FACTUAL, CONTEXTUAL AND SYMBOLIC GENERALIZATIONS IN ALGEBRA[1]

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Abstract: *The purpose of this paper is to investigate, from a socio-cultural semiotic perspective, novice students' pre-symbolic and symbolic types of generalization of patterns. The investigation is carried out in terms of the semiotic (linguistic and non linguistic) means of objectification that Grade 8 students display in the attainment of the goals of generalizing mathematical tasks. The results suggest that while rhythm and movement, as well as differentiated ostensive gestures (e.g. 'grotesque' and 'refined' pointing), play a central role in pre-symbolic generalizations, symbolic-algebraic generalizations require a desubjectification process ensuring the desembodiment of spatial-temporal embodied mathematical experience. In order to deepen our understanding of the cognitive and semiotic requirements underlying pre- and symbolic generalizations, in the last part of the paper, I discuss the desubjectification process in terms of the relation between the object of knowledge and the through-sign-knowing-subject.*

1. INTRODUCTION AND FRAMEWORK

In a certain sense, generalization is one of the more natural human semiotic processes. As John Mason remarked many years ago, in one of the sessions of the 20th PME Conference Algebra Working Group, if we were to communicate without being able to make generalizations, we would be restricted to be pointing to objects around us. Any word, in fact, is the result of a generalization: it applies to a range of objects (not necessarily present) and can be used in a variety of situations. Semiosis, as it is intended here (that is, as the use of words and other signs in human activity), allows one to go beyond pointing. Within semiosis (and only within semiosis), can objects be objectified in a process that goes from the use of signs (marks, names and the like) as pointers of attention to more and more complex presentation and re-presentation systems involving new signs, meanings and layers of generalization.

In this paper, I want to pursue my investigation of the students' processes of generalizing by looking into the way students deploy and mobilize signs (words, letters, etc.) to accomplish mathematical generalizations. In (Radford 1999) I focused on the way novice students, interacting with their teacher, underwent a process of dynamic and differentiated understanding allowing them to achieve the elaboration of a *meaning* for the general term of a pattern. In (Radford 2000) the analysis was brought further and several algebraic generalizing strategies were examined. This analysis was carried out in terms of the various meanings with which signs were endowed by the students and the semiotic role that students ascribed to signs as a way to convey relations between the particular and the general. One of the reported results was the identification of the nature of the signs that the students tend to use in the elaboration of the first algebraic formulas: it turned out that these signs appear genetically related to the arithmetical concrete actions and to the objectification of these actions in speech. More specifically, novice students often use algebraic symbols as *marks* or *abbreviations* of key words belonging to a *discursive* non-symbolic semiotic layer. Thus, the students' symbolic expression $n \times 2 + 2$ mirrors the utterance "The term times two plus two" previously produced

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during the students' discursive activity (Radford 1999, p. 95). Following Peirce's terminology, I suggested that the students' first algebraic signs were *indexical* in nature, inasmuch as they stand for their objects in such a way that, like pointers, they appear as *indicating* the place of the objects to which they refer.

Given the strong genetic connection between algebraic generalizations and generalizations achieved in previous discursive layers of mathematical activity, it seems then, that the investigation of the semiotic modes of functioning of the latter needs to be pursued further if we want to envisage some pedagogical actions to promote new meanings for algebraic signs in the classroom. In this line of thought, the purpose of this article is to offer an exploratory investigation of pre-symbolic types of generalization in patterns and to contrast them with the algebraic symbolic ones.

2. METHODOLOGY

To do so, I will interweave theoretical reflections that draw from Bakhtin's theory of speech (Bakhtin 1986) and Voloshinov's philosophy of language (Voloshinov 1973) with relevant passages coming from my classroom-based research (more details about the methodology can be found in Radford in press). I will present excerpts of the discussions held by one of the Grade 8 students' small-groups (the students will be identified as Josh, Anik and Judith) and I will make oblique reference to the work of other small-groups. The data mentioned here involve students in their very first contact with symbolic algebra and relate to a classroom mathematical activity designed in collaboration with the teachers to immerse students into the social practice of algebraic generalization. I shall focus here on one of the activities based on the classic triangle toothpick pattern (see *Table 1*). The activity included several tasks, among them the following: (a) to find the number of toothpicks required to make figure number 5 and figure number 25 (b) to explain how to find the number of toothpicks required to make any given figure and (c) to write a mathematical formula to calculate the number of toothpicks required to make figure number 'n'.

