Second Degree Equations in the Classroom: A Babylonian Approach*

Luis Radford
Université Laurentienne, Canada

and

Georges Guérette
Conseil de l'éducation de Sudbury, Canada

In this paper, we present a teaching sequence whose purpose is to lead the students to reinvent the formula that solves the general quadratic equation. Our teaching sequence is centered on the resolution of geometrical problems related to rectangles using an elegant and visual method developed by Babylonian scribes during the first half of the second millennium BCE. Our goal is achieved through a progressive itinerary which starts with the use of manipulatives and evolves through an investigative problem-solving process that combines both numerical and geometrical experiences. Instead of launching the students into the modern algebraic symbolism from the start—something that often discourages many of them— algebraic symbols are only introduced at the end, after the students have truly understood the geometric methods. The teaching sequence has been successfully undertaken in some high school classrooms.

1. The Babylonian Geometric Method

Before explaining the teaching sequence it is worthwhile to mention briefly some of the features of the Babylonian geometric method. The method to which we are referring was identified by J. Høyrup who called it *Naive Geometry.*[1] In order to show the method, let us discuss one of the simplest Babylonian problems, namely, problem 1 of a tablet preserved at the British Museum and known as BM 13901.

The statement of the problem, which seeks to find the length of the side of a square, is the following:

> The surface and the square-line I have accumulated: $3/4$.

As in most of the cases, the scribe states the problem using a very concise formulation. He is referring to the surface of a square, and the square-line means the side of the square. Thus, the problem is to find the side of the square, knowing that the sum of the area of the square and the side is equal to $3/4$. The method of solution is not fully explained in the text. Indeed, the text shows only a list of instructions concerning a sequence of calculations that allows one to get the answer.

The instructions, as they appear in the tablet, are the following:

> 1 the projection you put down. The half of 1 you break, $1/2$ and $1/2$ you make span [a rectangle, here a square], $1/4$ to $3/4$ you append: 1, make 1 equilateral. $1/2$ which you made span you tear out inside 1: $1/2$ the square-line. (Høyrup, 1986, p. 450)

Of course, the Babylonians should have had a method on which the numerical calculations were based. For some time it was believed that the Babylonians somehow knew our formula to solve second-degree equations. However, this interpretation has been abandoned because of the multiple intrinsic difficulties that it implies, one of them being the well known lack of algebraic symbols in Babylonian mathematical texts and the related impossibility for the Babylonians to handle complex symbolic calculations without an explicit symbolic language (details in Radford 1996 and Radford in print).

Based on a philological and textual analysis of the Babylonian texts, J. Høyrup suggested that the solution of problems (such as the preceding one) was underlain by a geometrical configuration upon which the oral explanation was based. In the case of the previous problem, the scribe thinks of an actual square (Fig. a). However, the side is not seen as a simple side (Fig. b) but as a side provided with a canonical projection that forms, along with the side, a rectangle (Fig. c). The duality of the concept of side is based on a metrological equality: the length of the side and the area of the rectangle that it forms along with its canonical projection have the same numerical value (see Høyrup 1990a). Keeping this in mind and coming back to problem 1, BM 13901, it appears that the quantity $3/4$ refers then to the total area of Fig. 1. Next, the scribe cuts the width 1 into two parts and transfers the right side to the bottom of the original square (see Fig. 2).

Now the scribe completes a big square by adding a small square whose side is $1/2$ (Fig. 3). The total area is the $3/4$ (that is, the area of the first figure) plus $1/4$ (that is, the area of the added small square). It gives 1. The side of the big square can now be calculated: that gives 1; now the scribe subtracts $1/2$ from 1, he gets $1/2$: this is the side of the original square.

