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## THE HISTORICAL ORIGINS OF ALGEBRAIC THINKING

## INTRODUCTION

Barely some fifteen years ago, a new and completely different interpretation of Mesopotamian mathematics arose due, on the one hand, to new philological analyses and, on the other, to the development of new methods in the historiography of mathematics; these have been increasingly incorporating anthropological and sociological categories in the understanding of the mathematics of the past. The new results have been changing the traditional view of Ancient Near East mathematics and their influence on Greek mathematics and have led us to review old questions and to raise new ones. One of these questions dea's with the transition from arithmetic thinking to algebraic thinking. Another question is the role played by language and symbolism in algebraic thinking. In this paper we will deal with these two questions ${ }^{1}$.

Concerning the first question, we shall attempt to show that (numeric) algebraic thinking emerged from proportional thinking as a short, direct and alternative way of solving 'non-practical' problems. In order to do so, in the first section of the chapter, we briefly describe some of the features of Proportional Mesopotamian thinking. The second section is devoted to the elaboration of our thesis. The historical technical evidence and its argumentation is presented in following sections. Since a certain knowledge of Diophantus' algebraic methods is required, we give an overview of Diophantus' monumental treatise Arithmetica, and deal with Babylonian geometric algebra.

The second question with which we deal in this paper - that of the role of language and symbolism in algebraic thinking - is submitted to the same methodological approach that we used in the previous sections deriving from our perception of what mathematics is: we see mathematics as a semiotic manifestation of the culture in which mathematics is practised. Consequently, we suggest that the role of symbolism in algebraic thinking needs to be studied through the social meaning of alge-

[^0]bra and through the different forms of symbolisation that the culture under consideration uses to symbolise its objects (regardless of whether or not these are mathematical). This is why, in many instances, we shall refer to some sociocultural components of the world of Mesopotamian scribes. The final section of the chapter contains some pedagogical remarks.

## MESOPOTAMIAN PROPORTIONAL THINKING

Mathematics arose in ancient civilisations intimately linked to the social, political and economic development of the cities. For instance, ancient historical records suggest that the numerical cuneiform signs used in Mesopotamia were, in fact, the achievement of an important semiotic phenomenon preceded by the commercial counting activities first based upon incised bones and later on tokens (SchmandtBesserat, 1992; Jasmin and Oates, 1986; Le Brun et Vallat, 1978). The expansion of the cities required new and more elaborate forms of internal organisation which led to new and complex problems -e. g. how to calculate the area of a certain piece of land, how to calculate the interest of a loan, how to solve inheritance problems, eg. how to calculate the price of different commodities ${ }^{2}$. Nevertheless, we also find 'non-practical' problems, i.e. problems that, although being formulated in terms of the concrete semiotic experience of everyday life, have no direct relation to practical needs ${ }^{3}$.

Practical and non-practical problems appear in two of the most important mathematical currents observed in ancient Egyptian and Babylonian civilisations: a geometrical one and a numerical one. In the first current, we find problems which deal with the calculations of area and perimeter of geometrical figures while, in the second, we find problems about (contextual) numbers ${ }^{4}$. In the two previous categories, problems are often solved using proportional thinking, which was, in fact, one of the most important areas developed in Mesopotamian mathematical thought. Not only is it attested to by the Babylonian tables of reciprocals, which were a byproduct, but also by the clever ancient methods based upon proportional tools that were developed to solve problems. One of the three oldest known problem texts (ca. third millennium BC), which was found in 1975 when the Italian Archeological Mission discovered the Royal archives of the city of Ebla, is the text TM.75.G. 1392 which contains a type of problem whose statement and problem-solving procedure reveal some features of early proportional thinking. The problem (concerning the assignment of cereals) is stated (according to Fribergs' reconstruction) as follows:

Given that you have to count with 1 gú-bar for 33 persons, how much do you count with for 260,000 persons? (Friberg, 1986, p. 19).

The problem obviously leads to division. However, the division is not easy to perform within the sexagesimal system. Therefore, the text shows a sequence of
calculations: 3 gú-bar for 99 persons, 30 gú-bar for 990 persons, and so on (ibid, p. 20). The sequence is based upon proportional tools which allows the scribe to obtain an (approximate) answer.

The most sophisticated proportional problem-solving procedure deals with the methods of false position which appear as the greatest achievement of Mesopotamian arithmetical thinking. These methods are based upon the idea of assuming some false values for the sought-after quantities and then adjusting them through a 'proportional adjusting factor' which allows one to modify -in a proportional waythe false values in order to transform them into the true values.