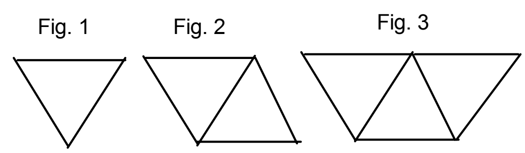

Table 1 . Toothpick pattern

3. RESULTS AND DISCUSSION

3.1 Factual Generalizations

In this episode, after finding out the number of toothpicks in figure 5, the students turned to the next question that asked to find the number of toothpicks in figure 25. Josh notices the following pattern:

1. Josh: It's always the next. Look! (*and pointing to the figures he says*) 1 + 2, 2+ 3 […]
2. Anik: So, 25 plus 26...

This led them to write the answer as 25+26=51.

As evidenced by this passage, the students did not have much trouble calculating the number of toothpicks in the concrete figure 25. What is more important, they did so *not* by counting the number of toothpicks, figure after figure up to figure 25, but by a process of *generalization*. The generalization thus achieved is what I want to call a *factual generalization*, that is, a generalization of numerical actions in the form of an *operational scheme* (in a neo-

Piagetian sense) that remains bound to the numerical level, nevertheless allowing the students to virtually tackle any *particular* case successfully. Their objectification takes the form of a process of *perceptual semiosis*, i.e. a process relying on a use of signs dialectically entangled with the way that concrete objects become perceived by the individuals. In this process, the mathematical structure of the pattern is revealed and ostensively asserted by linguistic key terms in the students' utterances. This is what Josh does in using the term 'the next' –a term reflecting the perceived ordered position of objects in the space. Another key term is the presence of adverbs like "always" (line 1). As noticed elsewhere (Radford, in press), these adverbs underpin the *generative functions of language*, that is, the functions that make it possible to *describe* procedures and actions that potentially can be carried out reiteratively.

The semiotic means to objectify factual generalizations are varied. In another small-group, one of the students sums up her group discussion by saying: "O.K. Anyways, Figure 1 is plus 2. Figure 2 is plus 3. Figure 3 is plus 4. Figure 4 is plus 5", and she points to the figures on the paper as she utters the sentence.

Here, the objectification is accomplished in a different manner. In this case, we do not find adverbs and spatial-positional terms. Actually, to obtain a similar generalizing effect, the students rely on the *rhythm* of the utterance, the *movement* during the course of the undertaken numerical actions and the ostensive correspondence between pronounced words and written signs. Rhythm and movement here play the role of the adverb "always".

Although rhythm and movement are also present in Josh's utterance ("Look! 1+2, 2+3") we would say that, in the second group, rhythm and movement create a cadence that, to some extent, dispenses the students from using other explicit semiotic linguistic means of objectification and also provide room for a type of social understanding based on a great deal of implicit agreements and mutual comprehension. We saw how the students understood Josh's brief utterance and agreed upon the numeric actions to be performed. In the second group, the students understood that "Figure 1 is plus 2" means "the total number of toothpicks in Figure 1 is equal to 1 plus 2", etc. However, factual generalizations, without having or being objectified by more specific linguistic terms or specialized symbols, cannot gain a more general status – they remain context-bound.

To further objectify the factual generalization through language is not an easy task, as we shall see in the next subsection.

3.2 Contextual Generalizations

The next task of the mathematical activity required the students to write an *explanation* of how to calculate the number of toothpicks for *any* given although non-specific figure. The characteristics of the required explanation in the mathematical activity introduced two new elements– a social-communicative one and a mathematical one.

The social-communicative element: In this task, the social aspect of understanding was shifted. Indeed, the explanation presupposes an addressee who is tacitly thought of as being absent from the actual scene in which the students' small-group activity unfolds. Implicit and mutual agreements of face-to-face interaction had to be replaced by objective elements of social understanding demanding a deeper degree of clarity in the communication.

The mathematical element: In addition to the social-communicative element, a new abstract object has been introduced into the discourse: the question, in fact, asks for *any* although non-specific figure. The two new aforementioned elements led the students to move into another layer of discourse (for some specific difficulties that students usually encounter in understanding this level of generality see Radford 1999). As in the previous episode, we shall present here excerpts of the students' dialogue. After a relatively long period of discussions and arguments, the students arrived at the following formulation:

1. Anik: Yes. Yes. OK. You add the figure plus the next figure … No. Plus the … […]
2. Anik: (*she writes as she says*) You add the first figure …
3. Josh: (interrupting and completing Anik's utterance says) … [to] the second figure.

