On the relevance of Semiotics

in Mathematics Education[1]

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0. Introduction

Let me start by addressing this Discussion Group with a question of interest:

Does semiotics have something to offer to mathematics education?

Obviously, since mathematics relies on an intensive use of different kinds of signs (letters, signs for numbers, diagrams, formulas, etc.) the answer is yes. So, the real question is:

What can semiotics offer to mathematics education?

Naturally, the answers here are far more complex and they can be different. The reason is that there are different semiotic approaches. Some of them may offer interesting avenues for the problems with which we deal in mathematics education while others may be less useful. For instance, a formal theory of signs can be of little interest to mathematics education.

I would like to mention a few points in which educational reflections and enquiries can be further enlightened by having recourse to a certain type of semiotics.
It is not an exhaustive list. This list, rather, reflects my own experience as a math educator who has ventured himself, for several years now, in the domains of semiotics.

1: The role of signs in Cognition

The first point is related to the role that signs and tools play in cognition.
Probably, one of the best statements made in connection to this point is one made by Peirce, who said, in *Some Consequences of Four Incapacities* (a paper published in 1868), that we do not have the power of thinking without signs.

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Of course, there are different ways to theorize the manner in which signs and tools relate to cognition.
One option in which to theorize the relation between signs and cognition is to conceive signs to be *helpers* of thinking. We find here Leibniz and the whole philosophical tradition that goes up to Frege and Analytic Philosophy. It is within this context that the research program of a universal language has to be placed (see Radford 1998a). The language should be as clear as possible, starting with the simplest terms in order for the ideas to be conveniently dressed and exposed. The classic itinerary is to start with signs for operations, signs for variables, propositions, predicates, rules for construction of sentences, etc.

In contrast to the conception of signs and tools as helpers or facilitators of thinking, we have those that consider signs and language as the origin of cognition.
The difference between these two poles is hence the following. In the first case, it is considered that signs derive from a mind that, in its cognitive endeavour, tries to find the correct signs to express itself.
In the second case, cognition derives from Language and signs. The received interpretation of the so-called Sapir-Whorf thesis is, in fact, that language offers conceptual categories through which we see our world in a form that remains locked within the confines of the cultural conceptual categories[2].
Among other approaches we have the structuralist tradition in which signs appear as revealing a hidden set of cultural structures of opposition and differentiation (Levy-Strauss, Saussure, etc.).

Semiotic approaches can also be distinguished in terms of the role of communication in cognition.
On the one hand, we have those that claim that cognition arises out of acts of communication. For instance, Harré and Gillett (1994, p. 22) say that "[t]he idea that the mind is, in some sense, a social construction is true in that our concepts arise from our discourse and shape the way we think."
On the other hand we have those who, without denying the role of communication in cognition, leave it in the background: thought, here, emerges independently of communication. This is the case of the *ego* conceived of as the *solitary ego*. Although it is a banality to provide examples of this, let me note that, in this case, communication is underpinned by the idea of a language that includes only the speaker and the object of the speech. The listener is thereby reduced to a passive participant. The paradigmatic example here is the Processing Information Theory and the kind of cognitivism that arose along with it.

I am more inclined to see the potential of semiotics in a rather Vygotskian perspective, that is, signs as psychological tools, or as prostheses of the mind, or even (but this is no longer Vygotsky) as the external locus where the individual's mind works.
In this theoretical perspective, communication plays a central role. But communication is not seen as a disinterested communication. The individuals communicate between

[2] It has been argued, nevertheless, that the Sapir-Whorf thesis does not state that language determines thinking but only influences it (see, e.g., Lloyd, 1993, pp. 206-207).

themselves to carry out goal-oriented activities having culturally motivated goals. Actually, in my cultural-semiotic perspective, I see signs playing a dual role in cognition: they allow individuals to move along in two interrelated directions: (1) the "technical" one, as a means to deal with the object of knowledge (much in the same sense as Vygotsky's early idea of signs and his comparison to labor tools: see Radford 1999a), and (2) the other one, that I want to call the "social" direction –where we find the niche of *meaning*-- in which individuals communicate with each other.

