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SIGNS AND MEANINGS IN STUDENTS' EMERGENT ALGEBRAIC THINKING: A SEMIOTIC ANALYSIS *

ABSTRACT. The purpose of this article, which is part of a longitudinal classroom research about students' algebraic symbolizations, is twofold: (1) to investigate the way students use signs and endow them with meaning in their very first encounter with the algebraic generalization of patterns and (2) to provide accounts about the students' emergent algebraic thinking. The research draws from Vygotsky's historical-cultural school of psychology, on the one hand, and from Bakhtin and Voloshinov's theory of discourse on the other, and is grounded in a semiotic-cultural theoretical framework in which algebraic thinking is considered as a sign-mediated cognitive *praxis*. Within this theoretical framework, the students' algebraic activity is investigated in the interaction of the individual's subjectivity and the social means of semiotic objectification. An ethnographic qualitative methodology, supported by historic, epistemological research, ensured the design and interpretation of a set of teaching activities. The paper focuses on the discussion held by a small group of students of which an interpretative, situated discourse analysis is provided. The results shed some light on the students' production of (oral and written) signs and their meanings as they engage in the construction of expressions of mathematical generality and on the social nature of their emergent algebraic thinking.

KEY WORDS: semiotic-cultural approach to algebraic thinking, generalization, signs and meanings, symbolization, social means of semiotic objectification

INTRODUCTION

Since Davis (1975), Clement (1982), Clement et al. (1981) and other pioneer studies on the learning of algebra, a significant number of didactic works has focused on the investigation of students' algebraic thinking and the search of pedagogical means to enhance it. Some of these works have dealt with generalization – a topic that several curricula around the world encourage as a route to algebra (see e.g. MacGregor and Stacey, 1992). The research results have stressed the fact that generalization of numerical patterns and the symbolic formulation of relations between variables raise specific problems for novice students (see e.g. Arzarello, 1991; Arzarello et al., 1993, 1994a, 1994b; MacGregor and Stacey, 1993; Mason, 1996;

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Lee, 1996; Rico et al., 1996). Although particular difficulties experienced by students have been reported by the aforementioned works and many others, it has been recently argued that more research is still needed (Sasman et al., 1999, p. 161). To go further, we want to add, we need to deepen our own understanding of the nature of algebraic thinking and the way it relates to generalization.

In this paper, we want to contribute to such an enterprise by interweaving classroom observations and theoretical reflections. To do this, we shall adopt a social and semiotic perspective having a central focus of attention on the students' understanding and constitution of the meaning of signs as used in algebra (see e.g. Nemirovsky, 1994; Meira, 1996; Vile and Lerman, 1996). In more specific terms, the purpose of this paper is:

(1) to investigate the way in which students use signs and endow them with meaning during their very first tasks related to the algebraic generalization of patterns[1] and
(2) to provide accounts about the students' emergent algebraic thinking.

Our approach is underlain by an anthropological view about the nature of thinking. As such, it relates thinking to the social practices from which it arises. There are, of course, several forms in which to theorize such a relation (see e.g. Chaiklin and Lave, 1993). Ours elaborates this relation in a way that thinking is conceived as a form of social sign-mediated cognitive *praxis*. It is against this anthropological background that, in what follows, we shall investigate the two aforementioned objectives. We will scrutinize the use of signs and their meanings in the students' emergent algebraic thinking in the intersecting territory of individual subjectivity and the social means of semiotic objectification. In doing so, we depart from epistemological evolutionary postures and their correlated view of thinking according to which thinking appears as an *a*cultural, natural, teleologically necessary process of abstraction in the intellectual development of the students.

The very semiotic nature of thinking and knowledge is addressed in Section 1, where we discuss some problems about signs and their meanings in light of our semiotic-cultural theoretical framework. Section 2 deals with the description of our classroom research program and the different phases underpinning its ethnographic methodology. Sections 3 to 5 are devoted to the analysis of a classroom activity about generalization where we investigate the novice students' expression of the general term of a geometrical-numeric pattern, focusing on the discussion held by a small group of students of which we provide an interpretative, situated discourse analysis. In Section 6, the results of the previous sections are re-examined at a more general level in order to discuss the emergent students' algebraic thinking through the prism of our anthropological approach and its semiotic-cultural theoretical framework.

1. SIGNS AND THEIR MEANING[2]

Teaching algebra seems to have been a pedagogical problem since Antiquity. In the introductory note to his monumental *Arithmetica*, written ca. 250 AC, Diophantus of Alexandria mentions the discouragement that the students usually feel when learning what we now term 'algebraic techniques' to solve word-problems. Contemporary mathematics curricula try to offer students some help to develop the algebraic ideas and to acquire and make sense of signs. For example, in the new Ontario Curriculum of Mathematics (Ministry of Education and Training 1997), students are introduced to a kind of 'transitional' language prior to the standard alphanumeric-based algebraic language and are asked to find the value of '$*$' in equations like: $* + * + 2 = 8$ or the value of '$\square$' in equations like $32 + \square + \square = 54$. This 'transitional language' approach, as any pedagogical approach for teaching algebra, relies on specific conceptions about what signs represent and the way in which the meaning of signs is elaborated by the students.

What does a sign represent? This question – also discussed by Mason (1987) – is at the center of the semiotic problem concerning the relation between the sign and its signified. Some psychological approaches, following the distinction of expression vs. content or surface structure vs. deep structure made in structural linguistics in the 1970s, have elaborated this problem in terms of the relation between internal (or mental) representations and external representations (seen as e.g. manipulation of signs or symbols). The link between both kinds of representations has often been described in terms of the property of a postulated mapping process according to which the external manipulations somehow *reflect* the structures of mental functioning. Long before the rise of cognitive psychology, Frege clearly expressed this idea. He said that the structure of propositions appears as a kind of mirror of the structure of thinking (Frege, 1971, p. 214). This position, however, is problematic in that it leaves unanswered the question of the nature of the mapping between the sign and the signified. Furthermore, this position is reductive in that, as Meira (1995) pointed out, signs, symbols and mathematical notations are seen as mere *accessories* for the construction of concepts.

Theoretical approaches based on the dichotomy internal/external representations are also problematic in the views that they offer concerning the *objects* (e.g. a physical entity) to which the external representations refer. The object (or what is sometimes called the *referent* and that Stoic philosophers called *tynchanon*) is reified and abstracted from its historical and cultural context and is conceived as standing in front of the subject as if the object was a mere item in the ecology of the subject.

The last remark has important consequences in the way meaning construction is conceptualized. Indeed, as long as the relation subject/object is seen as a non-culturally mediated, direct one, meaning construction appears to be the result of the relation that the isolated subject entertains with the *a*historical object (for a detailed discussion see Radford, 2000).

In the particular example of the ‘transitional language’ approach mentioned previously – as well as in other variants of structural approaches to meaning making in elementary algebra with epistemological and theoretical commitments to the information processing psychology (Kirshner, 2001) – meaning is expected to be extracted by the student (at least to some important extent) from the syntactic structure of the algebraic language and endless tasks of symbolic manipulation. The understanding of the syntactic structure of algebraic language and the meaning of signs is, however, a long process in the students’ ontogenetic trajectory. Students, at the very beginning, tend to have recourse to other experiential aspects more accessible to them than the structural one. In their study about students’ understanding of algebraic notations, MacGregor and Stacey (1997, p. 5) notice that some novice students associate letters with numbers according to the position in the alphabet (e.g. ‘h’ with 8) and suggest that this is due to the students’ experience in puzzles and translation into codes activities. We take examples like this as a clear indication that novice students bring meanings from other domains (not necessarily mathematical domains) into the realm of algebra. Hence, it seems to us, one of the didactic questions with which to deal is not really that of the elaboration of catalogues of students’ errors in algebraic manipulations, which may be interesting in itself, but that of the understanding how those non-algebraic meanings are progressively transformed by the students up to the point to attain the standards of the complex algebraic meanings of contemporary school mathematics[3].