Here, particular cases have been displaced and put in abeyance. Rhythm and ostensive gestures have also been excluded. What then are the semiotic mechanisms of objectification that the students display? And what are the epistemological and conceptual consequences?
First, notice the insertion of the addressee through the personal pronoun "You" (see line 1). Second, the addressee becomes interwoven with the new mathematical object: the addressee will indeed perform an *action* ("You *add* …") not on concrete numbers but on *abstract objects* ("You add *the figure* …). Abstract objects hence not only become abstract objects *per se* but become related to the actions required by the task and to the subject performing the actions. Third, the emergent abstract objects are objectified here by expressions like 'the figure', 'the next figure'. Such terms indicate a contrast with their surrounding; they have this kind of semiotic power to fix the students' attention (in the sense explained in our Framework: see Section 1). Through them, the students provide themselves with the capacity to achieve a fixity of reference much in the same way as 'deictic' or 'demonstrative terms' like 'that' and 'this' do in speech (see a clear example in the protocol analysis given in Radford 2000, p. 86).
We see then, that in terms of objectification, instead of a grotesque pointing, the abstract object appears as being objectified through a refined term pointing to a non-materially present concrete object through a discursive move that makes the structure of relevant events visible thereby creating a new perceptual field.

As a consequence of this linguistic objectifying process based on a refined but still ostensive way of functioning, the abstract objects are contextually conceptualized in reference to the particularities of the concrete mathematical objects. The latter stamp characteristics such as the spatial position of the sequence and a temporal sequencing action on the former, as clearly indicated in the utterance "You add the figure *and the next* figure", an expression that reveals indeed *tense* and *spatial aspects of contiguity*. The abstract objects are hence abstract while bearing at the same time contextual and situated features that reveal their very genetic origin. Their genesis also relates them to the individual who performs the actions on them. Because of the specific mode of objectification, subject and object bear an almost invisible but extremely powerful contextual dimension that allows the subject to perspectively see the emergent mathematical object.

All in all, without using letters and capitalizing on factual generalizations (which function as a guiding structure), the students hence succeeded in objectifying an operational scheme that acts upon abstract —although contextually situated— objects and indicates mathematical operations with them, ensuring thus the attainment of a new level of generality. These objects, belonging to a non-symbolic language, are not genuine mathematical objects in the traditional sense of the word. However, these objects abound in classroom discourses, where they become

part of the ontogenetic process of construction of the latter. This is one of the reasons to pay careful attention to their genesis and their functioning.
Let us call these types of generalizations, performed on conceptual spatial-temporal situated objects, *contextual generalizations.*

Contextual generalizations differ from algebraic generalizations on two important related counts. First, algebraic generalizations involve objects that do not have spatial-temporal characteristics. Algebraic objects are unsituated and atemporal. Second, in algebraic generalizations the individual does not have access to a perspectival view of the objects. As Bertrand Russell noticed, in the world of mathematics (and of pure physics), space and time are seen impartially "as God might be supposed to view it". And to emphasize the non-subjective character of space and time in mathematical descriptions, he then added that, in such descriptions, "there is not, as in perception, a region which is specially warm and intimate and bright, surrounded in all directions by gradually growing darkness." (Russell 1976, p. 108).

How then will the students proceed to the des-embodiment of their spatial-temporal embodied situated experience? How are they to produce the voiceless symbolic algebraic expressions? This is an extremely complex problem impossible to exhaust in the few remaining pages. I will, however, focus on one of the elements of the des-embodiment of spatial-temporal embodied experience, namely, a cognitive/semiotic dimension involving what I want to term the subject's *desubjectification* process –a process that stresses changes in the relation between the object of knowledge and the through-sign-knowing-subject.

3.3 Symbolic generalization

3.3.1 Bypassing the 'positioning problem'

In the next passage, the students did not symbolize the factual generalization based on the pattern "the figure plus the next figure" that we discussed in subsection 3.2. Actually, they worked out a different algebraic symbolization, as shown in the following excerpt:

1. Josh: It would be n + n...
2. Annie: n + ... OK. Wait a minute! ... n ...
3. Judith: Yes. n plus ... yeah it's n .... […] plus n plus 1.
4. Annie: Yes! n + n+1! (*that is, (n+n) + 1, as it will become clear later*) […]
5. Judith: Yes. Because, look! Look! ...
6. Annie: Your first figure is 'n' right? Plus you have n because it's the same number...
7. Judith: Because, look! Look! 4 + 4 = 8 + 1.
8. Josh: n plus n plus 1.
9. Annie: Bracket plus 1. (*they write* '$(n+n)+1$')
10. Judith: OK. Let's try it. Example... [*Josh says: 4 + 4 = 8 and Judith adds: 4 + 4 = 8 + 1 = 9*].

