacher: It may be fourteen …

4:14 (56) student 2: *(interrupting)* any number?

4:15 (57) teacher: *(continuing the explanation)* … it may be 18, it may be 25…

4:18 (58) student 1: Oh! That can be any number?

4:19 (59) student 2: *(interrupting)* The number that we decide!

4:20 (60) student 1: OK then, *(taking the sheet)* OK, n can be … uhh…

4:26 (61) student 2: Twelve.

4:27 (62) student 1: Yeah.

4:28 (63) teacher: But … yeah. What were you going to write?

4:31 (64) student 1: 12.

4:32 (65) student 2: 12.

The teacher's attempt failed. The proposed meaning for n as being "any number" is interpreted as an arbitrary but *concrete* number ("informing" utterances, lines 61, 64-65). The teacher tries to give meaning to the expression conveying the generalization by re-investing the students' arithmetic point of view in a way which is still coherent with the global plan to introduce the general term in the context of the classroom setting. The teacher's voice hence acquires a specific tone made up of the pedagogical plan and the students' contextual voices. The Bakhtinian text in which generalization is being written appears to be heterogeneous in its meaning. Realizing that things had not turned out as expected, in the next line the teacher launches a rescue mission from where the wanted meaning could properly arise:

4:33 (66) teacher: And if you leave it to say any number. How can we find … how can we find the number of circles for any term of the sequence (*making a sign with the hands as if going from one term to the next)*

4:51 (67) student 2: Figure n? There is no figure n!

4:54 (68) student 1: (*talking to student 2*) He just explained it! N is whatever you want it to be.

4:57 (69) student 2: *(talking when student 1 is still talking)* What is it?

5:01 (70) student 1: OK. Umm … seven. *(writing on the sheet)*

5:10 (71) student 2: Not on top! It's seven circles *(taking the sheet and looking at the figures)*

5:13 (72) student 1: Yeah! And in the bottom is 5 circles!

5:21 (73) student 2: *(writes the answer and starts reading the next question)* How many circles would … *(inaudible)* … 12 circles *(writing the answer)*.

Speech does not unfold alone. Speech unfolds accompanied by other semiotic systems, for instance systems of gestures that we make with our hands and arms (see e.g. Leroi-Gorhan 1964). When we make gestures, the hands can be used to produce signs by e.g. sketching objects (Kendon 1993), while in certain cases concrete objects can be used as metaphors of absent objects (an instrumental strategy generally employed and which becomes a cornerstone in the development of sign systems with deaf children). Gestures form a sign system with its own syntax and meaning that afford the production of texts. In previous activities, we frequently saw our students pointing to a concrete figure (the third figure of a pattern, for instance) to refer in fact to the 100th figure. And in the case of the episode that we are discussing here, the teacher makes an intensive use of gestures (line 66) to try to complement the sense of the expression "any number" —an expression whose even most forceful utterance cannot reach the students' understanding yet. Indeed, the students keep the arithmetic meanings for the relation between "n", "figure n", and "any number". We should note at this point that student 2 is clearly uncomfortable with their general understanding of "figure n". There is something that does not fit the modes of meaning generation as being used in their classroom culture. In the subjective understanding of student 2, the order of discourse (Foucault 1971), as legitimized by the discursive practices of the classroom cultural institution and instantiated here by the teacher's remarks seems to point to a different way to interpret "n". We may say that, for him, if n is meant to be any concrete intended number, as student 1 is proposing, then, according to the classroom culture, the teacher could have stated this clearly instead of using such a complicated phrasing. Student 2 is doubtful and this doubt appears as something very important for the future of the meaning negotiation process. Notice that the conflict between students 1 and 2 seen in lines 27-28, 32-34, arises differently here. In line 68, student 1 re-interprets the teacher's previous explanations as confirming his own arithmetic interpretation and challenges student 2 with an authoritative argument ("[the teacher] just explained it!"). Seeing this, the teacher decides to intervene again:

5:42 (74) teacher: So, uh… *(looking at the sheet)* Wait, wait, wait! But for any number…. There you did it for seven circles, but if seven… for any …

5:52 (75) student 2: *(showing the sheet with his pencil)* You add 2 to the number on the bottom… subtract….oh no, you add 2 to the number on top. If it is seven, the number like this what I … *(inaudible)*

As the dialog suggests, the opening towards a new understanding is not made possible through a discussion on a concrete example but through the *prise de conscience* of an action previously undertaken (in solving question *a* and *b* but also in many lines of the dialog presented here, e.g. lines 71, 72). The action is now formulated *not* as a *concrete action* within arithmetic (which would give as a result a concrete number, as in lines 71 and 72) but as a *potential action* in the metacode of natural language. As we can see, the new mathematical object is constructed with words: "You add 2 to the number on the bottom…". What we call "generality" is trapped here in the expression "the number on the bottom" —an expression that keeps all the sensuality of the figures in the space— and the operation of adding ("You add 2") to which is submitted this unutterable number ("the number on the bottom") within the elementary semiotic system of school arithmetic.