The theoretical position that we are taking in our research program concerning, on the one hand, the conceptual aspect of signs (i.e., the sign-signified relationship) and, on the other hand, their signifying feature (i.e., the one related to their meaning) is based on a general semiotic-cultural conception of cognition having its roots in two basic ideas. The first one is the Vygotskian idea according to which our cognitive functioning is intimately linked, and *affected* by, the use of signs[4]. We take signs here not as mere accessories of the mind but as concrete components of ‘mentation'. Following Vygotsky (1962), Vygotsky and Luria (1994), Zinchenko (1985), Geertz (1973), Bateson (1973) and Wertsch (1991), instead of seeing signs as the reflecting mirrors of internal cognitive processes, we consider them as tools or prostheses of the mind to accomplish actions as required by the contextual activities in which the individuals engage[5]. As a result, there is a theoretical shift from what signs *represent* to what they *enable* us to do[6].

The second basic idea on which our framework is based deals with the meaning of signs and stresses the fact that the signs *with* which the individual acts and *in* which the individual thinks belong to cultural symbolic systems which transcend the individual *qua* individual. Signs hence have a double life. On the one hand, they function as tools allowing the individuals to engage in cognitive praxis. On the other hand, they are part of those systems transcending the individual and through which a social reality is objectified. The sign-tools with which the individual thinks appear then as framed by social meanings and rules of use and provide the individual with social means of semiotic objectification.

In this line of thought, the conceptual and the signifying aspects of signs need to be studied in the activity that the signs mediate in accordance to specific semiotic configurations resulting from, and interwoven with, social meaning-making practices and cultural forms of signification (Radford, 1998a).

As a result of our theoretical requirements and our didactic purposes, the classic semiotic triangle, having as its vertexes the sign, its object and its signified, cannot suitably account for the conceptual relations of signs and the aspects related to their meaning. Such semiotic triangles often isolate the subject, the object and the act of symbolizing from the other individuals and their contextual activities[7].

Within the context of the previous discussion, in our classroom research program we consider the learning of algebra as the appropriation of a new and specific mathematical way of acting and thinking which is dialectically interwoven with a novel use and production of signs whose meanings are acquired by the students as a result of their social immersion into mathematical activities.

Naturally, by this we do not mean that students' knowledge appropriation is achieved through a kind of crude transfer of information coming from the teacher[8]. As we see it, knowledge appropriation is achieved through the tension between the students' subjectivity and the social means of semiotic objectification. It is in this sense that we will proceed in the next sections to examine the activity of a small group of students dealing with a task about generalization of patterns. In this didactic task, the different psychological, epistemological, semiotic and historical perspectives, which serve as the basis of our framework (elaborated in detail in Radford, 1998b), play a different role. While we retain the idea of signs as the concrete components of mentation, as suggested by cultural and Vygotskian studies, we borrow from Bakhtin (1986) and Voloshinov (1973) important elements of their theory of dialogue to understand the role of students' discursive actions and interactions. In this perspective, speech, in the various formats that discourse lodges, is not seen as an exchange of information. Rather, we ascribe to speech an epistemological role in the individual's reconstruction and novel construction of knowledge. (For other – similar or different – views concerning the epistemological role of language and its relevance for mathematics education see, e.g. Boero et al., 1997, 1998; Bartolini Bussi, 1995, 1998; Bauersfeld, 1995; Cobb et al., 1997; and Steinbring, 1998)[9].

2. ABOUT THE METHODOLOGY

The research reported in this paper is part of a three-year longitudinal classroom research dealing with the students' learning of algebra. Our research is based on an ethnographic qualitative design (Goetz and LeCompte, 1984) having the following three-phase architecture: (i) a first phase, that may be termed *ingénierie didactique* (Artigue, 1988), in which the design of the teaching settings is elaborated, (ii) a second phase, in which the implementation of those teaching settings in the classroom occur, and, (iii) a third phase, where the research analyses are carried out. The three phases take into account the specificity of the curriculum of mathematics which functions as a reference point for learning assessment and further development.

The first phase includes the elaboration of the classroom activities with the teachers. The activities have been elaborated in such a way that the students (who are presently in Grade 8) work together in small groups. Then, the teacher conducts a general discussion allowing the students to expose, confront and discuss their mathematical methods and solutions. The second phase consists of the implementation and video-taping of the activities on which the teaching settings are based[10]. Drawing some methodological elements from Discourse Analysis (mainly from the works of Fairclough, 1995; Moerman, 1988 and Coulthard, 1977), the third phase addresses the discussion of the video-tapes, their transcription and interpretation. Finally, the discussions between the teachers, researcher and research assistants provide feedback for the instructional design of the up-coming algebraic curricular units.

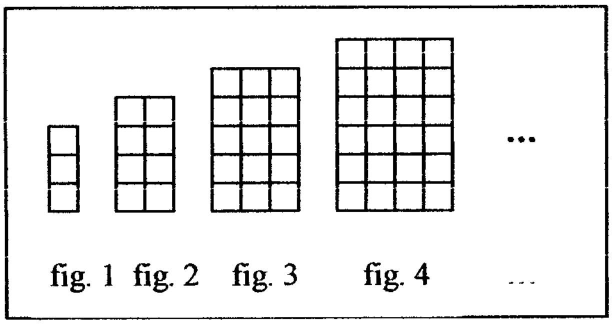

Figure 1. Problem discussed by the teacher and the students.

The classroom observations are of two different types: a) the videotaping of two to three co-operative groups of two to four students (depending on the mathematical activity) and b) quizzes administered to the whole class at the end of the teaching units[11]. Four classes are participating in the longitudinal research program. The four classes come from two different schools (which will be identified as school A and B) situated in northern Ontario and may be considered as typical schools of medium size cities in Canadian standards. Although some efforts have been made by the provincial government over the last few years to improve the quality of teaching and learning in mathematics (particularly the recent revamping of the curricular teaching content as a result of Ontario's poor scores in national and international mathematics tests), it is still too early to know what impact this will have.

The schools A and B have an independent curricular schedule, which allows us to continually generate feedback from one school to the other. For instance, after realizing the enormous difficulties that students in school A had in understanding what was expected from them in the algebraic task of writing the n-th term of a pattern (Radford, 1999), in school B we decided to open the classroom activity about generalization with a typical example that the teacher discussed with the students before they went to work in small groups. An overview of the introductory example and an analysis of the small-group activity are provided in the next section.

3. THE CLASSROOM ACTIVITY

The typical example that the teacher discussed with the students, prior to the small groups classroom activity, consisted of a sequence of rectangles (see Figure 1). After counting the number of squares in the first rectangles, the discussion revolved around finding a way to calculate the number of squares in rectangles such as the 25^{th}. The students associated the number of squares to the area of a rectangle and, with the help of the teacher, identified the area with the numerical sequence 1×3, 2×4, 3×5, 4×6.

The teacher then asked for a formula that would work for any rectangle of the sequence. Through an interactive question-answer process in which many students participated, the link between the number of the rectangle in the sequence and the number of squares on its length, on the one hand, and the number of squares on its width, on the other hand, became apparent. This led them to the symbolic expression $n \times (n + 2)$, which was then applied to a few concrete cases.

Following the classroom discussion of the introductory example, the students engaged in a small group classroom activity. In order to explore the discursively- and symbolically-based students' semiotic means of objectification in generalization tasks, the activity was divided into three steps:

(i) an arithmetic investigation,
(ii) the expression of generalization in natural language (in the form of a message), and
(iii) the use of standard algebraic symbolism to express generality.

On the first day, the students were presented two problems. We will focus on the first one (see Figure 2).