2. Conceptions of meaning and understanding

Another point in which semiotics may shed light on some current problems in mathematics education is the one related to the concepts of meaning and understanding.

It is clear that a conception of language that adopts a position focusing on the speaking individual and what he or she says, leaving in the background the addressee, will provide individualist concepts of meaning and understanding.

It is worthy to quote the following passage from Bakhtin:

> The "signification" of discourse and the "understanding" of this signification by the other (or by others) … exceed the boundaries of the isolated physiological organisms and presupposes the interaction of several organisms, which implies that this third component of verbal reaction has a sociological character. (quoted by Todorov, 1984, p. 30)

For Bakhtin, our relation to meaning and understanding is always dialogic. This is why meanings and understandings are always in movement. They are not passive processes but the encounter of several voices. This was made very clear in Bakhtin's reflections about our understanding of other cultures when he said:

> A meaning only reveals its depths once it has encountered and come into contact with another, foreign meaning: they engage in a kind of dialogue, which surmounts the closedness and one-sidedness of these particular meanings, these cultures. (Bakhtin 1986, p. 7)

A question of great importance for Mathematics Education is the question of meaning acquisition/formation. Again, the answers will vary depending on the chosen framework. At any rate, contemporary approaches will probably agree that the individuals have to become deeply engaged in actions leading to the formation of meaning.

3. The sign-ified mind

The fourth point that I want to mention, concerning the potentials of semiotic theorizations for mathematics education, is the following. Traditional cognitive

psychology has been one of the champions in conveying the idea of signs and tools as facilitators of thinking.
In this perspective, signs and tools (calculators, computers, etc.) that the individuals use in their cognitive activity do not alter or modify their cognitive functions. As it were, cognition remains in a bunker beyond the effects of signs and tools.

On this point, I follow Vygotsky and Luria, who said that our cognitive functions become modified by the use of signs and tools.

> By being included in the process of behavior, the psychological tool alters the entire flow and structure of mental functions. It does this by determining the structure of a new instrumental act just as a technical tool alters the process of a natural adaptation by determining the form of labor operations. (Vygotsky, 1981, p. 137)

4. The culturally patterned use of signs and tools

From what I said previously, it results that through the use of signs, artefacts and tools, the individual modifies his or her cognitive functions.
But tools and signs are not objects *per se*. By this I mean that signs and tools are elaborated and used according to cultural meaning-making practices.
A tool or a sign does not mean anything in itself.
To illustrate this point, let me refer to an interesting passage concerning the conquest of America in the early 16th Century. Todorov mentions this episode in his semiotic investigations about the Other. During an exchange of gifts, the conquistadors gave the natives a needle. The latter could not make sense of this cultural artefact, which clearly appeared as useless within the system of pre-Columbian activities. Then, to overcome embarrassment, the Spaniards promptly suggested that needles could be used to remove a splinter from the skin or to clean the teeth (Todorov, 1982, p. 45).

The point is that signs, tools and artefacts are elaborated and used in direct relation to the activities of the individuals.
This is an extremely important point in my research. It distinguishes my cultural-semiotic approach from other approaches that put aside the historical and phylogenetically constituted nature of activity focusing on speech and discourse thereby reducing cognition to a discursive practice.
There is a lot of discursive practice in cognition, but discursive practice cannot account for all the particulars of cognition.
One of the risks in reducing cognition to a discursive practice is that we may easily fall into verbal behaviourism.
In my view, the problem with this semiotic approach is that the focus of attention becomes the way individuals interact discursively with each other without paying sufficient attention to the fact that discourse follows a flow that is embedded in cultural traditions underpinned by specific modes of inquiry that define certain relations between the subject of knowledge and the knowing subjects. In marginalizing these issues, the

object of knowledge, the individuals and their relations are sunk into oblivion and language and discourse become endowed with a kind of supernatural creative power.
In a previous article (Radford 2000a), I quoted the following passage from Mikhailov. This passage encapsulates very well the idea that I want to convey:

> When formally analysed, language hangs in the air, as it were, is deprived of its roots and becomes an independent object of research; the individual, whose tongue makes language a living thing, is pushed into the background and forgotten." (Mikhailov 1980, p. 221)

5. Cultural Semiotic Systems: Pythagorean Pebbles versus Euclidean lines

So to understand the sign-ified mind we have to look for the way signs are used, for the way they are embodied by what I called, in a previous article (Radford 1998b), the *cultural semiotic systems*, that is, those cultural systems which provide signs with a symbolic dimension and that make available varied sources of meaning through specific social signifying practices.