But what is it that finally made possible the negotiation of meaning? The answer resides not in the students' suddenly grasping the teacher's intentions but in the teacher's continuous (polite, encouraging but always clear) rejection of the students' solutions and the students' will to search for alternative understandings. The construction of the potential action with words is pushed further by the teacher in order to end up with a mathematical formula:

6:01	(76)	teacher: OK. Could you put this in a formula?
6:04	(77)	student 3: Uhhh …
6:05	(78)	teacher: … using n.
6:06	(79)	student 3: Uhh … it's the term times two plus two.
6:10	(80)	student 2: The term times two plus two?
6:12	(81)	student 3: (*showing the figures with his pencil)* Uhhh … 2 times 6 … 2 times 3 is 6, plus two …
6:21	(82)	teacher: Could you say that again, please?
6:23	(83)	student 3: Yeah. The term times two plus two *(student 2 writes the explanation)* / […]
6:37	(86)	student 1: *(reading the answer)* OK. The term times two plus two.

The students finish by writing: "n2 + 2 = ".

4 Concluding Remarks

The word "term" (which emerges as a sign representing the previous term "the number on the bottom") is not used correctly by the students from a mathematical point of view. There is a confusion between the term and its rank. Nevertheless, the attempted meaning was functionally clear. It is worth noticing that the word "term" comes first to be used by the students as a tool that allows a refinement in the construction of the object. The use of words seems to be similar to that of concrete tools in apprentices. At first the tool (in terms of the "specialist's norms") is used awkwardly and only later can one use it with progressive mastery.

The concept of general term appeared as a potential action bearing the concrete characteristics of actions previously carried out in the social plane undergoing an internalization (in the neo-Vygotsky's sense given in Radford 1998) *through* and *by* signs (in this case words and mathematical signs, whether iconic or arithmetic ones). Such a potential action —which seizes the actual form of the generalization— is the particular expression of concrete actions as afforded by the students' mediated activity (not only by speech but by writing and the related cultural artefacts allowing it, e.g. the sheet and the pencil, the latter functioning as a key instrument in deictic gestures, as in the crucial line 75) arising in the course of their reflections to solve the problem. The students' reflections and their understanding and production of signs are embedded in discursive schemes and discourse orders prevailing in the classroom according to its own culture.

As we have seen, the potential action making possible the overstepping of concrete arithmetic thinking and the reaching of generalization finds expression in the semiotics of the concrete actions and the mode of thinking thus produced. Contrary to the traditional idea, generalization is not something dealing with the abstract and its evacuation of the context but a different contextual semiotic expression of previous actions, which afford the potential action (for instance, giving sense and virtually existence to it). It is enlightening to remark at this point of our discussion that Euclid's proposition quoted in the introduction also bears this distinctive trait of generalization as a potential action that, figuratively speaking, still has the sent of the concrete Pythagorean actions from where it emerged. Generalization is not a mere act of abstraction from the concrete; indeed, generalization keeps a genetic connection to the concrete according to the mediated system of individuals' activities and the epistemic and symbolic structure of these. In turn, as paradoxical as it may seem, the generalizing potential action, even without being *there*, is already producing the concrete actions. Indeed, without being explicitly there, the potential action is already present, making possible that the sixth, seventh or any other term be investigated in the very same form. Beyond their synchronic temporal dimension, the concrete and the abstract bear a dialectical relation, in which they mutually condition each other within the limits traced by the historical and cultural rationality of the individuals and the semiotic systems that the individuals are continually re- and co-creating.

As we saw, the web of possibilities from where generalization takes place is co-formed and revealed in the texture of the text that the teacher and the students deploy in their search for meaning. In the particular case studied here, we saw how meaning shifted from “figure n” to “n” to “any number” to “any arbitrary but concrete number” until they overcame the “multiple representation problem” and reached an algebraic “public” (Ernest 1998) standard meaning. We also saw that this was done by using different utterance genres ranging from reading (“R” e.g. line 22), requesting (“Q” e.g. line 36), confronting (“C” e.g. line 33), explaining (“E” e.g. line 44), acquiescing (“A” e.g. line 42), informing (“I”, e.g. line 45)[2] . The students’ production of signs in the formula was mediated by speech and its written form. After uttering the formula, the students wrote it in natural language as “the term x 2 + 2” and then as “~~n x 2~~ n2+2 = ” . It is worthwhile to note that Vygotsky suggested that “ [u]nderstanding written language is done through oral speech, but gradually this path is shortened, the intermediate link in the form of oral speech drops away and written language becomes a direct symbol just as understandable as oral speech.” (1997, p. 142). The fate of the students’ understanding of algebraic language seems to be the same, that is, it will be couched in speech (and the accompanying semiotic systems) and only later will it become a kind of autonomous semiotic action. Indeed, signs (like “n” in the students’ formula), we would like to insist in closing this paper, are but the result of semiotic contractions of actions (concrete or intellectual as outer or inner speech) previously carried out in the social plane.

Notes:

1. A research program funded by the Social Sciences and Humanities Research Council of Canada, grant number 410-98-1287.
2. The number of occurrences of types of utterances are as follows (notice, however, that a same utterance may belong to more than one category depending on its pragmatic dimension).

R	Q	C	E	A	I
3	22	7	9	5	15

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