The following problem is an example of a non-practical problem solved by a false position method. It is included in a tablet from Susa, probably slightly after the 1st Babylonian Dynasty, i.e. at the end of the 17 th century BC. (see Bruins and Rutten, 1961, pp. 101-103) ${ }^{5}$. The problem is to find the sides of a rectangle whose width is equal to the length minus a quarter of the length, and the diagonal is $40^{\circ}$. (Note that in a real situation, anybody who knows the stated relationship between the sides of the rectangle obviously already knows the actual sides of the rectangle. This problem is merely a riddle). To solve the problem, the scribe assumes a false quantity for the length. He says: 'You, put one for the length'. He then calculates the width, by subtracting $1 / 4$ (that is $15^{\circ}$ ) from $1^{\circ}$, which gives $45^{\circ}$. He calculates the square of both false sides, which gives $1^{2}=1$ and $\left(45^{\circ}\right)^{2}=2025^{\prime \prime}=33^{\prime} 45^{\prime \prime}$. The sum of the squares is $1^{\circ} 33^{\prime} 45^{\circ}$. He then takes the square root of $1^{\circ} 33^{\circ} 45^{\prime \prime}$ which is $1^{\circ} 15^{\prime}$. This is the value of the false diagonal. The true diagonal, $40^{\circ}$, is less than the obtained value. He then calculates the inverse of $1^{\circ} 15^{\prime}$, which is $48^{\circ}$; he multiplies this number by $40^{\prime}$; the result is $32^{\prime}$. This is the 'proportional adjusting factor' by which he multiplies the false length (i.e. $1^{\circ}$ ) and the false width (i.e. $45^{\circ}$ ): the scribe then obtains $32^{\circ} \times 1^{\circ}=32^{\circ}$ and $32^{\circ} \times 45^{\circ}=24^{\circ}$. Therefore, the resulting numbers $32^{\circ}$ and $24^{\circ}$ are the true length and width, respectively.

In modern terms, the problem requires one to find the length, $x$, and the width, $y$, of a rectangle whose diagonal is a given number d . From the relation $d=\sqrt{x^{2}+y^{2}}=\sqrt{x^{2}+\left(\frac{3}{4} x\right)^{2}}=\frac{5}{4} x$ we can deduce that $x=\left(\frac{5}{4}\right)^{-1} \cdot d$. In contrast, the scribe actually assumes a false length, $\mathrm{x}_{0}$, which gives a false width $y_{0}=x_{0}-\frac{1}{4} x_{0}=\frac{3}{4} x_{0}$. He calculates the false diagonal using $d_{0}=\sqrt{x_{0}^{2}+\left(\frac{3}{4} x_{0}\right)^{2}}=\frac{5}{4} x_{0}=\frac{5}{4}$ (because $x_{0}=1$ ). The proportional argument underlying the procedure leads the scribe to calculate the inverse of $d_{O}=5 / 4$ and to multiply this inverse by the given diagonal $\mathrm{d}=40$.

This problem clearly shows the functioning of the ancient false position method and will suffice to elaborate our historical reconstruction of the transition from arithmetic to algebra in Mesopotamian land. However, before going on to our next
stop in our historical journey, we need to make the following cultural and epistemological remark: the idea of using a 'false quantity' to start the false position method, leans on a deeper and more complex idea: at the beginning of the problem, the 'true quantity' (i.e. the exact solution of the problem) may be legitimately thought of through another quantity. 'False quantities' thus appear as metaphors of 'true quantities'. Furthermore, this is not a phenomenon restricted solely to mathematics: Mesopotamian thinking is full of metaphors. Odes, epic poems, literary and religious texts, for instance, show an intricate system of metaphorical expressions (see e.g. Wilson, 1901). Algebra, we shall suggest, was couched in such a system.

## ALGEBRAIC THINKING AS A METAPHOR OF THE FALSE POSITION METHOD

As we shall see in later sections of this chapter, where we focus on some technical details, the influence of false position methods in the emergence of algebraic ideas can be discerned through some important structural similarities between false position reasoning and early algebraic thinking. One of the studies of the ancient connection between the Babylonian false position method and algebra was made by François Thureau-Dangin (1938a). Following the trends of the old interpretation of Babylonian mathematics based on the possibility of translating the calculations shown in many of the tablets into modern algebraic symbolism, he noted a strong parallelism between the calculations done in some problems solved by false position methods and those of the modern algebraic methods ${ }^{6}$. He then claimed that, indeed, some Babylonian procedures were algebraic procedures. Thureau-Dangin's main idea was supposedly supported by the fact that, in some problems, the scribe takes the number one as the false solution (such an example could be the problem discussed at the end of the previous section) and when, according to Babylonian procedures, we replace the number one by our modern unknown ' $x$ ', the problemsolving procedures look much like the modern algebraic procedures. He, (as well as others, e.g. Vogel, 1960), claimed that the number one was actually taken as a representation of the unknown and if we cannot, straight out, see the unknown, it is simply because the scribes did not have a symbol with which they could represent it. However, the idea that Babylonians developed an 'invisible algebraic language' (i.e. a genuine algebraic language without symbols) has been abandoned (see Radford, 1996a, pp. 39-40). Effectively, there is no clear and safe argument supporting the statement that the Babylonian scribes actually thought that the number one represented the unknown in an algebraic sense (see Høyrup, 1993b, p. 260). On the contrary, this peculiar numerical choice for the unknown seems to have allowed the scribes to systematise the numerical problem-solving methods and hence to reach an important step in the conceptual development of ancient proportional thinking. In
fact, when ancient problem-solving procedures may be safely identified as «algebraic», which is the case of the problem mentioned in footnote 10 , the unknowns are not represented by the number ' 1 ' or anything else for that matter; instead, the unknowns bear their contextual name (e. g. the length and the width of a rectangle).