Cultural semiotic systems do not appear out of the blue. They are related to the activities of the individuals, the social distribution of knowledge, and cultural ontological and epistemological stances.

Let me illustrate this point through an example from Greek Mathematics. I will take the case of the Theory of Even and Odd Numbers. Euclid's Elements, book IX, proposition 21 reads as follows:

> If as many even numbers as we please be added together, the whole is even.

And the proof is the following:

> For let as many even numbers as we please, AB, BC, CD, DE, be added together; I say that the whole AE is even.
> For, since each of the numbers AB, BC, CD, DE is even, it has a half part; [VII. Def. 6] so that the whole AE also has a half part.
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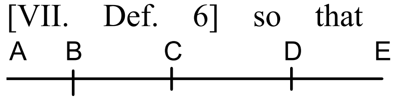
> 
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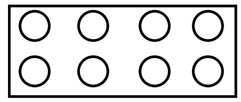
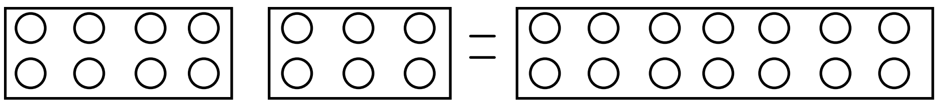
> But an even number is that which is divisible into two equal parts [id.]; therefore AE is even (Heath 1956: 413).

This proposition was very well known prior to Euclid's time and formed part of the Pythagoreans' mathematical results. But before the rise of the Euclidean tradition, the proposition was proved through a technique based on the use of stones.

Oskar von Becker (1936) has suggested a reconstruction that would go like this:
Even numbers are those that can be halved, and can be represented as such:



The proof could be displayed as follows:



If we could ask Euclid why he was not satisfied with the Pythagorean proof, he would probably have said that the proof actually proved that the sum of the two even numbers, 8 and 6, is an even number and that the Pythagorean proof does not prove the statement in its entire generality.
A Pythagorean would have retorted that you do not have to count the stones but just look at their *shape*. Had this Pythagorean known Peirce's terminology, he or she would have said that you just have to take the stone-formation as an iconic sign.

As you see, the same sign (here the stone-formation representing a generic even number) is perceived differently by Pythagorean and Euclidean mathematicians. The same sign signifies different things.
Of course, this is not the only point of disagreement. The Pythagorean mathematician could have his Euclidean colleague noticing that he has not done much better, since he proved the mathematical proposition for only 4 numbers. Then, again, the Euclidean mathematician could reply something like: "No, you don't have to see it that way. It is true that I wrote 4 numbers but you have to think as if I had considered more numbers… You see 4 numbers but you have to *see* like there were more!", and so on[3].

The difference in the significations and the use of signs in the process of generalization is accounted for by differences in mathematical practices backed by their corresponding distinct cultural semiotic systems. While in the Pythagorean practice the world of the sensible objects was seen as an arena to investigate the nature of things, in the Euclidean mathematical practice, the world of the sensible objects was not to be trusted as a source of knowledge.

The specific way to learn to see something as something else in a very particular way among the vast arsenal of perceptual and conceptual possibilities is, indeed, one of the effects of a cultural semiotic system and the practices that such a system legitimizes.

[3] In a recent book, Netz (1999) deals with the concept of deduction in Greek mathematics. Unfortunately I cannot comment here on Netz's interesting work.

A classroom episode

My imaginary dialogue between the Pythagorean and Euclidean mathematicians makes apparent the fact that differences concerning the “right” way to use and understand signs in the establishment of the generality of a mathematical statement is an instance of differences in cultural intellectual practices.
Differences of this sort do not necessarily happen when we cross cultural borders of historical periods. It may also happen in specialized practices within a same culture. In this case, the problem is that of being introduced into a practice while, at the same time, being immersed (probably on an unconscious level) in the corresponding web of cultural significations.
Let me refer to a classroom episode. It happened when a Grade 8 class was working on the topic of generalization.

