Students' Processes of Symbolizing in Algebra:

A Semiotic analysis of the production of signs in generalizing tasks

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> OK. You can say …you make ... OK you add the figure...oh my God, how do you say it [in algebraic symbols]… The figure plus the next figure? (Annik, Grade 8 Student)

Abstract*: In this paper we present some results concerning the students' production and use of signs in the elaboration of the general term of a pattern. Considering the students' production of signs as a process embedded in the activities that the signs mediate, the investigation reported here focuses on a semiotic analysis of the students' strategies seen as a set of goal-oriented heuristic actions displayed by the students in the attainment of the objective of the generalizing activities. The analysis was carried out in terms of a two dimensional grid whose purpose was to shed some light on two key elements in the mediating role of signs. The first one concerned the meanings with which signs were provided by the students. The second centred on the manner in which the students semiotically articulated the relation between the general and the particular. The results (conducted through an interpretative protocol analysis managed with the NUD·IST program for qualitative research) suggest that novice students tend to conceptualize signs as indexes (in Peirce's sense) having a range of specified indexical meanings supported by different views of the relation general-particular.*

1. Framework and Preliminary Remarks

In our ongoing longitudinal research program about students' processes of symbolizing in algebra[1], we are tracking a cohort of students for three years in order to understand their acquisition of the algebraic language. By students' processes of symbolizing we mean the ways students understand, produce and use signs. Our interest in investigating the students' processes of symbolizing in algebra is related to the need to better understand the difficulties that novice students usually encounter in mastering the algebraic language –difficulties systematically reported in literature since the pioneer studies of Davis (1975) and many other studies conducted in the 80's (e.g. Matz (1985) and Kieran (1989)) up to the more recent works (e.g. MacGregor & Stacey (1997) and Kirshner (in press)).

Our work is embedded in a theoretical perspective which puts forward the intimate epistemological link between signs and thinking as stressed in Vygotskian approaches to the mind. While, in most of the analytical and structural traditions in the philosophy of language, signs appear as *aiding things* to think, Vygotskian and some recent socio-cultural and anthropological approaches, in contrast, attribute to signs a constitutive epistemological role in that signs are seen as external cultural 'tools' *imbricated* in and *integrated* into the individual's conceptual functioning. Furthermore, the production of signs, according to our framework, is dialectically related to the activity (in Leontiev's sense) that the signs mediate. In this line of thought, we dealt, in a previous work (Radford 1999), with the different meanings with which

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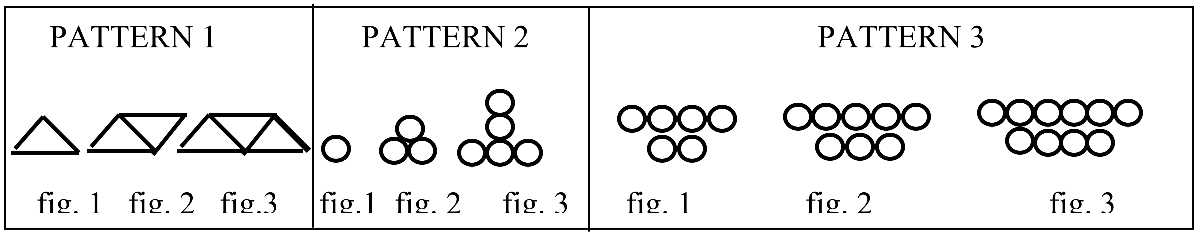
students provided signs in order to *understand* them in the algebraic context of generalization of patterns. In this paper we want to go further and to investigate in a more precise manner the students' *production* and *use* of signs in activities whose objective is the elaboration of a symbolic algebraic expression for the general term of a geometric-numeric pattern. Our approach takes into consideration a specific semiotic problem related to the construction of the general term of a pattern: a problem of *denotation* which can be stated as follows. Since the different elements of a pattern are characterized by the ordinal position they occupy in their well ordered sequence, the elaboration of the expression for the general term of a pattern requires that such a term be referred to a position which cannot be arithmetically expressed (at the modern algebraic level, this position, of course, is commonly denoted by 'n' or another single sign-letter of the alphabet or even a more complex assemblage like u_n). We shall call this particular denoting act 'indexing'. From a cognitive point of view, the indexing act brings forward several problems related to (1) the choice of the indexing signs, (2) the meaning of the indexing signs and (3) the way in which the conceptual-semiotic relation between the general and the particular is ideated. The problems posed by the indexing act in expressing generality are attested to by the history of mathematics, where we find different conceptions about the way the particular and the general are related. These conceptions, of course, are culturally framed as is the choice of the sign systems to express generality –sign systems of which the history shows us a rich variety, such as segments and non positional letters like η, λ, μ, χ in Antiquity (see Radford 1995, p. 47 ff.) or some 17^{th} and 18^{th} century AD additively based sign systems such as x', x'', x''', x'''', x''''' (see Radford 2000). The problems arising from the indexing act are also attested to by the difficulties that contemporary students encounter when trying to elaborate general expressions in patterns. The investigation of the semiotic nature of these students' difficulties is the purpose of this paper.