There is an aspect of the desubjectification process in which the students have succeeded so far, namely, the insertion of a speech genre based on the *impersonal voice*. This is evidenced by the students' utterances produced in lines 8 and 9. Thus, "Your first figure" in line 6 becomes "n" in lines 8 and 9. Furthermore, in contrast to the subjective utterance in line 6, lines 8 and 9 no longer make any allusion to an individual owning or acting on the figures. And with this, the traces of subjectivity start fading in a process where personal voices (e.g. "*I* add", "*you* put*"*) and the general deictic objects (e.g. "*this figure*"), underpinning the previous mathematical experience, have to shift to the background thereby providing room for the emergence of objective scientific and mathematical discourse.

But there were other aspects of the desubjectification process that proved to be more difficult to confront. To understand this, we have to raise the following question:

> *Why did the students not symbolize the generalizing strategy based on 'the figure plus the next figure' that they objectified before?*

As we shall see later, when we turn to the teacher's intervention, the change in strategy is related to the students' difficulty in symbolizing 'the next figure', something that requires finding a way to forge a symbolic link between the figure and the next figure and their corresponding ranks. This problem, previously referred to as 'the positioning problem' (Radford in press), results from the dramatic change in the mode of denotation that the disembodied algebraic language brings with it, caused by the exclusion of linguistic terms conveying spatial characteristics (e.g. 'the next') and their links with the now vanishing acting individual ("I", "You", etc.). As such, the 'positioning problem' is part of the desubjectification process that the mastering of the algebraic language requires and its presence here is a token of the difficulties that the students encountered engaging in this desubjectification process.

3.3.2 The teacher's intervention

When the teacher came to see the students' work, she noticed the discrepancy between the students' explanation (written in the previous task) and their current algebraic expression. She decided to further immerse the students into the objectifying process by commenting that the symbolic expression did not say the same thing as their explanation in natural language so she asked if they could provide a formula that would say the same thing. Josh continued:

1. Josh: That would be like n + a or something else, n + n or something else.
2. Anik: Well [no] because "a" could be any figure […] You can't add your 9 plus your … like ... […] You know, whatever you want it has to be your next [figure].

When the students reached an impasse, the teacher intervened again: Teacher: "If the figure I have here is 'n', which one comes next?" Then Josh, thinking of the letter in the alphabet that comes after n, says: "o".

The teacher's utterance shows how her attempt to help the students overcome the 'positioning problem' is underpinned by the spatial-temporal dimension of the general objects alluded to in the previous section (e.g. the figures are dynamically conceived of as coming one after the other). It is an open research question whether or not the mathematical meanings required to understand the denoting actions underlying the 'positioning problem' need to be imported (at least to some extent and probably within some variants) from previous non-symbolic contextual semiotic activities, as the teacher did here. If meaning is not seen as living in self-contained systems, the answer would be yes. At any rate, the teacher's intervention helped to refine Josh's understanding and to align it with the one required in the social practice of algebra. Finally, after reworking the case of figure 5, the students noticed that 6, that is, the number of the figure that 'comes next', can be written as 5+1, which was then reinterpreted as 'n+1'. In an attempt to recapitulate the discussion, the teacher asked:

1. Teacher: This would be …? (*referring to the expression '(n+1)' that the students had previously written on their page*)

2. Anik: It's the next [figure]!
3. Teacher: (*approvingly*) Ah!
4. Anik: OK! There, now. I understand what it is I'm doing.
5. Judith: OK.
6. Anik: You put your 'n', 'n' is your figure, right?
7. Judith: Yes.
8. Anik: OK. So, what we can do is n equals the figure … […] n + 1 equals the next figure, right?
9. Judith: Right. (*Anik writes the answer* $(n+1)+n$).

The teacher's intervention made it possible to overcome the positioning problem. It does not mean, however, that the students definitely secured new modes of denotation. The logic of signification behind the algebraic language requires a deeper engagement in the process of desubjectification. If subjective voices are no longer on the surface of the new students' mathematical discourse genre (see line 8 in the last dialogue), they are not far from it either (see line 6). Furthermore, the relation between the acting subject and the object upon which s/he acts is still perspectival in nature. It has not reached God's unperspectival view –to borrow Russell's metaphor. This is why students insist so tenaciously that brackets have to be written, as in line 9 here and in line 9 subsection 3.3.1. This is why the expression reached here, that is, $(n+1)+n$ and the expression $(n+n)+1$, reached in the beginning of section 3.3 are seen as different by the students –the reason being that they refer to two different actions.