Rather, the algebraic idea of unknown seems to have been thought of as a metaphor of the 'false quantities' used in the ancient false position method. It happened, we suggest, when scribes stopped thinking in terms of false quantities upon which the false position method is based and, looking at the false quantity metaphorically, began to think in terms of the sought-after object itself, accepting to consider this object as a number (i.e. an operable, manageable number) regardless that it was not yet known ${ }^{7}$. This could happen in solving non-practical problems previously solved by false position proportional methods like the following (Tablet YBC 4652, No. 7), where the method of solution is unfortunately omitted:
'I found a stone, but I did not weigh it; after I added one-seventh and added one eleventh (of the weight and its one seventh). I weighed it: 1 ma-na. What was the original weight? The origin(al weight) of the stone was $\frac{2}{3}$ ma-na, 8 gin, (and) $22 \frac{1}{2}$ $\overline{\mathrm{s}} \mathrm{e}$ ' (Based on Neugebauer and Sachs' reconstruction; 1945, p. 101).

In modern notations, this problem reads as follows ${ }^{8}$ : $x+\frac{x}{7}+\frac{1}{11}\left(x+\frac{x}{7}\right)=1$.
The algebraic way of thinking could have even been conceived when ancient scribes faced an even simpler problem. For example, a problem of this type ${ }^{9}: x+\frac{1}{11} x=1$.

Let us suppose that this problem concerns the weight of a stone. The false position method is as follows: we assume (according to the usual line of thought in Babylonian mathematics) that the sought-after quantity is 11 ; then, the stone and one eleventh of its weight is 12 . However, we should have 1. This means that we need to reduce the 'false position', that is the false value that we assumed at the beginning (i. e. 11). To reduce it, an elementary proportional argument shows that we need to take one 12th of our initial assumption, so the answer to the problem is $11 / 12$ (or $55 / 60=$ $55^{\prime}$ in the sexagesimal system).

To see the metaphor that we are suggesting at work, let us, instead of beginning by assuming a false position or false solution, start the problem-solving procedure by reasoning on the exact unknown sought-after quantity. In this case, the calculations unfold in a different way: first, we multiply both sides of the 'equation' by 11 ; thereby transforming the 'equation' into an 'equation' without fractional parts. This leads us to an equation that we would write as $12 \mathrm{x}=11$. Now, following a recurring Babylonian procedure, we just need to find the inverse of 12 , which is $5^{\prime}$, and to multiply this inverse by 11 , which gives us the answer $55^{\circ}$. (Note that the procedure
of multiplying both sides of the 'equation' by a number is attested to in many Babylonian problems, e. g. Textes Mathématiques de Suse ${ }^{10}$ ).

The type of problems that we have just discussed were frequent in ancient civilisations. For instance, one problem of this type is found in the Egyptian Rhind's Papyrus; another is found in Babylonian tablets. This is the case of problem No. 3 of tablet YBC 4669 which could be translated into modern notations as follows ${ }^{11}$ : $x-\frac{2}{3}, \frac{1}{3} x=7$.

The conceptual connection between false position ideas and algebraic ones can also be found in post-Greek mathematics. It can be retraced to some mediaeval mathematical treatises. It is particularly enlightening that, in the false position methods, mathematicians, at the beginning of the problem-solving procedure, used to refer to the action of choosing the false numbers as 'making a position'. In the same way, when a problem is solved by algebra, the introduction of the unknown is sometimes referred to as 'making a position'. For instance, in Filippo Calandri's Una racolta di ragioni ( 15 th century), problem 18 deals with a problem that we may translate into modern notations as: $\frac{x}{x+1}=x-1$
('Trouva U numero, che partito per uno, più ne vengha un meno'. Santini (ed.), 1982, p. 19).

Solving this problem through algebra and by calling the unknown the thing ('la cosa', according to the Italian mediaeval tradition) Calandri says: 'Farai posizione che quel numero sia una co(sa)' (I will make a position so that the (sought-after) number is a thing). An early example (14th century) is found in problem 6 of Mazzinghi's Trattato di Fioretti. In this problem Mazzinghi says: 'El primo (modo) è che si faccia positione che lla prima parte sia 5 et una chosa’ (Arrighi (ed.), 1967, p. 23).