2. Methodology

The general methodology of our longitudinal research program was sketched in Radford (1999, in press). For the purposes of this paper let us mention that the students of the 4 classes that we are following up for three years worked on activities which included (among others) the three patterns given below. They worked in small groups (usually comprised of 2 or 3 students) and, at the end of the activities, the teacher conducted a collective discussion. Before asking the students to find an expression for figure n, they were asked to perform an arithmetical investigation (e.g. to find how many circles are in figure 10, figure 100).

The data mentioned in this article come from the first year of the field research (1998-99, when the students were in Grade 8, i.e. in their very first year of learning symbolic algebra). The data was processed following an interpretative, descriptive protocol analysis (details in Radford 2000) and was managed using the *Non-numerical Unstructured Data Indexing Searching and Theorizing* (Nud·Ist) program for qualitative research. The analysis of the production and use of signs in generalizing tasks was conducted in accordance to the *goal* of the activity as it was devised by the students. The *goals* (in Leontiev's sense 1984, p. 113 ff.) gave rise to three main categories of actions which oriented the students' heuristic processes in the attainment of the objective of the activity, namely, the construction of the general term of the pattern. We called the heuristic oriented actions 'strategies' and examined them in light of a two dimensional grid whose axes are related to the understanding of the mediating role of signs in the accomplishment of the activity, as pointed out in the framework discussed in Section 1. The first dimension concerned the meanings with which the students provided signs in the indexing act of denotation. The second dimension focused on the manner in which the students perceived the semiotic relation between the general and the particular. Section 3 deals with the description of some of the students' strategies. These strategies have already been mentioned in one way or another by other researchers in previous works on the learning of algebra. Our contribution is to be found at the level of the semiotic interpretation that we offer for the strategies –something which we do in Section 4– as well as at the level of the conclusions that we reach in Section 5.



3. Description of students' strategies

There are three important strategies followed by the students when they try to elaborate a symbolic algebraic expression for the general term of a geometric-numeric pattern. It is important to notice that in solving the problem, the students did not necessarily keep the same strategy. In the course of the activity the goal to reach the foreseen objective could change and so the actions and the whole heuristic process. The reasons leading to a change of strategy are usually related to the interaction between students and between students and teacher. Even though they are very important, they will not be considered here for the limitations of space. However, the reader may consult Radford (1999, 2000, in press).

Strategy 1: The first one is based on the idea of formula as a procedural mechanism in which letters (say 'n') are seen as the designation of a place to be taken by numbers. The usual general heuristic procedure is based on a kind of quasi-trial-and-error method which can sometimes become sophisticatedly controlled but whose success will depend on the complexity of the pattern.

<u>Example 1 (P38A1Ca)</u>: (The code of the examples refers to their Nud·ist identification only)

This point can be illustrated with reference to the classic toothpick Pattern 1. In one of our Grade 8 classroom groups, the students found the formula '$f \times 2 + 1$' (where 'f' stands for 'the number of the figure'). When asked by the teacher to explain why they added '1' to '$f \times 2$', the student who proposed the formula said: "Uh...because it works!" and proceeded to show it through many numerical examples.

<u>Example 2</u> (P18A2Co): Another example, concerning pattern 2, is the following.