In what follows, we shall concentrate on one of our small groups of students. This group (like the others) was made up of 3 students – Anik, Josh and Judith – (naturally not their real names for deontological reasons).

A general overview of the role of transcript analysis and its place in our longitudinal classroom research was given in Section 2. For the purpose of the specific aspect of the research that we are reporting here, we need to add that the interpretative transcript analysis, inspired by Coulthard' work (Coulthard, 1977), was carried out in three steps. In the first step, all utterances were treated equally not paying attention to context, intention, and so on. In the second step, the rough material resulting from the first step was put into categories (see below) then refined into salient segments (Côté et al., 1993) and then contextualized by adding a social interactionist dimension (captured through interpretative comments that we insert in italics in the dialogue, emphasizing communicative aspects in relation to the research problem at hand). In the third step, the cadence of the dialogue was inserted by indicating pauses and verbal hesitations. The whole of the three aforementioned steps leads to what we want to term a *situated discourse analysis* whose elementary unit (i.e. the unit of analysis) was constituted by the refined (i.e. contextualized and cadenced) identified salient segments which were managed with the *Non-numerical Unstructured*

Using the bingo chips provided, reproduce the following sequence:

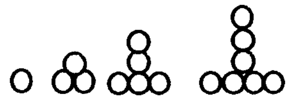

fig.1 fig. 2 fig.3 fig. 4

Continue the sequence up to and including figure 6.

a) How many circles would figure number 10 have in total?

b) How many circles would figure number 100 have in total?

c) You are now going to write a message to another Grade 8 student from another class clearly explaining what s/he must do in order to find out how many circles there are in any given figure of the sequence.

Message:

d) Find a formula to calculate the number of circles in figure number "n".

Figure 2. Problem presented to the students during the small group activity.

Data Indexing Searching and Theorizing (Nud·ist) program for qualitative research (Gahan and Hannibal, 1998).

In accordance to our theoretical framework and research purpose, the discourse analysis seeks to provide explanations about how students come to use signs and appropriate their meanings in the course of their initiation into the social practice of algebra. Although it is not possible to list the ingredients of meaning in advance, for, as Moerman (1988, p. 7) notices, '[t]he interpretations that conversants make [of the other conversants' utterances] are, in fact, *post facto*', something which makes the territory of the encounter between subjectivity and the social means of objectification very flexible, we will scrutinize the students' dialogue in light of their handling of the relationship between the particular and the general, its semiotization in natural language and in standard algebraic signs.

4. Expressing generality in natural language

A quick indication of the difficulty that students experienced in dealing with the construction of generality in natural language is provided by the length of the dialogue (measured as the number of lines). While questions a) and b) were solved in 3 or 4 lines, question c) generally took about 140 lines. In the following excerpt, we see how question b) was dealt with:

Line	Dialogue/comments
54	Judith: OK *(Reads)* ‘How many circles would figure number 100 have in total?’ That would be 100 . . . (*reflecting*)
55	Anik: . . . (*interrupting*) 199 because you have 100 on the bottom and 99 on top.
56	Josh: (*inaudible*)
57	Judith: Yes *(Writes the answer)* . . . OK.

Given the length limitation of this article, it is impossible to offer a transcription *in extenso* of the students’ dialogue concerning question c). Thus, in the next part (Subsection 4.1) we will present and discuss only some of the salient segments[12], focusing on the way the students linked the general and the particular.

4.1. *The general and the particular: the role of the metaphor*

Here is an excerpt of the students’ dialogue:

63	*(Josh silently reads the problem. Anik and Judith are looking at him. Then he says . . .)* I don’t understand.
64	Anik: (*Talking to Josh*) OK. You just have to explain to someone, let’s say me (*pointing to herself*), I don’t know how to do it, like I don’t know . . . like what this is here (*pointing to a figure on the page*). You have to show me how I would know how many chips to put, like if it said: in figure . . . uh . . . I don’t know . . . 120, OK? You would have to explain really well why, I mean, how I would . . . /[. . .]
71	Josh: How would I say that?
72	Anik: OK. Alright, look. You say how one has to add (*pointing to a figure on the paper*) . . . you always add 1 to the bottom, right? Then you always add 1 to the top (*referring to the rule to go from one figure to the next*). (W*hen she utters the word ‘top’ her hand moves to the top of the figure on the paper.*) So it’s always 1,2 . . . wait a minute. Do you know what I’m talking about?
73	Josh: Yes.
74	Anik: Do you know what it is that I want to say?
75	Josh: So, you start by adding . . . how I . . . exactly in a sentence?
76	Anik: You could say, uh, the figure . . . OK, say: Let’s say that in figure . . . um . . . Figure 12 (*moving her hands on the desk near the 6 first figures made up of bingo chips, as if she were touching the hypothetical Figure 12*). You’d put 12 chips . . .

77 Josh: on the bottom . . .
78 Judith: on the bottom . . .
79 Anik: That would go . . .
80 Josh: Not on the bottom.
81 Judith: Horizontal? /[. . .]
84 Josh: OK. Miss (*raising his hand*) . . . how would I explain it? I know the answer, it’s just that . . . /[. . .]
87 Anik: (*after being reassured by the teacher about the correct use of the word ‘horizontal’*) OK. So you’d put the number of chips horizontally with the figure.

In line 64, Anik rephrases with her own words the instructions about the message to be written. At the end of line 64, she hypothetically takes the role of the addressee (‘You would have to explain really well why, I mean, how I would . . . ’). Interestingly, in this move, consisting in taking the place of others and which is essential in social understanding (Astington, 1995), she omits the linguistic expression conveying the generality, i.e., that the message must say what to do to know how many circles are in *any figure*. Instead, she takes a concrete figure – Figure 120 – as an example to talk about the general. In line 71, Josh overtly asks how to *say* it. We now see the enormous difference in talking about obtaining an answer for a specific figure and saying it for any figure in general (as the student acknowledges in line 84, when the teacher comes to see the group’s work).

To talk in general terms, they hence take a specific figure, which is Figure 12 from line 76 onwards. Notice, however, that Figure 12 (as well as the aforementioned Figure 120) is not among those made with colored plastic bingo chips that the students *materially* have in front of them. Thanks to its ‘unmateriality’, Figure 12 fits the purpose of their reasoning about the general very well.

Nevertheless, Anik and her group-mates are not really talking about the *particular* Figure 12, something emphasized by the hypothetical expression ‘Let’s say’ (line 76). This is why they are not strictly counting the number of circles in Figure 12. We may say hence that Figure 12 is not taken literally but *metaphorically* by the students. In discursively taking an absent albeit specific figure, they talk metaphorically about the general *through* the particular[13].

4.2. *Talking about the metaphor: Deictic and generative functions of language*

Of course, Figure 12 is not enough to deal with mathematical generality. The students will now display a range of subtle discursive resources for their objectifying process of generality through the metaphorical figure. A close look at the dialogue shows that structural elements in reasoning about and expressing generality are based upon two key categories of words

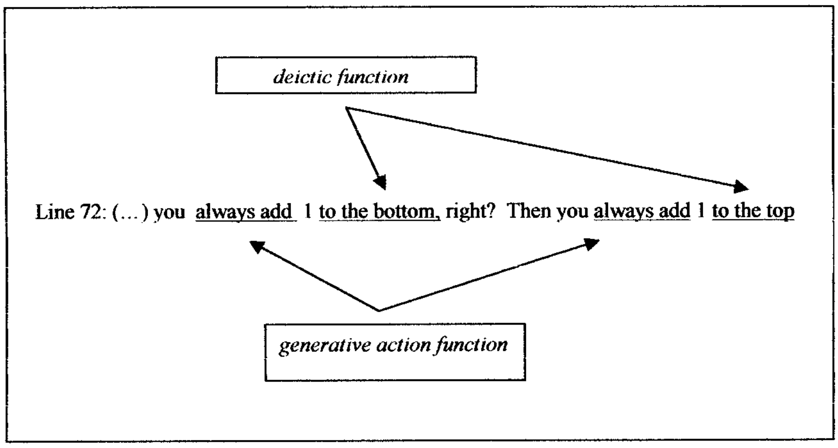

Figure 3. Generative action function and deictic function.

having two different semiotic functions: a *generative action function* and a *deictic function* (see Figure 3).