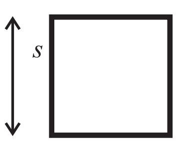


FIGURE a (square)

FIGURE b (side)
length = s

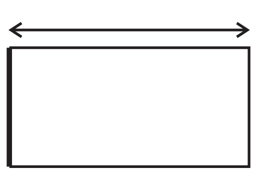


FIGURE c (side with canonical projection)
area = s

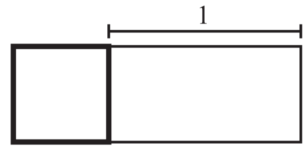


FIGURE 1

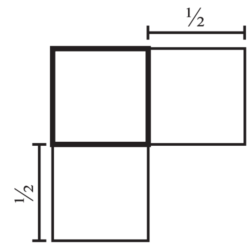


FIGURE 2

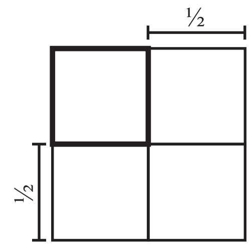


FIGURE 3

This is the same type of transformation that seems to be the basis of the resolution of many problems found in a medieval book, the *Liber Mensurationum* of Abû Bekr (probably ninth century), whose Arabic manuscript was lost and which we know of through a twelfth century translation by Gerard of Cremona (ed. Busard, 1968). In fact, many of these problems are formulated in terms similar to those of the *Naive Geometry.*

Let us consider an excerpt from one of the problems of the *Liber Mensurationum* (problem 41; Busard 1968, p. 95):

> And if someone tells you: add the shorter side and the area [of a rectangle] and the result was 54, and the shorter side plus 2 is equal to the longer side, what is each side?

As in the case of Babylonian texts, the steps of the resolution given in the *Liber Mensurationum* indicate the operations between the numbers that one has to follow.[2] In all likelihood, the calculations are underlain by a sequence of figures like figures 4 to 10 hereinafter. The initial rectangle is shown in Fig. 4. The shorter side, x, (placed at the right of the figure) is provided with a projection equal to 1 (Fig.5), so that the length of the side is equal to the area of the projected rectangle, as in the case of the Babylonian problem discussed above. Given that $y = x + 2$, the base 'y' (bottom of Fig. 5) can be divided into two segments 'x'

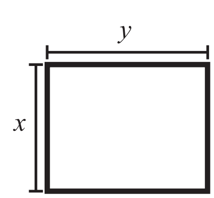


FIGURE 4

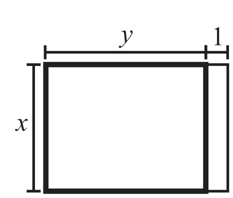


FIGURE 5

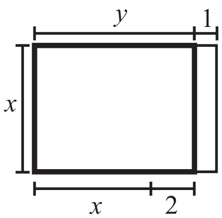


FIGURE 6

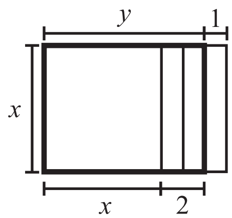
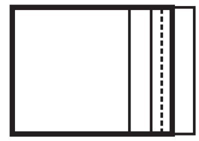


FIGURE 7



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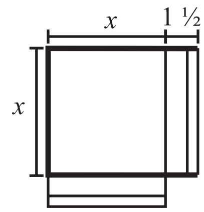


FIGURE 9

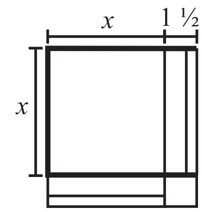


FIGURE 10

and ‘2’ (see Fig. 6). Now two small rectangles are placed inside the original rectangle, as shown in Fig. 7.

The next step is to divide into two the set of the three equal rectangles (Fig. 8); one of these parts (that is, a rectangle and a half) is placed at the bottom of the remaining figure. As a result of this transformation, we now have Fig. 9, which is *almost* a square.

The key idea in the resolution of these types of problems (and which appears in an explicit manner in Al-Khwarizmi's *Al-Jabr* (ed. Hughes 1986)) is to complete the current figure (Fig. 9) in order to get a square. The completion of the square (Fig. 10) is achieved then by adding in the right corner a small square whose area is equal to $\left(1\frac{1}{2}\right)^2 = 2\frac{1}{4}$. The final square then has an area equal to $54 + 2\frac{1}{4} = 56\frac{1}{4}$, so that its side is $\sqrt{54\frac{1}{4}} = 7\frac{1}{2}$. The shorter side, x, of the original rectangle is then equal to $7\frac{1}{2} - 1\frac{1}{2} = 6$, so the longer side is 8.