The connection between false position ideas and algebraic ones is more explicit in an anonymous abacus treatise of the 14th Century: Il trattato d'algibra. In this treatise, the unknown is defined just as a position: '...in prima noi diremo che sia questa chosa, onde dirò che non sia altro se non è una posizione che si fa in molte questioni...' (first of all we shall say what this thing is, where I shall say that it is no more than a position that one makes in many questions; Franci and Pancanti, eds. 1988, p. 3, my translation).

We can go one step further in our connections between algebraic and false position ideas by referring to a book written in 1522: Francesco Ghaligai's Prattica d'Arithmetica. However, in this case, algebraic ideas have been developed enough to be taken as the explanatory substratum in which the false position ideas are set up. Ghaligai says:
> 'We can notice that the position is a concept assimilated to the thing that is chosen according to the knowledge of the intellectual realm. Speaking in the case of a thing not known to you, right away the mind will think as if it already knew and say: position is a
quantity placed according to the case (the problem) and even though there are two positions, sometimes with only one position the case can be solved and one finds that which is necessary.' (Ghaligai, 1548, p. 62; my translation).

Ghaligai's Prattica d'Arithmetica shows then that the conceptual development of the unknown has completed a loop, now changing the roles of ideas: at that time, the metaphorical-analogical process was reverted and one explains the false position ideas in terms of algebraic ones. methods have to arise, thereby making it possible to handle the algebraic unknown. The metaphorical-analogical process underlying the passage from arithmetic to algebra will map or induce, as is the case in many metaphors (see Lakoff and Núñez, in

To end this section let us stress the fact that the new algebraic object of unknown does not come to life alone: it emerges along with new methods. False position methods deal with numbers only. So, new print), important structural features of the first domain - here the arithmetical one - into the second domain - here the algebraic one. This metaphorical induction is very clear in many passages of Diophantus' Arithmetica. Indeed, many of Diophantus' algebraic methods are hardly understandable without linking them to the ancient false position methods, as we will see in the next section. In order to understand this specific aspect of problem-solving methods, we now need to examine the place of Diophantus' Arithmetica in the development of algebraic ideas.

## ALGEBRAIC IDEAS IN DIOPHANTUS' ARITHMETICA

As we know, Diophantus' Arithmetica (written circa 250), is made up of 13 books (3 of them are lost) dealing with the resolution of problems about numbers. Book 1 contains a short introduction in which a division of numbers into species or categories is presented: the squares, the cubes, the square-squares, the square-cubes, the cube-cubes. Each category contains the numbers that share a similar form or shape (the same eidos). Instead of being merely riddles like the Babylonian nonpractical problems, the problems of the Arithmetica were formulated in terms of the mentioned species. For instance, problem 10 from Book 2 reads as follows: 'To find two square numbers having a given difference' (Heath, 1910, p. 146).

Undoubtedly, within the philosophical principles of Classical Greek thinking (where the search of origins and rational organisation was a starting point), Babylonian numerical word-problems and all the subsequent numerical activity surrounding similar problems in the post-classic Greek period, an activity particularly attested to by the Anthologia graeca (Paton, (ed.), 1979), did not find a suitable niche to prosper ${ }^{12}$. By transforming the Babylonian numerical word-problems into problems about abstract Greek species and other ancient well-known riddles that Diophantus supposedly disguised in abstract terms in his Arithmetica (e.g. Book I, problems 16-
21), Diophantus elevated this unscientific discipline to a scientific one ${ }^{13}$. This was not the only important new aspect incorporated by Greek algebraists. There is another one related to the introduction of indeterminate numbers to the mathematical realm. This was done through a new use of the concept of arithmos (ariqmoV), that is, the 'number'. 'The arithmos', says Diophantus, 'is an indeterminate multitude of units' (cf. Ver Eeck, 1926, p. 2) - although, in the solution of many problems, it can be an indeterminate multitude of fractional parts.

The subtle, yet fundamental, step made by Diophantus in introducing indeterminate numbers to the mathematical realm can be better appreciated if we see it within the heritage of the ancient philosophers. In this line of thought, it would be worthwhile to remember that, in one of the few extant fragments of the first Pythagoreans, Philolaus says: 'Actually, everything that can be known has a Number; for it is impossible to grasp anything with the mind or to recognise it without this (Number), ${ }^{14}$; and here, Number means a determinate multitude. Thus, by introducing the arithmos as an indeterminate multitude Diophantus extended the borders of what can be known. By the same token, the aforementioned concept of arithmos gave way to the creation of a completely new theoretical calculation on indeterminate amount of units (e.g., in modern notations, rules dealing with calculations like $x \times x^{2}=x^{3}$ or $\frac{1}{x} \times x^{4}=x^{3}$ ) that proved to be very powerful in the resolution of problems. It is important to note that these mathematical accomplishments at the end of the Antiquity were linked to an increasing (albeit not complete) abandonment of Greek classical principles and the spreading of neo-Platonistic speculations that made it possible to think in new, different and promising ways (see Lizcano, 1993).