Deictic terms are linguistic units (in our example, 'top', 'bottom') referring to objects in the universe of discourse (in this example, the figures of the pattern) by virtue of the situation where dialogue is carried out. It is the contextual circumstances which determine their referents. As such, deictic terms depend heavily on the context (see Nyckees, 1998, p. 242 ff.) and have a particular function in dialogical processes.

The linguistic terms ('top', 'bottom') through which the deictic function is carried out (lines 72, 77, 78, 80, etc.) – terms which are based on the geometric shape of the concrete figures – allow them to refer to specific parts of the general mathematical object which, from the students' point of view, is being socially and discursively constructed at this precise moment. The interrogative interjection 'right?' (line 72) – strategically placed at this point of the unfolding dialogue and the social process of semiotic objectification – serves as a discursive request for gaining approval.

We term 'generative action function' the linguistic mechanisms expressing an action whose particularity is that of being repeatedly undertaken in thought. In this case, the adverb 'always' provides the generative action function with its repetitive character, supplying it with the conceptual dimension required in the generalizing task. The relevance of generative action functions can be acknowledged by noticing that, in our example, generality is objectified as the *potential action* that can be reiteratively accomplished.

But the current discursive formulation still needs more elaboration. The students are not completely satisfied with it. The formulation is incomplete according to their idea of cultural communicative standards required in the practice of mathematics (cf. line 84). There is still a gap between the current product and the envisioned task discussed in line 64.

4.3. *Structural descriptors of the figure*

What is it that allows the students to go further? The insertion, in the dialogue, of another deictic term – the word 'horizontal' (line 81; 'horizontally' in line 87) – is crucial in the search of new paths. Equipped with the horizontal/vertical dichotomy (a dichotomy in which Lotman (1990) found one of the general traits in the way that human beings semiotically mark their environment, as can be observed in cultures around the world) the students' generalizing task acquires a sharp and more precise expression. In fact, the words horizontal/vertical function as a *structural descriptor* of the figure. In particular, the word 'horizontal' allows the students to go a step further. They identify the number of the figure with the amount of circles in its bottom row and start tackling the 'positioning' problem, i.e. the problem consisting in relating the rank of the term in a sequence to the numerical value of the term. (We shall come back to the 'positioning problem' in the next section.) This is clear in line 124:

124 Anik: You (*talking to Josh in paused segments although the talk seems more like a discursive strategy to organize her own thoughts with her words*) have the same number of chips . . . (*making a long horizontal gesture with her hand*) . . . horizontally, like . . . the number of the figure . . . The number of the figure is 12 . . . so you're going to have 12 on the bottom *(looking attentively at him as if scrutinizing any sign of acquiescence)*
125 Judith: Yes (*approving Anik's statement*)
126 Anik: How do you . . . ? (*still unsatisfied with what she said*)
127 Josh: The regularity (*making a gesture with the hands*)
128 Judith: (*Interrupting*) The figure, if you know . . . you'll have 12 on the bottom . . .
129 Josh: If you know the regularity of the figure, it'll be 12 on the bottom. Then it's always minus 1 for the top (*Pause of some 8 seconds in which Anik and Judith think about Josh's proposal*).
130 Anik: Yes . . . Yes . . . OK. Yes. I know. Yes, that's it. But like how do you say that?
131 Josh: Yes.
132 Anik: OK (*taking the page to read what they have done up until now*)
133 Josh: I don't know.

Line 129 contains the verbalized form of what will be the skeleton of the written message later. This line, as many others, exhibits the verbal/mental operations that the students perform on the metaphorical Figure 12. One may say that they 'touch' the figure with words and so words now become a new organ of sensation. Since the students are experiencing the general through the particular, the geometric quality of the pattern remains their focus of attention. Their thinking moves around the rows of the figure, through linguistic terms playing deictic and generative action roles. As a result, it is possible to see a fundamental contrast when it comes to express generality through the algebraic formula $n + (n - 1)$ or $2n - 1$ that cannot capture, in the same manner, the sensual-spatial nature of the figures of the pattern.

All in all, a substantial gain has been achieved but, again, no satisfactory elaboration is reached yet (cf. lines 126, 130, 131 and 133). The words 'top' and 'bottom' (which are structural descriptors too, even though here they convey a less precise descriptive meaning than the vertical/horizontal couplet) do not sound appropriate for them. A more precise expression is offered by Judith in line 142:

142 Judith: If it's the figure it'll always have the number ... like if we say it's Figure 12, you'll have 12 on the bottom and then you'll have one less on top vertically.
143 Josh: OK. Because ...

4.4. *The positioning problem*

Generalization, as we have seen in the students' dialogue, is undergoing a process of consecutive changes and refinements. While *what* to say seems more or less well agreed upon, *how* to say it is still under construction. To continue, they have to solve, in deeper detail, the *positioning problem* with which they were dealing before (line 124). By this we mean that up until now, the students succeeded in objectifying the numerical relationship between the number of circles in the horizontal and vertical components in specific figures of the pattern and in the metaphorical Figure 12. They now have to relate it to a non-specific figure, saying in *general terms* which position the figure occupies in the sequence.

157 Anik: (*Talking to the teacher who was checking up on the work of a neighboring group*) Miss, how do you say, like, the number of the figure? Like the ... you know, like ...
158 Judith: (*Talking to the teacher*) Since the same number of chips is the same number of the figure, you know? It's like in Figure 1 there's 1, so the Figure 2 has 2 on the bottom, then if it's the Figure 3, you'll have 3 ...
159 Anik: Then, like, we want to say like the number of the figure.
160 Teacher: This *(shows horizontal with her hand)* is your number on the?
161 Anik: (*interrupting*) Like the number of the figure is 4 here, right? (*Shows Figure 4 on the sheet of paper made up of bingo chips).*
162 Teacher: OK.
163 Anik: Well, there we want to say, we want to say that it's like, the number of the figure.

164 Teacher: OK. Well, the row on the bottom is the same as the rank of the figure (*emphasizing the word rank*).
165 Josh: So, the horizontal row . . . /[. . .]
180 Josh: The number of chips horizontally is the same . . .
181 Judith: the same . . .
182 Josh: the same . . . ?
183 Judith: it's the same as the rank of the figure. Rank (*and then, trying to spell it, she says*) I think it's rank, r-a-n-k /[. . .]
199 Anik: Then, after that, you have to say that you have less . . . you have one less vertically.

Lines 157, 158 and 164 show three different rhetorical forms embodying what Miller identifies as different genres of speech (Miller, 1984). Each one provides individual views of the problem at hand emphasizing layered aspects of it. In line 157, the problem is sketched in terms whose vague formulation invites Judith to re-voice and precise it further for the teacher – which is what Judith actually does in the following line. Anik's imprecise utterance is reflecting the difficulty the students have in expressing and objectifying generality and requires the engagement of the teacher to fulfil what is not materially said in the message and to complete it with the student's intended meaning. Strictly speaking, line 157 says, in fact, very little. One may say that the co-text (that which contains what is collateral and implicit) here says more than the text itself. Line 158 is, hence, an attempt to complete the message of line 157 and this is done in a more vivid and dynamic form. The question in line 157 (although invisible for the teacher) holds the historical traits of the students' previous joint experience in dealing with the problem. In line 164, the teacher offers the answer to the problem from the point of view of the fluent speaker. Then, with line 164, comes the process of individual appropriation and *(re)creation* (Hall, 1995) of the technical mathematical expressions. As we will see in the next subsection, in the *(re)creation* of the teacher's technical words, and in order to end up with the required message, the students will interweave the teacher's words with their own subjectivity and end up with a modified version of the meaning of rank.