The relation between $(n+1)+n$ and $(n+n)+1$ leads us to the relation between the signified object and its signifier. This was exactly the question that Frege asked in his article *On Sense and Denotation* (Frege 1971). Within Frege's semiotics, the alluded symbolic expressions are denoting the same mathematical object and the difference between signifiers account for differences in the modes of denotation and their respective senses. And he took sense as one of the ingredients of meaning, actually the only one related to the truly logical or mathematical aspect of the object to which the symbolic expressions refer. What the students' dialogues suggest is that to reach desubjectification and to end up with the objective kernel of the algebraic generalization, meaning has to be disembodied and become thus pure mathematical sense.

4. Synthesis and Concluding Remarks

In this article, we identified three types of generalizations related to geometric-numeric patterns. These generalizations appear as operational schemes relying on different semiotic means of objectification. While factual generalizations remain bound to a numerical level and their objectification is based on a process of *perceptual semiosis* stressing the patterned effect through adverbs of generative actions (e.g. 'always') or through articulated semiotic devices like *rhythm* and *movement*, contextual generalizations take, as their arguments, general non-specific numeric objects. These proto-mathematical objects displayed on a still non fully mathematized layer of discourse, are objectified through linguistic, non symbolic terms e.g. 'the figure', 'the next figure'. In doing so, in the course of a discursive practice, the students achieve a fixation of attention and extract from the undifferentiated horizon of objects certain elements that make apparent new objects that are beyond direct perception (indeed, the term 'the figure' *is not* 'figure 1' or 'figure 2' or figure 3, i.e. any of the figures *shown* on the activity page). Yet, the students' type of denotation is one that conveys the embodiment of the mathematical experience. It provides the students with a perspectival view of the emergent general objects. As a result, the proto-mathematical objects bear a very important characteristic: they remain *contextual* objects because of their spatial-temporal mode of being. They are abstract deictic objects.

The semiotic means of objectification underpinning these types of generalization shed some light on a question that has been tormenting me for the last couple of years. The question is related to the novice students' meaning of signs in algebraic generalization of patterns. As I noticed in the Introduction, and the phenomenon was again visible in the episodes seen in this paper, the students' signs in their first algebraic expressions bear the characteristics of associative indexes whose primary function is that of an abbreviation. The analysis presented here suggests that in this semiotic operation, the students succeed in accomplishing the devoicing of subjectivity. The suspension of subjectivity (related to objectivity) was recognized by Kant in his *Critique of Pure Reason* as one of the two conditions for knowledge. The second one that he contemplated concerned the exclusion of time (that Kant related to logical necessity). It is in this regard that the major cognitive and epistemological problems appear. Indeed, as we saw, the phantom of the students' actions still haunts the algebraic symbols. The difficulty of the effacement of the individuals in the action that they produce was noticed by Piaget (1979) during the course of his investigation of children's sensorimotor stage and talked about the individuals' *décentration* of their actions. One of the reasons for the persistence of the action as a link between subject and object may be that, as Vygotsky (1997) suggested, actions appear as a formidable source of meaning in the emergence of the child's semiotic activity.

The question of the individuals' actions and their semiotic objectification (discussed from other theoretical perspectives and in different contexts by Arzarello 2000 and Núñez 2000) appears as an important element in contemporary understandings of the ontogenesis of algebraic language. The analysis offered in this paper evidenced some tensions caused by a shifting in the relation between the knowing subject and the object of knowledge imposed by the cultural requirements of algebraic and scientific languages. As we saw, natural language accounted for close dialectical forms of relationship between subject and object. In algebraic language, the relationship between subject and object is shattered. The dual reference subject/object becomes lost and it is no longer possible to talk about e.g. "your first figure". The students now have to refer to the objects in a different way. Deprived of indexical and deictic spatial-temporal terms, the new objects have to be denoted in a layer of discourse where they bear a different kind of existence and where the subject denoting them has to become (to use a term from Lacanian theory of discourse) *decentred* (see e.g. Bracher *et al.* 1994). The epistemological and didactic understanding of the decentration of the subject urges us to reflect on and envision new dialogical and semiotic forms of action in the activities that we propose to students during their insertion into the phylogenetically constituted social practice of algebra.

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