We are not going to discuss here the historical arguments that support the reconstruction of the procedures of resolution for problems such as the preceding one in terms of the *Naive Geometry* (see Høyrup 1986 or Høyrup 1990b). We shall limit ourselves to simply indicating that the explicit appearance of these procedures in Al-Khwarizmi's work leaves no doubt that these procedures were well-known in the ninth century in certain Arabic milieus.

2. The Teaching Sequence

In this section we shall present the teaching sequence that we have developed in order to introduce the students to second degree equations and that culminates with the reinvention of the formula to solve these equations. The sequence is divided into 5 parts (whose duration may vary according to the students' background).

For each part of the sequence:

(i) we give the indications of the different steps to follow in the classroom;

(ii) we include an item called *particular comments*, which, through concrete examples, intends to shed some light on the issues of the teaching sequence according to our classroom experience.

Part 1. The introduction to the *Naive Geometry*

In part 1, the students are presented with the following problem:

> What should the dimensions of a rectangle be whose semi-perimeter is 20 and whose area is 96 square units?

Working in cooperative groups, the students are asked to try to solve the problem using any method. After they complete the task, the teacher, returning to the geometrical context of the problem and using large cardboard figures on the blackboard, shows them the technique of *Naive Geometry*.

This can be done through the following explanation: If you take a square whose side is 10, then its area is 100 (Fig.11). One must therefore cut out 4 square units of the square whose side is 10 (Fig. 11) to obtain a figure whose area is 96. This can be achieved (and that is the key idea of the resolution) by cutting out of the big square a smaller square whose side is 2 (see Fig. 12). In order to obtain a rectangle one cuts the rectangle shown by the dotted line in Fig. 13 and places it vertically on the right (Fig. 14). The sought-after sides then measure 12 units and 8 units.

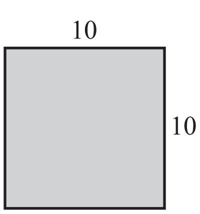


FIGURE 11

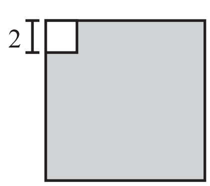


FIGURE 12

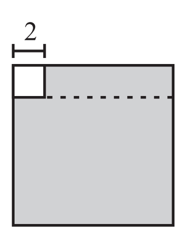


FIGURE 13

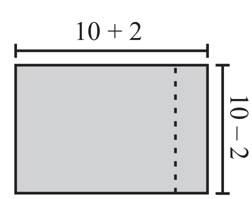


FIGURE 14

Once the technique is presented, the teacher gives the students other similar problems. In order to avoid a simple repetition, the parameters of the problem (i.e., the area and the semi-perimeter) may be chosen as follows: area of the rectangle $= 30$ and semiperimeter $= 12$. Problems like this are particular in that the area of the small square to be removed (Fig. 12) is not a perfect square. This led the students to reflect about the *Naive Geometry* technique on a deeper level.

In order to help students achieve a better understanding, the teacher asks them to bring a written description to class the next day outlining the steps to follow to solve this type of problem. They may be told that the written description or "message" should be clear enough to be understood by any student of another class of the same grade.

Particular comments.

(1) The idea of asking the students to solve the problem 1 using any method is simply to get them exploring the problem. As expected, usually, they use a trial-and-error method. Other students choose a rather numerical-geometric method; they choose a square of side equal to 10 (a solution motivated by the fact that the number 10 is half of the semi-perimeter 20). A less usual strategy is to take the square root of 96. In the last two strategies, when they try to justify their answer (sometimes at the request of the teacher), they realize that it is incorrect. The teacher may then ask for ideas about direct methods of solution (something that excludes trial-and-error methods).

(2) The geometrical resolution of this problem, a problem that can actually be found in a numerical formulation in Diophantus' *Arithmetica* (c. 250 CE) (Book 1, problem 27), is far from evident to the students. As we have quite often noticed, when we first show the *Naive Geometry* approach in the classroom, the visual seductive geometrical particularity of the resolution awakens a genuine interest among the students.