4.5. *The final message*

The final message was the following:

If I ask you to give me the amount of circles in Figure 12, there will be the same number of circle (*sic*) horizontally that would be the same as the rank of the figure. And to have the vertical rank, you have to subtract 1 from the number of horizontal circles.

Some remarks need to be made concerning this message.

First, the *structure*: the message is divided into two parts that reflect the geometrical configuration of the figures in the pattern. The first part explains how to find the number of circles in the horizontal line. The second part tells the addressee (who was incorporated in the text by the expression ‘If I ask *you*’) how to calculate the amount of circles in the vertical line. In this sense, the message keeps the *perpendicularity* of the components of the described object. In other words, the semiotic identified structure of the figures of the pattern induces the semiotic structure of their general description. There is still another way to say this. As mentioned in endnote 2, Peirce distinguished three kinds of signs: the icon, the index and the symbol. The icon is a sign that resembles the represented object. ‘An index is a sign which refers to the Object that it denotes by virtue of being really affected by that object.’ (Peirce, 1955, p. 102). Or, as Eco (1988, p. 75) says: ‘*L’Index* est un signe qui entretient un lien physique avec l’objet qu’il indique; c’est le cas lorsqu’un doigt est pointé sur un objet’. The symbol refers to its object in an arbitrary manner, or as Peirce said, ‘by virtue of a law’ (op. cit. p. 102). The analysis of the structure of the message written by the students and its structural *resemblance* to Figure 12 in the pattern makes it clear that the message, taken as a sign, belongs to Pierce’s icon category.

Second, the message relies on a particular example – that of Figure 12 – but, as previously stated, in a metaphorical way. It does not mean, however, that Figure 12 is merely an ornament. The allusion to Figure 12 in the message is a token of the students’ struggle to talk about and objectify the general term of the pattern. Figure 12 (which is no longer mentioned in the rest of the message and which could be omitted or replaced by the expression ‘any figure’ without affecting the comprehension of the text) is that which makes it possible to talk about the still unspeakable general term of the pattern in the students’ evolving technical language. Figure 12 is what ensures the students’ link between the particular and the general[14].

Third, the word ‘rank’, first mentioned by the teacher is appropriated and used by the students in a way that was not intended by the teacher. The students *ventriloquate* the teacher’s word. Ventriloquation is, according to Bakhtin, the integration of somebody else’s words in our own discourse. This allows us to *talk* and to *think* through the words of others. It is not a mere copy of them, rather we adapt the words in terms of our pragmatical needs and particular expressive intentions (see e.g. Bakhtin, 1981, pp. 299, 304, etc; Wertsch, 1991, p. 59; Todorov, 1984, p. 73). In their *ventriloquation* of the word ‘rank’, the students make it a *hybrid construction* (Bakhtin, 1981) supplying the teacher’s word with new meanings. Thus, the students talk about the rank of the vertical line. The confusion seems to arise when the teacher contextually equated the rank of a figure with the number of circles in its horizontal line. The rank then became sort of synonymous with 'the size of the line', that is, the number of circles in it.

It is important to notice here that our interest is not in finding what was 'right' and what was 'wrong' in the classroom. As Seeger (1991, p. 153) wrote, '[m]athematical precision may not be the best model available to steer the process of sense-making in the math classroom, especially in relation to instructional discourse.' Rather, we are trying to understand the discursive meaning-making movements that lead to certain shared understandings in the classroom. After all, although improperly used in terms of a fluent algebra speaking person, the word proved to be useful for the novice students, allowing them to refine and carve with more precision the elements to experience and express the generalizing task. It is clear that the word rank is also used in other human non-mathematical practices (e.g. in physical education competitions or music, where one talks about rankings). Its use in the study of patterns, however, has a different and particular goal. The 'transposition' from one practice to another requires an almost entire conceptual reconstruction. And this functions like the insertion of a new word in a language that one is starting to learn. First, you hear somebody using it and you want to do the same. You find that your new word sounds strange and when you begin using it the pronunciation surprises you more than your fluent speaking audience. Slowly, you dare putting it in a short phrase, then in another and again in another, sometimes in a faulty form, but you are still understood so you keep going. The teacher's utterance in line 164 (where the word 'rank' is first used) is echoed by Judith's utterance. In fact, right after, we find her saying: 'Then, it's the same as the rank ... the rank ... it's the same ... it's the same ... '.

Let us now turn to the expression of generality through standard algebraic symbolism.

5. Expressing generality in algebraic symbolism

In the last part of the activity about generalization (question d), the students had to provide a mathematical formula for the general term of the pattern. It is impossible to go through all the dialogue (it contains 94 entries), so let us highlight the salient segments, focusing, as in the previous section, on the use of signs and the appropriation of their meanings in the emerging students' algebraic thinking.

The students started discussing question d) in remarking that this is a question similar to the last question concerning the pattern about rectangles discussed that morning by the teacher – as we saw previously, a question which led to the formula $n \times (n+2)$ and that was still written on the blackboard:

271 Judith: (*after reading the question*) That's what is on the blackboard.
272 Josh: Yeah.
273 Judith: That would be 12 and 11.
274 Anik: I didn't understand what you said. *(Takes the paper to read the problem.)*
275 Josh: (*Explaining to Anik*) It's the same thing as what we did on the blackboard.
276 Anik: Wait a minute ... *(Takes the page from Judith and reads the problem)*
277 Josh: (*Continuing his explanation after looking at the blackboard*) but it's n minus 1 ...
278 Anik: OK, yeah (*and gives back the page to Judith*).
279 Josh: That's n minus 1 ...
280 Anik: (*repeating Josh's utterance*) n minus 1 ... umm ...
281 Josh: (*adding to Anik's utterance*) ... times n. OK. Then it's ...
282 Judith: (*interrupting abruptly*) No!
283 Anik: (*interrupting also*) It's not times. There is no times.
284 Josh: Uuh ... (*he then tries to find something else and adds, using a reconciliatory voice*) n plus ...
285 Anik: That would be ... 1 ...
286 Josh: No ... n minus 1 ... (*inaudible*)
287 Anik: Wait! Let's ask the teacher. *(Raises her hand to call the teacher to come over to the group.)*

Our interpretation of the episode is that the understanding of 'n' was not the same for all of the students. Josh and Judith seem to be satisfied with the formula '$n - 1$', but Anik does not. While waiting the teacher's arrival, the students kept discussing *how* to write the formula:

299 Josh: Yes. You can just put n minus 1. No?
300 Anik: Well, to have the ... to have the ... those that are ... um ... horizontal.

Anik is aware of the fact that the formula '$n - 1$' does not include the circles in the horizontal line, but she does not know how to include them in the symbolic expression. The teacher joins the group in line 310, when the students were still discussing in the spirit of lines 299–300:

310 Anik: OK. Miss? OK. On the other question, *(takes the sheet)* this here says ... this here (*showing the page to the teacher*) OK. We know what that means. But we don't know to say ... like ... figure ... ummm ...
311 Teacher: OK. Here the figure number n, all that that means there, is that it does not matter what figure. It could be Figure 1, 2, 3, 4, 5
312 Anik: (*interrupting and uttering with discouragement*) We know that
313 Teacher: ... (*continuing her explanation*) 58 ... this does not matter ... it is not important.
314 Anik: OK (*saying it without conviction*).

