OF COURSE THEY CAN!
A reaction to Carraher *et al.*'s paper: "Can Young Students Operate on Unknowns?"

Luis Radford
Université Laurentienne
Ontario, Canada

In their enlightening paper, D. Carraher, A. Schielmann and B. Brizuela ask several questions and mention some problems that have been at the core of the research conducted in the learning and teaching of algebra for many years. The central question that they ask –namely, Can Young Students Operate on Unknowns?– is rooted in the idea that novice students find it difficult to operate on unknowns. This idea, however, has been refuted experimentally and historically many years ago. Hence a more appropriate question and more in line with their research intentions would be: When can young students start operating on unknowns? Carraher *et al.* suggest that arithmetical operations bear an algebraic meaning and that an early contact with algebra can help infuse this meaning into the children's arithmetic. I will discuss in Section 2 of this reaction the possibilities of such an enterprise, when I will comment, from a semiotic-cultural perspective, on some salient aspects of the classroom episode. In Section 1, before mentioning some of the historical and contemporary experimental data that show that the operation on unknowns is not an intrinsic problem arising in the transition from arithmetic to algebra, I will argue that, in adopting a traditional view according to which algebra relates to arithmetic only, Carraher *et al.* restrict the scope of their endeavor and miss important chances to infuse arithmetic concepts from other fields and to talk about e.g. geometrified arithmetic.

1. Operating on the unknown

Without denying developmental prerequisites for the learning of algebra, Carraher *et al.* seem reluctant to accept that a gap set by developmental levels could be the reason for which children cannot operate on the unknown in algebra and that such a gap would be out of the sphere of influence of educators. They suggest that it is probably a question of teaching and that it is wrong to attribute the gap to developmental constraints.

Carraher *et al.*'s refusal of a teleological idea of development and its entailed kind of determinism ("that there must *always be* such a gap") is well tuned with current anthropological views on cognition and conceptual development, where closer attention has been paid to the role of the context and of the others in the conceptual growth of the child, leading to the elaboration of new approaches in which biological lines of development appear dialectically interwoven with cultural ones, so that development becomes inseparable from context (Radford 2000). In their reference to the history of mathematics, they leave without questioning, however, the fact that to operate on the unknown in a certain historical period is not necessarily equivalent to operate on the unknown on a different period. They failed to notice that mathematical concepts are framed by cultural modes of knowing and that several cultures have conceptualized numbers and unknowns in different ways –sometimes more arithmetically (as in Diophantus' work), sometimes more geometrically (as in Babylonian mathematics). In adopting the traditional view that algebra relates to arithmetic only, they reduced the scope of their endeavour and narrowed the possibilities to deal with what is one of their more important submitted problems, that

is, the infusion of new meanings into arithmetic. Many years ago, I presented a communication in a meeting held at the CIRADE, in Montreal. I titled my presentation: "Why does algebra not come from arithmetic?" (It later appeared with a slightly different title in a volume edited by Bednardz, Kieran & Lee: see Radford 1996). In that paper I wanted to show that algebra is much more than a generalized arithmetic and that the algebra that we know owes a lot to geometry too (to witness the term *square* root) and argued for a broader view in considering the students' introduction to algebra.
However, the main point that I want to discuss now is the operation on the unknown. Historico-epistemological research has evidenced that the operation on the unknown did not seem to have presented particular difficulties to past mathematicians. This is attested to in the Old Babylonian period, the Antiquity, the Middle Ages and the Renaissance. As concerning the rhetoric pre-Vietan period of the Renaissance, this can found in many abacus treatises. For example, in Raffaello Canacci's *Ragionamenti d'algebra,* we find different ways to operate on and with the unknown. In solving a problem with the means of rhetoric algebra, Canacci (a Florentine algebraist of the second half of the 15th Century) was led to an equation that, for brevity, we can put into modern notations as follows: $t+12=35t-60$. Canacci operated the unknown and easily solved the problem. In an earlier book, Fibonacci's *Liber Abaci* (1202), we find Fibonacci (also working within the representational possibilities of rhetoric algebra) solving the equation $12t-t^2$=54-9t. He transformed it into $t^2+54=21t$ and then solved it by canonical procedures (the problems are discussed at length in Radford 1995. Concerning the operation on the unknown in the Antiquity, see Radford 1991/92; and for examples in Babylonian mathematics see Radford 2001, p.35 footnote 36). In each case, the way the unknown was handled was different: it depended, in particular, on the concept of number.