(3) In one of our sessions, when confronted with the problem of area $= 30$ and semi-perimeter $= 12$, one group of students started assuming, according to the technique, that the sides were each equal to 6 (which meets the requirement of semi- perimeter $= 12$). Given that the area of this square is equal to 36, they realized that they needed to take away 6 square units. In order to avoid irrationals, they cut out a rectangle whose sides were equal to 2 and 3. Then they realized that in doing so it is not possible to end with a rectangle, as required by the statement of the problem. Fig.15b shows the non-rectangular geometric object to which one is led when one takes away a 2×3 rectangle (Fig. 15a) instead of a square whose sides are equal to $\sqrt{6}$). Then, they became aware that a square of area equal to 6 has to be removed and that they had to take away a side of length equal to $\sqrt{6}$.

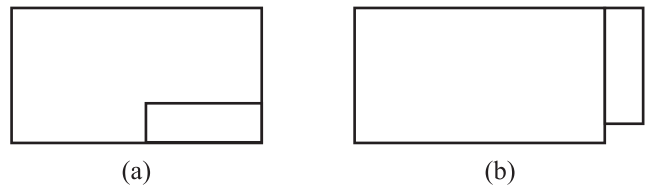


FIGURE 15

Part 2.

This part begins with a discussion of the messages containing all the steps required in the resolution of the problems seen in part 1. Working in cooperative groups, the students have to discuss and come to an agreement about the points which could cause a conflict or could give way to an improvement. When all the group members are in agreement, the teacher can choose one student of each cooperative group to present the work to the other groups. This allows certain students to better understand. Following this, the students are asked to *pose* problems themselves with the following restriction: the sides of the sought-after rectangle have to be expressed in whole numbers; then, as a second exercise, the sides of the sought-after rectangle do not have to be expressed in whole numbers. The students may even be asked to find fractional answers. A few of these problems would be used in the test at the end of the chapter.

Particular comments. We want to stress the fact that many students have some problems in writing the message in general terms, that is, without referring to particular numbers for the semi-perimeter and the area. Although they were not explicitly asked to write the message using letters (i.e., using an algebraic language), it was very hard for many of them to express, at a general level, the actions that they were able to accomplish when working with manipulatives (concrete rectangles on paper that they produced themselves using scissors) or drawings.

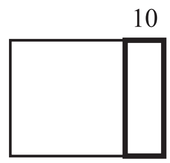
Despite the intrinsic difficulty of this task, it is important that students try to use the natural language to express their ideas. This will simplify the transition to the abstract symbolic algebraic language.

Part 3.

In part 3, the students are presented with a problem that requires a different use of the Naive Geometry technique.

The problem, inspired by that of Abû Bekr seen at the end of section 1, is the following:

> Problem 2: The length of a rectangle is 10 units. Its width is unknown. We place a square on one of the sides of the rectangle, as shown in the figure. Together, the two shapes have an area of 39 square units. What is the width of the rectangle?



The teacher asks the students to solve the problem using similar ideas as the ones used to solve problem 1. If students do not succeed in solving the problem by the *Naive Geometry* technique, the teacher may show the new problem-solving method as follows: Using large cardboard figures placed on the blackboard, the teacher cuts the initial rectangle vertically in two (Fig. 17), then takes one of the pieces and glues it to the base of the square (Fig. 18). Now the students notice that the new geometrical form is almost a square. The teacher then points out that the new form could be completed in order to make it a square. In order to do so, a small square, whose side is 5 (Fig. 19), has to be added. The small square has an area equal to 25. Thus the area of the new square (Fig. 19) is equal to $39 + 25 = 64$. Its side is then equal to 8. From Fig. 18 it follows that $x + 5 = 8$, which leads us to $x = 3$. Next, other similar problems are given to the students to solve in groups.

As in part 1, the students are asked to work on a written description or message of the steps to follow in order to solve this type of problem.

Particular Comments. The students soon realize that the problem-solving procedure used in problem 1 does not apply directly to problem 2. Here, the central idea (and it is important that the teacher emphasizes it) is the completion of a square, something that will be very important when the students work with algebraic symbols later.