When the teacher realizes that this explanation is not enough for the students, she continues:

315 Teacher: But you will always apply the same formula. On the board there, I wrote ... let's say, one formula ... then it would work for all the figures. So, here, I want you to try to come up with a formula that you could use and that would work for all the figures by replacing a letter. Okay? Then, let's say even to go up to [Figure]150 if you want. Because often you will require a formula ... I won't always be giving you bingo chips, you know. OK? So, with the formula, you will be able to find out how many bingo chips there are in Figure 25 without always having to write them all out.

The teacher is hence shifting from '*n*' as a dynamic general descriptor of the figures in the pattern to '*n*' as a generic number in a mathematical formula. This requires a double-view of '*n*' supported by different meanings: '*n*' as an ordinal number, and '*n*' as a cardinal number capable of being arithmetically operated. This is discursively sustained by referring to the formula on the blackboard and supplemented now with pragmatic arguments (e.g. 'often you will require a formula'). However, the central question in line 310 was not stated clearly so the teacher, unaware of the group discussion and its specific difficulty, started an explanation about what she imagined the difficulty to be. Again, we do not mean that all this was wrong. To formulate a clear question is to have almost answered it to some extent and, most of the time, this requires using the involved concepts with great skill.

The teacher's explanation in lines 311 and 315 then led Josh to propose '*n*' as the formula:

331 Josh: That would be n.
332 Anik: (*ignoring Josh*) So, you could say: To know the ... um ... how many chips to have vertically ... you would subtract 1 from how many chips ... *(at this point, the teacher, seeing that Anik is getting close, makes a gesture to encourage her to continue)* there are horizontally.
333 Teacher: OK. But now you have to say that without using words! Use letters! OK?
334 Josh: You have to do 1 n minus ...
335 Teacher: OK. You're getting it. *(She departs from the group.)*

A fundamental difference with the discussion held in question c) – a question to which we devoted the previous section – is that now no specific figure is included in the dialogue. Indeed, Figure 12 has now been completely evacuated from the discussion – apart from the ephemeral mention made by Judith in line 273 which was ignored by her group-mates. As we saw, the metaphorical Figure 12 was a key element in the students' objectification and construction of a conceptual relation between the particular and the general and a central source of sense-making in talking about the general in natural language. Now the deictic terms (horizontal, bottom, etc.) seem to take over the role of Figure 12 and to become the basis for the semiotization of the general in standard algebraic symbolism.

347 Anik: OK. (*After a moment of silent reflection, she takes a deep breath and says:*) We'll just say that n is like ... n is the chips that there are on the bottom (*making a horizontal gesture with the hand*) ... like ... this is the number of chips that there are on the bottom ... OK.
348 Josh: Yes ... well ... no ... well ... OK.
349 Anik: So (*talking with difficulty, and making gestures with the forefinger she clearly 'writes' in the air what she says*) ... n minus one equals n.
350 Josh: Yes.
351 Anik: That's it (*satisfied*).
352 Judith: (*who is in charge of writing the formula on the page, says*) n minus one equals n?
353 Josh: 'Cause n is the figure.
354 Anik: Well, do you understand what it is that we said there, though?
355 Judith: Yes.
356 Anik: n minus 1. No (*seeing that Judith is using brackets*) ... you don't have to put it in brackets.
357 Judith: Well ... no ... (*referring to the algebraic expression which she realizes is not completely right.*)
358 Anik: OK.
359 Judith: If n is the thing on the bottom ... well ... let's say that it's 4, well 4 minus 1 would be 3 ... that wouldn't get us back to n again. (*The student notices that the equality does not work.*)
360 Anik: Well ... we'll put another letter ... n minus 1 equals c. Makes no difference what letter. You just have to put a letter. You don't need to put it in brackets either because you're not putting another equation beside it.
361 Judith: OK.
362 Anik: You just have to write it next to it. You just have to (*inaudible*). OK? Write (*making a gesture with her hand as if she is writing*) n minus 1 equals a letter. OK?

The students' final formula '$(n-1) = a$' was crossed out and then rewritten as '$n-1=a$'.

Even though it was not easy, the students were able to produce a message in natural written language about the total number of circles in any figure. They could not, in contrast, produce the algebraic expression for the *total* number of circles in figure 'n'. As long as Anik kept herself from entering the territory of standard algebraic symbolism, she had the sense that something was missing from Josh and Judith's '$n-1$' proposal. When she finally crossed over to this foreign territory, she, like her group-mates, seemed to lose sight of the figure. And to fulfil the uncompleted symbolic formula, she transformed it into the algebraic expression: $n-1=n$. What, then, did 'n' mean for them?

In fact, the students provided 'n' with two different meanings. Josh and Judith took the formula that they had to construct as an object bearing a formal resemblance to the blackboard formula of $n \times (n + 2)$. In this sense, they started from the formula $n \times (n + 2)$ and conceived the new one as a formula endowed with a mode of representation which is characteristic of the *icons*[15]. In assuming that the formula must look like the one on the blackboard, they relied on an anticipated formal resemblance. The meaning with which 'n' was provided was underpinned by a metonymic semiotic process. That is, a process of substitution of terms considered similar in some respect, in this case $n + 2$ becomes $n - 1$, and as Josh suggested in line 281, the new formula would be $n \times (n - 1)$ instead of $n \times (n + 2)$. This is for the first meaning of 'n'. The second meaning is related to the students' functional use of signs as *indexes*. This meaning was endorsed by Anik.

To be able to explain this we have to refer back to lines 332, 334, 347 and 349. What we notice is that it is in the junction of the sign system of speech and the bingo-chips-artefacts that 'n' found a niche out of which to emerge. Indeed, in line 332 we found Anik saying: 'To know the ... um ... how many chips to have vertically ... you would subtract 1 from how many chips ... ' , which later (line 334) acquired a more contracted form when Josh ventriloquated Anik's utterance and said: 'You have to do 1 n minus'. Then, Anik, in turn, counter-ventriloquated Josh's utterance in line 349. Using Peirce's terminology, we can say that the letter 'n' is a sign of a specific kind: it is an *index* in the sense that it is pointing, like a gesture with the forefinger, to the linguistic expressions contained in line 332. There is, indeed, a match between 'you would subtract 1 from how many chips there are ... horizontally' (line 332) and the left side of the students' symbolic expression '$n - 1 = n$' (line 349, mediated furthermore by line 347: 'We'll just say that n is like ... n is the chips that there are on the bottom'). In other terms, the students' production of symbolic expressions appear here as contractions of words in speech[16]. Probably, because elementary school arithmetic is focused on finding answers (Kieran, 1989, p. 33), Anik was led to 'close' the symbolic expression '$n - 1$' by indicating its *total*. The sign 'n', which pragmatically functions as an index, serves to point to this result. Hence, the final formula '$n - 1 = n$'. It is because of the indexical nature of signs, as used by Anik, that the students found it completely legitimate to replace 'n' by 'another letter', since, as Anik said, it 'makes no difference' (line 360).

To sum up, the students' first meaning of signs was actually derived from a metonymic process seeking to end up with a formula whose semiotic characteristic was that of being an icon of the formula on the blackboard. The second meaning was related to the students' objectifying discursive activity in which 'n' (functioning from a semiotic viewpoint as an index) replaced some key words in the students' speech to generate algebraic symbolic expressions. In the second case, the construction of symbolic expressions was heavily supported by the students' discourse. Notice that Vygotsky found a similar phenomenon when children learn to write. Indeed, he remarked that the signs of the written language find support in the sign-words of speech and only later the signs of written language acquire a certain autonomy:

> Understanding written language is done through oral speech, but gradually this path is shortened, the intermediate link in the form of oral speech drops away and written language becomes a direct symbol just as understandable as oral speech. (Vygotsky, 1997, p. 142)

To conclude our remarks about the meaning of algebraic signs in Anik's group, it may be stressed that although both meanings, the iconic and the indexical, conflicted in lines 299 and 300, the students' reached a consensus regarding the final formula. This can be credited to the fact that Josh's suggested formula ('$n - 1$') appears formally embedded in the formula '$n - 1 = n$' (see e.g. Josh's approving expression 'Yes' in line 350). In subsequent classroom activities about generalization of patterns the conflict in this small group of students between meaning of signs was deeper and the search of a common understanding took much longer.