The students spent some time answering questions about the following pattern:

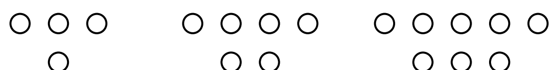


Figure 1 Figure 2 Figure 3

One of the goals of the classroom activity was to elaborate a formula indicating the number of circles making up Figure n.

The students were not shown any previous example. Hence, they had to make sense of the question according to their previous arithmetic experiences (details in Radford 1999a, 1999b).
The task was not evident. Thus, in one of the small-groups, a student, after looking again to the figures of the pattern, contended that there was no such thing as Figure n.

In a certain sense, the student was right in noticing that there is no material Figure n in the pattern. Regardless of how far you go down the pattern, figure after figure, Figure n is already an abstract object that is not located in the concrete field of plastic chips. The central question about Figure n is that it does not belong to the realm of concrete objects.

Where is it, then?
This is an ontological question whose answer places us in the center of a cultural semiotic system. I think nobody will reasonably claim that Figure n does not exist. It exists, but we are not able to point to it as we can point to, say, Figure 2 or Figure 5.

I would say that we point to it but not with our finger, we point to it with words, or to be more general, with signs. And in saying this we already crossed the gates of semiotics.
To point to an object with a finger is one of the more elemental modes of denotation (see Radford 2000b). If we follow Raymond Duval’s semiotic approach (Duval 1995) we should say that we are not yet in the realm of semiotics. He seems to be right in saying that for semiosis to start, the referred object has to be absent or out of sight. Otherwise

the sign has no *raison d'être*. I would say that we are on the border and probably even a bit inside of the territory of semiotics, in a region that can be called "perceptual semiosis" (see Radford, in press). This region is characterized not by the object being present or absent but by the emergence of a perceptual relation between the perceiving individual and the perceived object which the individual turns then into a sign in order for the object to become a matter of reflection. In the course of this relation between the perceived object and its transformation into a sign, a link is forged -out of which meaning is disclosed.
At any rate, to point or to refer to an object through signs is a semiotic problem. And the study of the modes of denotation, that is, those modes rendering the diaphanous mathematical objects sensible and hence capable of awareness is a problem for which semiotics can be of great help.

The mathematics lesson to which I am referring was intended as a way to initiate the students into a social, intellectual practice and into the specific forms of predication about certain concepts.

The analysis of the mathematics lesson shows that Figure n was intended, by the teacher, as a *genera* whose *species* would be the concrete figures of the pattern (here we are brought to a kind of Aristotelian ontology). One of the formulas proposed by the students (but not the only one; see Radford 1999b) to find the number of plastic chips in Figure n was the following: n+ (n+2).

This formula reflects the spatial perception of the shape of the eidetic Figure n: the *genera* is supposed to have the same shape as the *species*. This supposition justifies that the formula be written as n+(n+2), a formula that, in its more intimate semiotic nature, appears as an icon of the unperceivable Figure n (see Radford, in press).

What is not clear though is if the shape of the *species* is what it is because the *genera* has this form; or is it the other way around? In other terms, where does the shape originate? In the particulars or in the general? The answer to this question will lead us to different conceptualizations of denotation and hence into the realm of semiotics.

To close my discussion, let's notice that one of the strengths of semiotics, as I understand it, is that, on the one hand, semiotics deals with cultural and situated signs. The study of the cultural modes of signifying can help us to better understand a concept that, as I mentioned previously, has become central in current research in Mathematics Education, namely, the concept of meaning. But, on the other hand, since signs denote objects, semiotics is urging us to better clarify the very nature of the mathematical objects with which we deal in our classroom practices. The point is not necessarily to end up philosophizing in the mathematics classroom (although it would do no harm, I guess) but to better understand our own practices and the cognitive, epistemological and educational role of the semiotic systems that we are encouraging in the classroom.

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