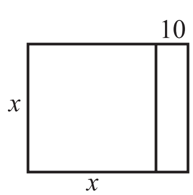


FIGURE 16

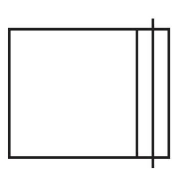


FIGURE 17

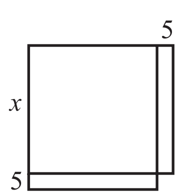


FIGURE 18

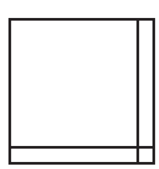


FIGURE 19

Part 4.

As in part 2, the students' written descriptions or messages are discussed. After this, the teacher asks them to pose some problems requiring a specific condition on the sides of the rectangle:

(i) the sides of the rectangle have to be expressed in whole numbers;
(ii) the sides of the rectangle have to be expressed in fractional numbers;
(iii) the sides of the rectangle have to be expressed in irrational numbers.

Part 5. Reinventing the formula.

In this part, the students will keep working on a problem of the same type as in parts 3 and 4. The difference is that *concrete numbers* are given neither for the base of the rectangle nor for the area that the two shapes cover together. The goal is to help students reinvent the formula that solves quadratic equations.

In order to do so, the teacher explains to the students that s/he is interested in finding a formula which will provide one with the answers to the problems seen in parts 3 and 4.[3] The teacher may suggest that they base their work on the written message produced in step 4 and to use letters instead of words. To facilitate the comparison of the students' formulas in a next step, the teacher may suggest using the letter "*b*" for the base of the rectangle and "*c*" for the area of both shapes (see Fig. 20). The equations are discussed in co-operative groups. The final equation is

$$x = \sqrt{c + \left(\frac{b}{2}\right)^2} - \frac{b}{2}.$$

The teacher may then proceed to translate the geometric problem into algebraic language: if the unknown side is 'x', then the area of the square is x^2 and that of the rectangle is bx; thus the sum of both areas is equal to c, that is: $x^2 + bx = c$. Now, in order to link equations to the formula, the teacher gives some concrete equations (like $x^2 + 8x = 9$,

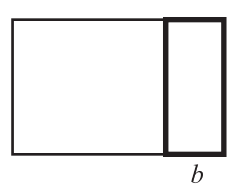


FIGURE 20

$x^2 + 15x = 75$) and asks the students to solve them using the formula.

The next step is to give the students the equation $ax^2 + bx = c$ and ask them to find the formula to solve this equation. The students might note that if this equation is divided by a (we suppose that $a \neq 0$), then we are led to the previous kind of equation. It suffices then to replace 'b' by 'b/a' and 'c' by 'c/a', in the previous formula, which gives the new formula

$$x = \sqrt{\frac{c}{a} + \left(\frac{b}{2a}\right)^2} - \frac{b}{2a}.$$

The last step is to consider the general equation $ax^2 + bx + c = 0$ and to find the formula that solves it. The formal link with the previous equation $ax^2 + bx = c$ is clear: we can rewrite this equation as $ax^2 + bx - c = 0$. Thus, in order to get the equation $ax^2 + bx + c = 0$ we need to replace 'c' by '$-c$' and to do the same in the formula.

When we replace 'c' by '$-c$' in the formula we obtain the general formula:

$$x = \sqrt{-c + \left(\frac{b}{2a}\right)^2} - \frac{b}{2a}.$$

Of course, this formula is equivalent to the well-known formula:

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a},$$

where in order to obtain all the numerical solutions one needs to consider the negative square root of $b^2 - 4ac$. This leads us to the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Particular comments.

(1) Usually, the students are able to provide the formula that solves the equation $x^2 + bx = c$ and to use it to solve concrete equations (as those mentioned above). That they can produce such a formula and realize the amount of work that the formula saves is appreciated very much by the students. This gives them a 'practical' sense of the formula.

(2) However, many students need some time in order to abandon the geometrical context and to limit themselves to the numerical use of the formula. Further, there are many students who prefer to keep thinking in terms of the *Naive Geometry* technique. The geometrical versus numerical preference may be caused by the specific students' own kind of rationality (something that is referred to in the educational field by the unfelicitous expression "styles of learning", an expression that hides more things than it explains!). Some students have the impression that they no longer understand if they merely use a formula. *Understanding*, for many of them, does not seem to mean simply 'being able to do something'.