The successful operation on the unknown has also been reported in contemporary students with no prior knowledge of algebra. This is what Pirie & Martin did in 1997. In light of their experimental research they suggested, referring to the operation with the unknown, that "Rather than an inherent difficulty in the solution of linear equations, the cognitive obstacle is created by the very method which purports to provide a logical introduction to equation solution." (Pirie & Martin 1997, p. 161). A similar conclusion was reached in a previous teaching setting inspired by the history of mathematics. The engineering of the lessons was based on a use of manipulatives that allowed the students to act on concrete objects and then undergo a progressive semiotic process affording the production of meaning and the elaboration of more and more complex representations of the unknown and its operation (Radford & Grenier 1996a, 1996b).

To sum up, the question of the operation of the unknown in algebra has received a positive answer for many years, from a historical and from an educational point of view. When can young students start operating the unknown is, in contrast, a new question. Carraher *et al*. relate this question to their idea that arithmetic can be infused with algebraic meaning in early mathematics education. I would like to comment on this idea in terms of the kind of algebraic meanings that novice students may attain and how it relates to their symbols and repertoire of representational tools. To do so, I will refer to the students' mathematical activity as provided in the paper.

2. Some remarks on the mathematical activity

The paper describes the teacher's attempt to bring the students into contact with some elements of algebra and the way the students gained insights and underwent a process of progressive understanding of key concepts of algebra. For space constraints I will limit my discussion to some aspects of the children's conceptualization, representation and operation of the unknown.

In general terms, the students seem to have reached a certain level of algebraic understanding. Time and movement were two vital ingredients in the activity. The problem itself was set in terms of steps, where amounts of money were changing. However, time and movement were intermingled with speech, gestures, written symbols, arrows and cultural artefacts –such as *geometrical* N-number line. These elements constituted the arena where the activity and the production of meaning unfolded.

2. 1 The concept of unknown, its representation and operation

The activity provided the students to conceptualize the unknown in a meaningful way. Indeed, the idea of using a piggy bank permitted the students to think of the unknown as a hidden amount of money. Yet this was not enough. A semiotic act still had to be accomplished: the unknown had to be named or represented. The representation of the unknown is a very important step because, through this representation, the students objectify a new mathematical entity that can be applied not only to the piggy bank context but to other completely different contexts as well. An 'all-purpose-or-so' name/sign was hence needed. In the teaching episode, we saw that many students suggested the letter N. Of course, it was not through an individual well-inspired creative act of thought that the students suggested N. The activity was preceded by other activities where the idea of using a letter to represent an unknown number was introduced. The choices, of course are many. Diophantus used the term *arithmos* (number), Al-Khwarizmi used *root* and the Italian algebraists used *res*, and later *cosa* (the thing). But what did the letter N represent? The dialogue suggests that for the students the difference between *any number* and *a-not-yet-known number* was not completely clear. Furthermore, even if the students realized that N is *a-not-yet-known number*, some of them showed a strong tendency in adjudicating to N one of the possible numbers in the range of possible values, as in the Monday episode. As to the children's operation on the unknown, strictly speaking, in the classroom episode there was no operation on the unknown. For instance, unknown terms were not added or subtracted. The operations were performed on numbers (3, 5 etc.).

2.2 The geometrico-algebrafied arithmetic and the rise of meaning

Carraher et al.'s idea of starting algebra earlier than usual is related to the infusion of algebraic meaning in arithmetic. If we were to answer the question: Where does arithmetic really become geometrico-algebrafied? I would suggest that it is in reaching the expression N+3. The *movement* along the *N*-number *line*, is crucial for the construction of meaning. And it acquires a more dramatic tone when the students arrived at N+3-5 and identified as N-2 (on the Tuesday episode) and arrived at N+3-