In most of the problems of the Arithmetica, the formal structure of their statement follows the same pattern: the problem is stated in terms of operations performed on some categories of numbers (the square numbers in the previous example).

At the beginning of Book I, Diophantus gives a description of the heuristics that one should follow in order to solve the problems. This is the very first explicit ancient description about how to solve problems on numbers that we know. For our discussion we will quote the following extract:

> ...if a problem leads to an equation in which certain terms are equal to terms of the same species (eidos) but with different coefficients, it will be necessary to subtract like from like on both sides, until one term is found equal to one term. If by chance there are on either side or on both sides negative terms, it will be necessary to add the negative terms on both sides, until the terms on both sides are positive, and then again to subtract like from like until one term only is left on each side." (Heath, 1910, p. 131$)^{15}$.

In modern terms, this passage tells us that if in a problem we are led to an equation of this type:
$\alpha$ numbers $\pm \beta$ squares $\pm \delta$ cubes $\pm \varepsilon$ square squares $\pm \ldots=\alpha^{\prime}$ numbers $\pm \beta^{\prime}$ squares $\pm \delta^{\prime}$ cubes $\pm \varepsilon^{\prime}$ square squares $\pm \ldots$
we have to add or subtract the terms sharing the same eidos in order to reduce the problem to the case in which a species equals another species, that is, an equation of the type $a x^{n}=b x^{m}$.

To solve the problems, Diophantus often chooses some quantities involved in the statement of the problem. For instance, in the previous problem, he chooses the difference equal to 60 . Then he chooses the root of one of the sought-after numbers equal to the arithmos, which plays the role of an algebraic unknown. The other sought-after number is chosen as the arithmos plus 3 . This leads him to the equation that could be translated into modern symbols as $(x+3)^{2}-x^{2}=60$ where he is able to obtain the equation $6 x+9=60$ and finally the solution $x=8 \frac{1}{2}$. So the sought-after square numbers are the fractional numbers $72 \frac{1}{4}$ and $132 \frac{1}{4}$.

Of course, these are not all the solutions of the general problem. This is why Diophantus' solutions are often seen as incomplete. However, we have to be very cautious at this point. In fact, finding out how to express (in an explicitly verbal or symbolic way) all the couple of members of the species of squares that verify the stated condition in the previous problem (or in another problem) is a rather modern problem and not an ancient one. The problems of the 'Arithmetic Books' of Euclid's Elements are not concerned with the task of describing, in an explicit form, each of the members of a certain class of numbers (e.g. the perfect numbers; see Euclid's Elements, Book IX, proposition 36). Another example can be the 'formula' to produce polygonal numbers found by Diophantus himself. This 'formula' does not describe the elements of a class but produces as many numbers as we want (triangular, square, pentagonal numbers and so on: see Radford, 1995a). By the same token, Diophantus' problem-solving methods do not aim to find nor describe all the solutions of a given problem (except, of course in the cases where the problem has only one solution) but to produce as many solutions as we want. The alleged incompleteness of Diophantus' solutions are relative to our modern point of view. Without taking into consideration Diophantus' own conceptualisation, Diophantus' Arithmetica becomes just a mere compendium of problems solved in a way that 'dazzles rather than delights' and Diophantus himself appears 'unlike a speculative thinker who seeks general ideas' but as someone looking only for 'correct answers' (e.g. this is the case of Kline's perception of Diophantus: see Kline, 1972, p. 143, from whom the quotations were taken). Certainly, for a Babylonian scribe, Diophantus' Arithmetica would be seen as the product of a genuine speculative thinker.

## THE TRACE OF PROPORTIONAL MESOPOTAMIAN THINKING IN GREEK ALGEBRAIC THINKING

We are now ready to technically tackle the first question raised in the introduction of this chapter. The essence of our manoeuvre consists in showing that the functioning
of the algebraic concept of unknown in Diophantus' Arithmetica is too closely related to the functioning of the concept of false quantities of the Babylonian false position method to be regarded as a mere accident. On the contrary, the structural coincidence in both concepts is fully understood on the basis of the idea that the algebraic unknown was conceived as a metaphor of its correlated arithmetical concept -the false quantities. Although there are many structural coincidences between the two concepts, here we shall refer to one of them: reasoning in terms of fractional parts.

Let us refer to problem number 18 from a tablet conserved at the British Museum (BM 85196) that goes back to the ancient Babylonian period. It concerns two rings of silver. $1 / 7$ of the first ring and $1 / 11$ of the second weigh 1 sicle. The first, diminished by its $1 / 7$, weighs just as much as the second diminished by its $1 / 11$ (See Thureau-Dangin, 1938b, p. 46). In modern notations, the problem can be stated as follows: $\frac{1}{7} x+\frac{1}{11} y=1 ; x-\frac{1}{7} x=y-\frac{1}{11} y$.