6. THE NATURE OF STUDENTS' EMERGENT ALGEBRAIC THINKING

What then can be said about the students' algebraic thinking? According to the semiotic analysis offered in the previous sections and in light of our framework, algebraic thinking, we want to suggest, is the specific way in which the students conceptually acted in order to carry out the actions required by the generalizing task. That which makes 'algebraic' the students' thinking is the distinctiveness of the mathematical practice in which they engaged, namely, the investigation and expression of the *general* term of a pattern – something that may not be required at the level of the arithmetic thinking. This is why, for us, the students were already thinking algebraically when they were dealing with the production of a written message (Section 4), despite the fact that they were not using the standard algebraic symbolism. But the particular way in which the students conceptually acted and that underpinned the emergence of students' algebraic thinking was regulated by a socially established mathematical practice where the teacher plays a central role. This role was that of immersing and initiating the students into the particularities of the signs and meanings in which the practice of algebra is grounded. The students' emergent algebraic thinking, as required in the generalization of patterns, appeared hence as the appropriation of a highly specialized kind of cognitive praxis requiring a social use of signs and the understanding of their meanings to achieve specific expressions of generality.

Naturally, from the point of view of the individual *qua* individual, every student brought forth his or her own unique contribution and achieved an understanding of signs and of algebraic techniques which, as evidence suggests, were different in some aspects from one student to the other. Thus, in the quiz following the teaching unit about patterns, Anik was able to comfortably express generalization in both natural and standard symbolic algebraic language. Her written message did not have recourse to any metaphorical figure and, in contrast to the formula discussed in this paper, her formula was now exact. This suggests that she was able to produce and understand signs better than the first day of the activity and to meet the learning demands as prescribed by the curriculum[17]. Josh, in contrast, produced a written message based on a metaphorical figure and, instead of providing a symbolic formula, he carefully explained how to calculate the total number of circles through a metaphorical example. Judith, instead of writing a message for the total number of circles for any figure of the given pattern, gave a message that described the actions to go from one term of the pattern to the next and her formula was built on the basis of signs as elementary indexes, much in the same form as the first-day activity discussed in this paper.

In accordance to our conception of algebraic thinking, we see the students' mathematical behaviour as *personal* attempts to engage with the concrete and the general in varied forms. Josh and Judith, who were able to find the exact answer for the 10$^{\text{th}}$ and the 100$^{\text{th}}$ figure of the pattern in the quiz, seem to be more comfortable experiencing the general through the semiotic strategy based on metaphorical figures and the use of signs that this requires. The signs with which the fabric of our intimate mathematical experiences is made up offer different semiotic possibilities and the unavoidable aesthetic experience that accompanies our personal and unique encounter with the general may be lived in different forms. However, the forms taken by the personal attempts to engage with the concrete and the general are not arbitrary. The strategy based on metaphorical figures and those leading to the alphanumeric algebraic formula, as we saw, were *jointly* objectified by the students during the small-group work (where, e.g., processes of ventriloquation were of great importance). It does not mean, of course, that the mathematical enculturation achieved through classroom activities is a straitjacket for the mind. Rather we take classroom activities as a world of possibilities in which our intimate mathematical experiences occur. As Mikhailov rightly pointed out,

> In the individual's mentality there is not a single phenomenon determined by social being that is not at the same time deeply personal. And, on the contrary, in the individual mentality each 'only' personal perception comes 'only' out of the social means of reflection, the chief of which is language. (Mikhailov, 1980, p. 198)

To summarize, instead of seeing algebra learning and the emergence of algebraic thinking as the direct imprint of the external environment on students' minds, as empiricist or behaviorist pedagogical approaches with their corresponding models of direct transmission of knowledge contend, we see the emergence of algebraic thinking as resulting from the encounter between the individual's subjectivity and the social means of semiotic objectification. The varied forms taken by the students' algebraic thinking in the mathematical activity about patterns are seen as evidence of a complex process in which the students mesh personal and interpersonal tones within the limits of the contextual possibilities to actualize the mathematical practices.

The dynamic process of knowledge acquisition happens to be led by the irreducible tension between the individual *qua* individual and the range of semiotic means of objectification offered by the historically constituted domain of culture allowing experience to occur. As Bakhtin wrote in one of his earliest works found many years later in a dark, damp room in Saransk, Russia:

> An act of our activity, of our actual experiencing, is like a two-faced Janus. It looks in two opposite directions: it looks at the objective unity of a domain of culture and at the never-repeatable uniqueness of actually lived and experienced life. (Bakhtin, 1993, p. 2)

7. Synthesis and Concluding Remarks

Following a relatively recent research field in Mathematics Education which draws from cultural semiotics and focuses on the understanding of students' use of signs and production of meanings, this article presented an anthropological approach to the didactic study of introductory algebra. Our study of the students' use of signs and production of meanings offers some new insights, both on the practical and the theoretical level.

On the practical level, our investigation of the students' use of signs (word-signs and letter-signs) sheds light on the processes of semiotic objectification by evidencing the manner in which the students semioticized and conceptualized the relation between the particular and the general. It was suggested that the objectification of generality was discursively elaborated as a *potential act* articulated on two key linguistic elements: the use of deictic terms and adverbs of generative action out of which a concrete example (here the Figure 12 of the sequence) functioned as a metaphor. The metaphoric Figure 12, crafted with terms like 'rank', 'vertical' and 'horizontal', allowed the students to accomplish an apprehension and a discursive representation of the general and provided them with a way for expressing the general through the particular. In addition, the objectification of the general in natural language proved to be fundamental to the rise of the symbolic formula in that the symbolic formula appeared as contracted or abbreviated speech. An important result was to have identified that the letter-signs resulting from this process of semiotic objectification corresponds to the category of *indexes.* Nevertheless, algebraic formulas as contracted speech have an intrinsic limitation: indeed, as we noticed in the students' activity, algebraic formulas cannot seize the sensual aspect of the figures and cannot be based on a paradigmatic example. In this sense, the passage from a non-symbolic to a symbolic algebraic expression of generality means two ruptures, one with the sensual geometry of the patterns and the other with the numerical feature of them. A deeper investigation of the very semiotic nature of indexes as used by novice students is an open didactic problem that needs to be pursued in order to better understand the students' first contact with symbolic algebra. A better comprehension of the semiotic nature of indexes should also enlighten the meaning of the equal sign that they tend to incorporate in the symbolic expressions.

On the theoretical level, in conceiving signs both as concrete components of mentation and vital parts of specific social semiotic means of objectification, our work provides some possibilities in which to elaborate new understandings about students' algebraic thinking. We sketched here one of those possibilities, based on the idea of thinking as a sign-mediated cognitive praxis. Within this context, the emergent students' algebraic thinking was seen as the students' insertion into an intellectual practice requiring a social use of signs and the understanding of their meanings to achieve specific expressions of generality. This kind of cognitive praxis was not homogeneous. The students displayed different mathematical behaviours that we took as *personal* attempts to engage with the concrete and the general in varied forms and which resulted in differences of mastery to elaborate algebraic symbolic expressions.