1. Madeleine: But, no, that would really be the number of the figure times 2 minus 1. Because, look! 2 times 2, 4, minus 1, 3. One times ... (…) Yes. That would work! 1 times 2, 2, minus 1, 1!
2. Carole: 1 times ... All the time, times 2?
3. Paul: Yes.

Strategy 2: The second general strategy consists in finding a general expression on the basis of certain numerical facts occurring between some terms of the pattern. In Pattern 1 and 2, it was often noticed by our students that the number of ' basic elements' (i.e. circles or toothpicks) in the next figure was two more than in the previous figure. In modern notations, this corresponds to the recursive formula $u_{n+1} = 2 + u_n$. Here is an example:

Example 3 (P18B1Co):

1. Jessy: Look ... n plus two ... (points to a place on the page) This is n. This plus two equals this. This plus two equals this. This plus two equals this. It's n plus two.

The problem with Jessy's recursive formula, applied here to Pattern 2, is, as Michelle noticed, that the formula does not provide one with the total of elements in the figure:

2. Michelle: But, if you want … if you want like the figure 200 ? (…) But, you want the figure 200, then they tell you n plus two equals the figure 200.
3. Jessy: Yes, it's 198 + 2. ... You would say that the figure before it is 198 ...
4. Michelle: How do you know that?

The arithmetic experience led sometimes the students to observe other numerical facts. For instance, concerning Pattern 1, in some groups it was noticed that the total of toothpicks in a figure equals the number of the figure plus the number of the next figure. Here are two examples:

Example 4 (P38A1Ca):

1. Guy: (*interrupting*) one plus two, two plus three, three plus four, four plus five, ...
2. Joe: (*interrupting*) five plus six. Oh! (realizing that Guy's idea works) O.k. (inaudible)

Example 5 (P18B2Co):

1. Josh: It's always the next one. One plus two, two plus three (…), three plus four ….

Of course, in these two examples, the students do not state the noticed numerical regularity in a general verbal form. Their understanding and shaping of the general occur at a numerical level. As a matter of fact, the analysis of the protocols shows clearly that students tend to talk about the general through the particular. As we shall see in the next section, this is crucial to the students' ways of symbolizing and expressing generality in symbolic language. Let us now turn to the third strategy.

Strategy 3: This strategy is based on the shape of the figures in the pattern. The main idea is to count the basic elements in each of the structural parts of the pattern and to combine the partial totals into a kind of grand total. In Patterns 2 and 3, the procedure consists in finding the total of circles in each of the branches of a figure and then to add those totals. The next example is related to Pattern 2.

Example 6 (P18B2Co):

1. Judith: If it's the figure it'll always have the number … like if we say it's figure 12, you'll have 12 on the bottom and then you'll have one less on top vertically.

When applied correctly, this strategy usually leads to the expression n +(n-1) for figure n in Pattern 2 and (n+1) + (n+1+2) or (n+1) + (n+3) for figure n in Pattern 3. The reasons why the students do not go further and group similar terms is related to the meaning with which they provide signs, as we will see in the next section.

4. Semiotic analysis of the strategies

Strategy 1.The production and use of signs in the three strategies presented in the previous section are underlain by different meanings that students ascribe to signs. In Strategy 1 signs are understood as a place to be taken by numbers; that is, they appear as parts of chains of operations functioning as mere emplacements where concrete numbers come to be logged in order to produce a numerical result. From the point of view of denotation, the sign is understood as denoting the number of the figure (see Example 2, line 1). But the arithmetic operational context framing the formula in which signs appear makes signs play another role: that of marks indicating the result of operations. Thus, in the discussion of Example 2, when it came time to put the formula into symbols, the discussion revolves around whether 'n' also has to be included in the total. The students say:

Example 2 (continuation):

1'. Carole: "n", after this, bracket, n times 2, minus 1.
2'. Madeleine: Equals "n".
3'. Carole: You don't have to write equals "n". Do we?
4'. Madeleine: Yes. You have to write it.
5'. Carole: Just ... we don't need "n".
6'. Paul: You need a formula.
7'. Carole: OK. "n" bracket "n" times 2, minus 1. (And she writes $n=(n\times 2-1)$)

The syntax of the final formula sheds further light on the actual meaning of signs. Instead of writing "$2\times n-1$" as it would be more in tune with the canonical syntax of the algebraic language, the students write "$n\times 2-1$". Why? The reason is that the sign "n" in the expression "$n\times 2-1$" is an *index* (in Peirce's sense). That is, "n" is pointing to the verbal utterance "the number of the figure times 2 minus 1" in line 1 of Example 2 shown in the previous section. (The dialogue analyzed in Radford 1999 exhibits also this same phenomenon). Of course, the sign "n" indicating the *total* in the students' formula is also an index –although with a different indexical meaning. At any rate the common indexical nature of the two signs "n" guarantees their common appearance in the same symbolic expression.