3+4 that they identified as N+4 (Wednesday episode). These experiences are impossible to reach within the confines of arithmetic. Let us analyze the meaning arising from these experiences. Concerning the first one, Arabian mathematicians would make sense of N-2 in thinking of N as being deprived of 2 units (and then, in the process that we now call 'the isolation of the unknown', they would have hurried up to 'repair' N that they would have imagined as a 'broken' segment. To 'repair' it, they would have then applied the rule of *al-gabr*, from where the name algebra derives). Concerning the second one (referring to N+3-5), a Babylonian scribe would have said that 5 is 'detached' from N+3, and would have associated the latter to the width of a field that he would have imagined in the mind or would have drawn on a clay tablet. There is a marvellous Mesopotamian problem in which the scribe arrives at a subtraction of equal terms, and to remove them he says: "not worth speaking about" (transcription and analysis in: Høyrup, 1994, p. 9). Talik, in the video-taped episode, thinking of 'n', '3' and '-3' in terms of money, and coordinating the symbolic expression with *movements* along the *N*-number line, says that +3-3 is not needed anymore. We see how cultural conceptualizations and their meanings, rooted in different semiotic systems, enacted in mathematical activities and objectified in speech, may be different. Still the point is that as different as they may be, the richness of the conceptualizations results from the variety of contexts and the management of varied semiotic resources (speech, gestures, drawings, etc.) to produce meaning. The students' grasping of certain algebraic ideas in Carraher *et al.*'s lesson is related precisely to the richness of the cultural representational repertoire with which the students were provided and to the students' and teacher's progressive integration of such a repertoire in the mediated space of interactions.

Conclusion: We saw that Carraher *et al*. started asking an incorrectly founded question. Their research, however, opens up new avenues. The idea of infusing algebraic meaning into arithmetic appears appealing. Yet it still has to be demonstrated how the learning of arithmetic is really enhanced or if it is merely a question of starting algebra earlier. Their work suggests that 8 and 9 year-old students can attain a certain understanding of the algebraic unknown. The scope of this understanding requires further research. Another point that deserves more reflection is the status of negative numbers. I don't want to venture saying that the students secured a strong concept of negative numbers. Nevertheless, it is clear that the beginning of a conceptualization was started in the course of the lessons.

References

Høyrup, J. (1994). *Old Babylonian Mathematical Procedure Texts*, Max Planck Institute for the History of Science, Reprint 3.

Pirie, S. E. B. & Martin, L. (1997). The equation, the whole equation and nothing but the equation! One approach to the teaching of linear equations, *Educational Studies in Mathematics*, vol. 34, pp. 159-181.

Radford, L. (1991/92) Diophante et l'algèbre pré-symbolique. *Bulletin AMQ (Association Mathématique du Québec)*, 31/32, (4/1), pp. 73-80. Reprint: *L'ouvert*, IREM de Strasbourg, No. 68, pp. 1-13.

Radford, L. (1995) Before the Other Unknowns Were Invented: Didactic Inquiries on the Methods and Problems of Mediaeval Italian Algebra, *For the Learning of Mathematics*, **15** (3), 28-38.

Radford, L. (1996) The roles of Geometry and Arithmetic in the development of Elementary Algebra: Historical Remarks from a Didactic perspective, in: *Approaches to Algebra: perspectives for research and teaching*, N. Bednarz, C. Kieran & L. Lee (eds.), Dordrecht /Boston/ London: Kluwer, 39-53.

Radford, L. (2000). The historical development of mathematical thinking and the contemporary student understanding of mathematics. Introduction. In: J. Fauvel & J. Maanen (Eds.) *History in Mathematics Education*. *The ICMI Study.*, Dordrecht /Boston/ London: Kluwer, 143-148.

Radford, L. (2001) The Historical Origins of Algebraic Thinking, in: *Perspectives in School Algebra*, edited by R. Sutherland, T. Rojano, A. Bell & R. Lins, Dordrecht /Boston/ London: Kluwer, pp. 13-36.

Radford, L., Grenier, M. (1996a) Entre les idées, les choses et les symboles. Une séquence d'enseignement d'introduction à l'algèbre, *Revue des sciences de l'éducation*, Vol. 22, pp. 253-276.

Radford, L., Grenier , M. (1996b) On the dialectical relationships between symbols and algebraic ideas, In: *20th PME international conference,* Vol. 4, pp. 179-186, Universidad de Valencia, Spain.