In order to conclude this synthesis, let us hence come back to the twofold aforementioned rupture underpinning the passage from a non-symbolic to a symbolic expression of generality, and to discuss it in light of the territory of encounter of individual subjectivity and the social means of semiotic objectification. This rupture encompasses a subjective experiential component that, as we want to submit, includes an aesthetical aspect that we cannot neglect. The metaphorical figure and concrete examples taken from the pattern provide vivid figures of speech that are at odds with the functioning of the sober algebraic symbolism. Both correspond to radically different ways with which we can semioticize our world. To become a fluent user of algebra (in the curricular sense) would then mean to acquire a set of tools and concepts allowing one to become conversant with the general, the unknown and the variable in ways embedding aesthetical experiences which are different from those based on concrete or metaphorical linguistic devices. For those students feeling comfortable with metaphorical figures of speech (like Figure 12), the rupture that we are mentioning may somehow be similar to the one that a person in an art gallery may live when, after experiencing paintings in the perspectival style of Piero della Francesca and other painters of the Quattrocento, he or she moves to the next room exhibiting non-lifelike cubist paintings of Picasso and Braque. Although in both cases one may be ‘talking’ about e.g. portraits, the differences in the respective sign systems (e.g. lines, gradient colors, forms, etc.) and in their semiotic organization ensure a different order in the ‘text’ and leads to different modes of expression and aesthetical experiences. Coming back to the didactic problem at hand, it may well be that in focusing on standard algebraic symbolism as the starting point to have the students thinking algebraically – whether in its alphanumeric form, in its disguised form as a ‘transitional language’ (see Section 1 above) or even in a related disguised manner of using some hands-on manipulatives (see a discussion of this in Radford and Grenier 1996) – we have narrowed down the possibilities for the students to converse with the general. By excluding other forms of ‘texts’ based on different figures of speech, it would seem that we have silenced some voices. The variety of algebras that we find throughout the history of mathematics (see e.g. Radford, 2001) and whose traces and sediments appear in one way or another in the phylogenetically constituted practice of contemporary algebra in school, on the one hand, and the individual intellectual trends of our contemporary students, on the other, suggest a variety of forms in which to semioticize the general. However, the celebration of cultural differences and the diversity of modes of thinking risk leading us to a sad picture of unrelated islands. In our conception, algebraic languages (as natural languages do) might allow the students to interact between themselves and, in doing so, to elaborate new mathematical meanings and understandings.

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NOTES

[1] Here, we intend 'signs' in a broad sense; they may be word-signs of written and oral natural language, letters (e.g. 'x', 'n') or even artefacts.

[2] In Semiotics some authors make a distinction between signs and symbols. For instance, in Peircean semiotics, a symbol is understood as a specific kind of sign bearing (in contrast to the icon or the index) an arbitrary relation to its object (see Peirce, 1955, p. 102; Parmentier, 1994, p. 6; Eco, 1988, p. 76). Vygotsky distinguished between signs and marks, reserving in some cases the first term for marks having been provided with meaning by the individual and often taking signs and symbols indistinctly (see e.g. Vygotsky, 1997, p. 129, 139, 141 *passim*). In this paper, we do not make any differentiation and take signs and symbols as synonymous.

[3] For a detailed critique of didactic approaches to algebra centered on the study of the syntactic aspect of the algebraic language see Nemirovsky (1994).

[4] Vygotsky stressed many times this altering function of signs and tools in the psychology of individuals. In one passage, he wrote: 'By being included in the process of behavior, the psychological tool alters the entire flow and structure of mental functions. It does this by determining the structure of a new instrumental act just as a technical tool alters the process of a natural adaptation by determining the form of labor operations.' (Vygotsky, 1981, p. 137).

[5] We take here activity in a neo-A.N. Leontievan sense (Leontiev, 1984), and we consider the relation sign/activity after A.A. Leontiev's elaboration (A.A. Leontiev, 1981).

[6] This conception of signs and other concrete artefacts leads to several practical implications. Since they are not conceived as sole accessories of the mind, educational practices (mostly at the primary level) no longer need to be oriented to the rapid abandonment of manipulatives and concrete artefacts to focus solely on mental or less concrete-based practices. This, naturally, does not mean that we will be confined to walk around with a calculator in our pocket or with an abacus under our arm. What this means is that a different attitude towards signs and artefacts is a result of the theoretical position we are advocating.

[7] Unfortunately, the isolation of the subject, the object and the act of symbolizing from the context encompassing the individuals' semiotic activity often leads to reified analysis of language and discourse. In doing so, the individuals and their relations are sunk into oblivion and language and discourse become endowed with a kind of supernatural creative power. As Mikhailov noticed, 'When formally analysed, language hangs in the air, as it were, is deprived of its roots and becomes an independent object of research; the individual, whose tongue makes language a living thing, is pushed into the background and forgotten.' (Mikhailov, 1980, p. 221).

[8] A critique of this misleading and oversimplified interpretation of knowledge acquisition often ascribed to cultural approaches can be found in Waschescio, 1998.

[9] We argued elsewhere (Radford, 1998a), that the minimal and peripheral epistemological role, that up to recent times has been given to speech, has a long-standing tradition in Western thought, where speech was considered (by Frege, 1971 among others) as something obstructing the space between the mental concept and its representation. Since ideas and concepts were conceived as a faculty of the solitaire mind, speech was considered too noisy, ambiguous and illogical to reflect and express the – supposedly *aphonic* – nature of the concept and its silent paradise from where voices and utterances were banished since the beginning. In the tradition of Western thought, the idea, the concept, the logos, are mute, voiceless, and can, at best, be well dressed by signs if suitable dressings are found: this is *grosso modo* the story of the quest of the *Universal Language* from the 17$^{\text{th}}$ century onwards.

[10] The teacher is in charge of the instruction in the classroom.

[11] Although the reason for the quizzes is primarily related to our need of keeping track of the mathematics classroom achievement (as typified by the curriculum) in order to gain feedback for the design of the next teaching units, the quizzes bring forth complementary information to understand the problems related to meaning making and sign use.

[12] Those places where some lines are omitted will be indicated by '/[...]'. Although the duration of a pause in speech may obey different reasons, to give an idea of the cadence of the dialogue we will use '...' to indicate a pause of 3 seconds or more, and we will use ',' or '.' to indicate a pause of less than 3 seconds.

[13] Metaphors have been recognized as important elements in the elaboration of new concepts (see e.g. Sfard, 1994). A recent account is given in Presmeg (1998). See also Núñez 2000, Núñez et al., 1999.

[14] There was a Greek mathematician by the name of Hypsikles who wrote a treatise called *Anaphorikos* (Manitius, 1888). This treatise – which had some influence on Diophantus' work (see Radford, 1995) – has a few propositions about arithmetical progressions, and what is so particular about it is that Hypsikles proved them in a deductive manner: *Anaphorikos*, in fact, contains the first deductive known proofs about arithmetical progressions and polygonal numbers. Hypsikles, in the abstract line-diagram upon which the deductive proof was based, inserted numerical examples that may be considered redundant or superfluous. Yet, the recourse to concrete numbers may also be taken, as in the case of our students, as an evidence of Hypsikles' efforts to talk about the general.

[15] Algebraic expressions were among the usual examples that Peirce gave of icons (see Peirce, 1955, p. 104 ff).

[16] Notice that the emergence of mathematical signs in the late Western Middle-Ages and the Early Renaissance followed a similar course. For example, the sign for the square-root was preceded by the word 'root' which was first written *in extenso* and later abbreviated by a stylized form of its first letter 'r' (see Cajori, 1993). Notice, however, that we do not take this action in the students' process of symbolizing as a recapitulative instance of phylogenetic development. We take it as a token of the importance of indexes in semiotic strategies of objectification.

[17] Although there is no room in this article to provide explanations about the growth of Anik's understanding, let us notice that her improvement in sign use and their meanings may be related to her (direct and indirect) participation in the classroom general discussions held after the groups completed their co-operative work.

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