As to the relation between the particular and the general, it is framed by the operational conception of the symbolic expression. As a result, the particular (v.gr. the prior arithmetical investigation of some concrete figures such as figure 10, figure 100) plays a little role (if any) in informing the form and structure of the algebraic symbolic expression for the general term of the pattern. Hence, when confronted with the question about how many circles in total figure 10 in Pattern 2 has, Carole suggested to find out the general formula first and then to apply it to figure 10 when she said: "If we figure out the formula first then we calculate it, that would be easier than just thinking." The relation particular/general is very restricted in that, on the one hand, the particular serves only to check the validity of the symbolic expression; the general, on the other hand, appears as making the economy of the analysis of particulars.

Strategy 2. In this strategy signs have a different semiotic function. They have to express numerical regularities involving two or more terms which need a semiotic articulation. The difficulties arising here may become very complex in terms of denotational requirements, as clearly illustrated in the next example:

Example 5 (continuation):

1'. Annik: OK. You can say … you make...OK you add the figure ...oh my God, how do you say it [in algebraic symbols]… the figure plus the next figure?

In Example 3, as we saw, Jessy's 'recursive formula' was stated verbally as: "… This plus two equals this …") and a symbolization was also provided: "It's n plus two". When the students came back to this recursive formula after failing to find a non-recursive, direct one, they continued:

Example 3 (continuation):

1'. Jessy: It's always n + 2.
2'. Michelle: Yes, Yes. Sure.
3'. Jessy: n + 2 equals figure 5.
4'. Michelle: n + 2 ...
5'. Jessy: Figure 4. It's like figure 4 + 2 equals figure 5.

Here Jessy proposes to symbolize such a numerical fact as 'n+2'. Thus, at an implicit level, the sign 'n' is seen as denoting the number of circles in a figure which remains unspecified. We see how the 'indexical problem' and the denoting process in which such a problem is embedded would require the differentiation of referents. It is important, in fact, to distinguish: (i) the figures, (ii) their position in the pattern and (iii) the number of circles that they have. Natural language equips the students with a whole arsenal of deictic terms that Jessy indeed exploits to his advantage in line 1 of Example 3:

Deictic terms

1. Jessy: (…) This plus two equals this.This plus two equals this.This plus two equals this …(…)

This is not possible within the realm of the sign system of algebra, which requires a clear differentiation of referents. The lack of such a differentiation is often accompanied by a common idea of sign-letters as conveying *indeterminacy*. This is made clearer in the following example:

Example 4 (continuation):

1'. Noemi : So, you want to have n plus the next number.
2'. Joe: (*writing the answer*) n plus n?

In this passage, as in many others, the students symbolize 'the next number' as 'n'. And in fact, for many students, all that is unknown is designated by 'n'. Thus, when the teacher went to see the work done by one of our small groups and tried to help them to simplify their symbolic expression by saying, "Then, n+n is equal to …?", the students promptly answered "n". However, in other instances, as we will see later, it was recognized that a different sign was required to symbolize 'the next number'. As for the relation general/particular in Strategy 2, we see that the particular (through the articulation of numerical facts) offers a powerful tool in the heuristic process. The particular is much more than the realm where to check the correctness of the symbolic general expression.

Strategy 3: In this strategy signs are used in intimate relation to the form and the parts of the figures of the pattern. The relation general/particular plays here a central role in that the referent is clearly emphasized (for instance, having recourse to a geometrical mean). Furthermore the relation general/particular is crucial to the modelling role that the particular will play in the construction of the symbolic algebraic expression. Indeed, the structure of symbolization of the general term of the pattern reflects the structure of the numerical actions of the students. And the particular is often taken as a *metaphor* for the general (for a detailed example of this see Radford 2000). The following example is related to Pattern 3.