The scribe's reconstruction of the solution given by Thureau-Dangin (1938a, pp. 74-75) suggests that «the first ring diminished by its $1 / 7$ » is transformed into «6 times the $1 / 7$ of the first ring». By the same token, «the second ring diminished by its $1 / 11$ » is transformed into «10 times the $1 / 11$ of the second ring». The reasoning is then carried out on the above-mentioned fractions (i.e. «1/7 of the first ring» and «1/11 of the second ring»). These fractional quantities are in a 10 to 6 ratio. Therefore, by employing the false position method the scribe assumes 10 for the $1 / 7$ of the first ring and 6 for the $1 / 11$ of the second. He then adds the false assumed values and gets 16. However, he was supposed to get 1. The canonical Babylonian proportional process leads to the question of finding a 'proportional adjusting factor' which, in this problem, corresponds to the inverse of 16 . The scribe finds that the inverse of 16 is $3^{\prime} 45^{\prime}$. To find «1/11 of the second ring», he multiplies the false value (i.e. 6) by the 'proportional adjusting factor', $3^{\prime} 45^{\prime}$, and finds $22^{\prime} 30^{\prime}$. Next, to find $« 1 / 7$ of the first ring», he multiplies $3^{\prime} 45^{\prime}$ by 10 and gets $37^{\prime} 30^{\prime}$. He multiplies $22^{\prime} 30^{\prime}$ by 11 and gets $4^{\circ} 7^{\prime} 30^{\prime}$ the weight of the second ring. He multiplies $37^{\prime} 30^{\prime}$ by 7 and gets $4^{\circ} 22^{\prime} 30^{\prime}$; the weight of the first ring.

Let us now examine the Greek counterpart. In problem 6 of Book 1, Diophantus tackles the problem to divide 100 into two numbers such that $1 / 4$ of the first exceeds $1 / 6$ of the other by $20^{16}$.

This problem cannot be solved by the Babylonian false position method ${ }^{17}$. However, Diophantus' method of solving the problem begins by following the Babylonian pattern seen above: the reasoning is based on the fractions of the sought-after numbers. Diophantus takes the $1 / 6$ of the second part as the unknown (which he calls the arithmos, that is, the number and represents it by the letter V). Thus, the second number becomes 6 times the number. 'Therefore, he says, the quarter of the first number will be 1 number plus 20 units; thus, that the first number will be 4
numbers plus 80 units. We want it so that the two numbers added together form 100 units. Therefore, these two numbers added together form 10 numbers plus 80 units which equal 100 units. We subtract the similar terms: 10 numbers equal to 20 units remain and the number becomes 2 units (...)'. (Ver Eecke, 1926, p. 12-13; my translation). Having found that the unknown is 2, Diophantus finds that the sought-after numbers are 88 and 12 .

Problem 5 of Book 1 of the Arithmetica shows also another example of reasoning performed on fractions of the sought-after numbers.

The structural coincidence between the algebraic concept of unknown and the false quantity of arithmetical, proportional thinking can be traced to a time preceding Diophantus, as we will see in the next section.

## THINKING IN TERMS OF FRACTIONAL PARTS: EARLY HISTORICAL EVIDENCE

The preserved fragment of a Greco-Egyptian papyrus, dated circa the first century and called Mich. 620, contains three mathematical problems with the following being one of them:
'There are four numbers, the sum of which is 9900 ; let the second exceed the first by one-seventh of the first; let the third exceed the sum of the first two by 300 , and let the fourth exceed the sum of the first three by 300 ; find the numbers' (According to Robbins' reconstruction, 1929, p. 325).

Our modern notations allow us to write the problem in question as shown in the rectangle:

$$
\begin{aligned}
& a_{1}+a_{2}+a_{3}+a_{4}=9,900 \\
& a_{2}-a_{1}=\frac{1}{7} a_{1} \\
& a_{3}-\left(a_{1}+a_{2}\right)=300 \\
& a_{4}-\left(a_{1}+a_{2}+a_{3}\right)=300
\end{aligned}
$$

The first part of the solution is not completely preserved, but it can be reconstructed from a kind of tabular arrangement or 'matrix' placed at the end of the solution. It is used to display the calculations and functions as an aid to help solve the problem. The 'matrix', which is comprised of 4 columns divided by a vertical line, suggests that the choice of the unknown, which the scribe represents as $\varsigma$, like Diophantus did in his Arithmetica to designate 'the number', (arithmos), is $1 / 7$ of the first number. It is, therefore, also the same pattern found in Babylonian mathematics.