(3) Most of the students are able to find the formula which solves the equation $ax^2 + bx = c$. However, some students may experience some difficulties. The main problem is that here, as in the subsequent steps, the geometrical context is being progressively replaced by a symbolic one.

(4) To end these comments, we want to stress the fact that our approach cannot avoid or conceal the problems that are specific to the mastering and understanding of algebraic symbols (see section 4 below). Our approach aims to provide a useful context to help the students develop a meaning for symbols. It is worthwhile to mention that the use of manipulatives and geometric techniques in order to derive the formula were appreciated by our high-school students. A girl, for instance, said: "I better understand with the drawings, I find this a lot more interesting and fun than the other mathematics."

3. About the duration of the teaching sequence

The teaching sequence that we discussed here may vary in time depending on the background of the students and their familiarity with classroom research activities. In one of the first times that we undertook it, we allowed a period of 80 minutes to each step. However, it is possible to reduce the time and the steps of the sequence. A variant of the sequence, that we undertook in an advanced mathematics high school course, consisted of steps 3 to 5. This can be done in two periods of 80 minutes each.

4. A concluding (theoretical) remark about the use of symbols in mathematics

The passage from numbers to letters does not consist of a simple transcription, as we have noted. In fact, the symbol must, in part 5, summarize the numerical and geometrical experiences developed in parts 3 and 4. The encapsulation of these experiences includes a stage of generalization and of reorganization of the actions which opens up on a much more ample description of mathematical objects.

Of course, the new semiotic category (that is, the category in which the algebraic symbolism is embedded) offers new challenges to the students (see the students' reinvention of the formula shown in Fig. 21) as it did for past mathematicians (see Radford, in print). For instance, operations with symbols need to be provided with new meanings. In this sense abstraction does not seem to proceed to a *detachment* of meanings or to more "general" ideas. Indeed, contrary to a general interpretation, abstraction does not mean to take away some features of a given object but to *add* new ones and to be able to focus our attention on the features required by the context. This was suggested by the transference from geometric to syntactic algebraic symbolism and vice-versa that students showed when solving second-degree equations at the end of the sequence (for example, $2x^2 + 12x - 64 = 0$), after having reinvented the formula.

These considerations lead us to the following intriguing idea: abstraction is a contextually based operation of the mind.

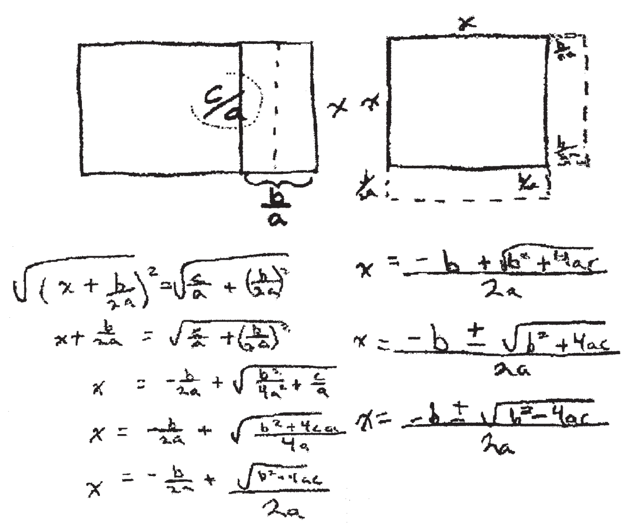


FIGURE 21

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Endnotes

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[1] His main work on Naive Geometry is (Høyrup 1990b).

[2] The solution is given in the Liber Mensurationum as follows:

> The way to find this is that you add two [to one], such that you have 3. Now you take half which is one and a half and multiply that by itself so you get two and a quarter. So, add 54 to this and you get 56 and a quarter; take the root and subtract 1 and a half; you are left with 6 and that is the smaller side; add to it 2 and you will have the longer side, that is, 8. However, there is a method to find this according to the people of the al-gabr. . .

[3] The students are already familiar with the concept of formula: not only have they seen formulas in mathematics, e.g., the formulas for the areas of regular geometrical figures, but in the sciences as well.