Example 6 (continuation):

In this example, the students noticed that the top line always has two more circles than the bottom line. They first built an expression for the number of circles on the bottom line and then added two to it.

1'. Anik: n + 1 in brackets plus...
2'. Judith: Plus 2.
3'. Anik: Plus 1 at first. Look! You do this, then …
4'. Jeff: Yes. Then after this it's plus 2.
5'. Judith: In brackets.
6'. Anik: Yes. Plus 2. [The formula given is: $(n+1)+2=a$]

This example clearly shows how, in this strategy, actions precede symbolization and how the latter is but an expression of the former. The terms "at first" (line 3) and "then after" (line 4) order the temporal numerical sequence of actions at the symbolic level. The syntax of the expression is even dictated by the order of the actions. This is why the student in line 5 says that brackets have to be written. Curiously, in the process of symbolizing, the symbolization of the bottom line of the figure is sometimes left out. In fact, as it appeared during the collective classroom discussion of the activities, some students do not see the need for writing again 'n+1'. They see the formula as indicating a calculation process progressing from bottom to top in a cumulative way. Following with our discussion of the particular/general relation, we see that in this strategy the particular informs in a significant manner the construction of the general expression. This is why the way in which the particular is read somehow anticipates the advent of the general (Radford 1999). Consequently, in this strategy, the particular is not systematically called up to check the validity of the formula. Like in Strategies 1 and 2, in terms of denotation, the sign 'n' is also seen as an *index*. The sign 'n' is pointing to the bottom line of the figure. And, as the first sign 'n' in Example 2 (continuation) line 7', the sign 'a' here is pointing to the result. The whole symbolic expression '$(n+1)+2=a$' can be considered as an *icon* (again, in Peirce's sense) of the concrete figures. Indeed, within its own semiotic space, the symbolic expressions are 'reproducing' the shape of the figures of the pattern.

5. Summary and Conclusion

Our investigation into the students' ways of symbolizing was carried out in terms of the students' production and use of signs as required in activities concerning the algebraic generalization of geometric-numeric patterns. The students' heuristic oriented actions (or strategies) were examined in light of a two dimensional grid (meanings and the general/particular relation). Although these two dimensions cannot account for the whole range of phenomena required to investigate the students' ways of symbolizing (see e.g.

Radford 1999 and 2000 for an analysis of another important dimension related to the students' discourse), the results, nevertheless, shed a new light on the semiotics of generalization. When the three strategies were scanned with the aid of our grid, it appeared that these strategies entail different articulations between the general and the particular. Furthermore, it became apparent that in these strategies the students tend to use signs of a particular sort –*indexical signs*. However, the meaning of signs was different. Indeed, given that indexical signs can signify in a variety of forms, they may bear different indexical meanings. It is true that the fluent algebra user employs indexical signs too. The difference is that the fluent user, in contrast to the novice, can provide the indexical sign with non-indexical meanings. We saw, for instance, how impossible it was to successfully add "n+n" for a group of our students. The difficulty resides in that indexical signs cannot be added. As long as they are still pointing to their objects, one cannot collect them and merge them into a single new symbolic expression. As seen in our discussion of Example 6 (Continuation), the token occurrence of indexical signs unfolded in the realm of an experience sequentially framed in which the signs remained contextually anchored. The algebraic expression is seen as a mnemonic device reflecting the actual course of the flow of calculations. In this sense the algebraic expression functions as a 'performative comment' imbued with the subjectivity of the students' symbolic code which –given its still non-cultural accepted conventional status– requires a 'compiler' to decode it. The possibility to imbue the indexical signs with new meanings stands in need of the creation of new semiotic experiences taking into account these indexical signs. This semiotic experience resides, in part, in that the indexical signs will become the 'objects' of which one thinks, talks and writes. In other terms, they have to become part of a metasemiosis; they have to become part of a new language-game (in Wittgenstein's sense). As for patterns, the question of successfully playing the drama of algebra seems hence to be related, to some important extent, to the possibility of providing the indexical signs with non-indexical meanings.

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