The first sought-after quantity, which appears at the left of the first column of the table (that is, at the left of the first vertical line; see below), is equal to seven $\varsigma$ (which is an abbreviation of the whole expression «7 numbers»); the second number (found to the left of the second vertical line), is equal to eight $\varsigma$. From that, the scribe finds that the third number is 300 plus fifteen $\varsigma$ and that the fourth number is 600 plus thirty $\varsigma$. The sum of the numbers then is 900 plus sixty $\varsigma$, which must equal 9,900 . The scribe gives the answer 150 , which corresponds to $\varsigma$, and then arrives at the sought-after quantities: the first one is 1,050 , the second one is 1,200 , the third one is 2,550 and the fourth one is 5,100 .

|  | $1 / 7$ | 300 | 300 | 9900 |  |
| :--- | ---: | ---: | ---: | ---: | ---: |
| $\varsigma_{7}$ | $\varsigma_{8}$ | $\varsigma_{15}$ | 300 | $\left[{ }^{\varsigma}\right] 30$ | 600 |
| 1050 | 1200 | 2550 |  | $[5100]$ |  |

(Table appearing in the Mich. 620 papyrus according to the reconstruction of Frank Egleston Robbins, 1929, p. 326).

It is worthwhile to note, at this point, that the separation of numbers into columns, allows the scribe to divide each number into an unknown part (found to the left of the vertical line) and a known part (found to the right of the vertical line). This suggests an explicit and systematic way of dealing with the first literal symbolic algebraic language. Notice that this is basically the same pattern which is used to carry out calculations with symbolic expressions some fourteen or fifteen centuries later (e.g. Stifel, 1544).

Nevertheless, we should note a difference: in the first case - that of the Mich. 620 papyrus - the algebraic language is seen as a heuristic tool (one calculates with symbolic expressions); in the second case - that of late mediaeval and early renaissance mathematics - the algebraic language begins to be seen as a quasiautonomous object leading to a new theoretical organisation (one calculates on symbolic expressions). Many treatises will then begin to display rules on how to carry out calculations on symbolic expressions (see the excerpt from Stifel's Arithmetica Integra, page 239).


## THE BABYLONIAN NAÏVE GEOMETRY

So far, our work has dealt with the numerical origins of algebraic ideas. We claimed that the Babylonian mathematical proportional thinking provided the conceptual basis for the emergence of elementary numerical algebraic thinking. There is, however, another Babylonian mathematical current which leads to another kind of 'algebra'. In fact, J. Høyrup, through an in-depth analysis of the linguistic sense of the terms occurring in Babylonian mathematical tablets, has suggested that a large part of problems was formulated and solved within a geometrical context, using what he calls a 'cut-and-paste geometry' or 'naïve geometry' (e.g. Høyrup, 1990, 1986, 1993a, 1994). In particular, this is the case of problems that have been traditionally seen as problems related to 'second-degree equations'. We cannot discuss here, at length, the Babylonian Naïve Geometry. For our purposes, we shall just look at two examples of the new interpretation of the second-degree Babylonian algebra (see also Radford, 1996a).


Problem 1 of the tablet BM 13901 deals with a square whose surface and a side equal $3 / 4$. The problem is to find the side of the square ${ }^{18}$.

Then, the scribe cuts the width 1 into two parts and transfers the right side to the bottom of the original square.


Now, the scribe completes a big square by adding a small square whose side is $1 / 2$. The total area is then $3 / 4$ (that is, the area of the first figure) plus $1 / 4$ (that is, the area of the added small square). It gives 1 . The side of the big square can now be


The basic idea is that of bringing the original geometric configuration to a square-configuration. However, not all the problems can be solved by cut-and-paste methods alone. For instance, problem 3 of tablet BM 13901 deals with a square whose area less a third of its area plus a third of its side equals $20^{\circ}$.

fig. 1

fig. 3

fig. 2

fig. 4

As Høyrup suggests, the procedure followed by the scribe is that of removing a third of the original square (fig. 1). After that, a rectangle of Width 1 is projected over the side, obtaining a configuration like fig. 2, A third of the projection is kept, which leaads to the next configuration (fig. 3).

Finally, 'in order to obtain a normalised situation (square with attached rectangle), the vertical scale is reduced with the same factor as the width of the square, i.e., with a factor $2 / 3(\ldots)^{\prime}$ (Høyrup, 1994, p. 13). Now the scribe can apply the procedure which solves problem 1 from Tablet 13901 discussed above.

We have discussed this last example because it shows how proportional thinking also permeates the cut-and-paste geometrical thinking. Changing the scale is, in fact, a proportional idea.

On the other hand, it is important to note that cut-and-paste procedures involve known and unknown quantities in a very particular way. Firstly, in Naïve Geometry, semantics plays a strong role throughout the problem-solving procedure. In Numerical Algebra, rooted in proportional thinking, the original Semantics is lost once the equation is reached (cf. Mich. 620 and Diophantus' Arithmetica). Secondly, in Numerical Algebra the unknown is directly involved in the calculations. For instance, in problem 6, Book 1, mentioned above, (and translated here into modern notations in order to abbreviate our account), Diophantus performs the following calculations: $6 x+4 x+80=100$, and gets $10 x+80=100$. He then operates on the unknown within a side of the equation. In other problems he performs calculations involving the unknown in both sides of the equation (e.g. Book 1, problems 7-12. In problem 7, for instance, Diophantus solves the equation $3 x-300=x-20$. See Ver Ecke, 1926, p. 13 ff.).

In contrast, the algebraic concepts rooted in cut-and-paste geometry (i.e. Naïve Geometry), do not seriously involve the unknown quantities in direct calculations (see Radford, 1995b, footnote 6). For instance, in problem 1, tablet 13901 seen above, the projection, and not the unknown-side, is halved.

The previous discussion suggests that very different conceptualisations underlie the Algebra embedded in Naïve Geometry and the Numeric Algebra. As seen above, their methods and their concepts are essentially different. The difference can also be seen in terms of problems. Most of the problems in Naïve Geometry deal with problems beyond the tools of first-degree or homogeneous algebra. However, it is possible to detect some interactions between both kinds of algebras. In fact, problem 27, in book 1 of Diophantus' Arithmetica, is a classical problem stated in the realm of Naïve Geometry. Although stated in a numerical form, Problem 27 has the traces of its old geometrical formulation (see Høyrup, 1985, p. 103) and appears then as a numerical reconceptualisation of the old cut-and-paste technique (we shall return to this point in the section, below).

However, some connections between the cut-and-paste technique and first degree algebra could happen even within Babylonian mathematics themselves: this is what the solving procedure of problem 8 of tablet 13901 suggests. In fact, using Høyrup's notations, we can represent the squares by $\mathrm{Q}_{1}$ and $\mathrm{Q}_{2}$ and their side by $\mathrm{s}_{1}$ and $\mathrm{s}_{2}$, respectively. The problem can then be formulated as follows:

$$
\begin{gathered}
Q_{1}+Q_{2}=21^{\prime} 40^{\prime \prime} \\
s_{1}+s_{2}=50^{\prime}
\end{gathered}
$$

The solution begins by taking the half of the sum of the sides. Taking a new side which is equal to the half of the sum of the sides is a recurrent idea in many of the Babylonian geometrical problem-solving procedures. The 'half of the sum' idea also appears in Babylonian numerical problem-solving procedures (see our discussion of tablets VAT 8389 and 8391, in Radford, 1995b). This suggests an early link between geometrical and numerical algebraic ideas.

## LANGUAGE AND SYMBOLISM IN THE DEVELOPMENT OF ALGEBRAIC THINKING

In this section, we would like to make a first attempt at exploring the problem of the development of early algebraic thinking with regards to language and symbolism. First of all, it is important to stress the fact that it is completely misleading to pose the problem of the development of algebraic thinking in terms of a transcultural epistemological enterprise whose goal is to develop an abstract and elaborate symbolic language. Indeed, language and symbols play an important role in the way
that we communicate scientific experiences. Nevertheless, their use is couched in sociocultural practices that go beyond the scope of the restricted mathematical domain. A more suitable approach to the study of the relationships between symbols and language on the one hand, and the development of algebraic thinking on the other, might thus be to analyse language and symbolism in their own historical sociocultural semiotic context. The case of Mesopotamian scribes may help us to illustrate this point. In this order of ideas, it is worthwhile to bear in mind that the oldest tablets suggest that they were first of all seen as a complementary tool to record information. The signs that formed a 'text' in the proto-literate periods known as Uruk IV and III ( $3300-2900 \mathrm{BC}$ ) reflected the key words of the messages inscribed on the clay tablet without any «syntactic relations» (Nissen, 1986, p. 329). The meaning was often suggested by the pictographic form of the sign; this is the case, for instance, of the sign SAG, 'head', where the sign shows the profile of a head with the eye, the nose and the chin ${ }^{19}$. The archaic pictographic writing was later replaced by cuneiform writing - in all likelihood related to the needs arising from the transactions of the Sumerian administrative bureaucracy and the emergence of a new technological artefact: an oblique new stylus leaving the impression of 'nails' (cunei) on the clay. The cuneiform writing was increasingly used to reproduce the oral language and when the Semitic Akkadian language became the spoken language, Akkadian was written following the cuneiform syllabic tendency. These changes brought about two important modifications: first, the way to convey the meaning of the text changed radically; it no longer relied upon pictographic insights. Second, there was a radical diminution in the number of signs. While by 3200 BC a writing system had some 30 numerical signs and some 800 non-numerical signs (Ritter, 1993, p. 14), by the first half of the second millennium Akkadian could be written with some 200 cuneiform signs reproducing the spoken language with very little ambiguity (Larsen, 1986, pp. 5-6).

This point connects us with two of the most salient particularities of ancient Mesopotamian mathematical texts which have very often puzzled historians and mathematicians who attempt to understand Babylonian mathematics from the perspective of modern mathematics: firstly, the texts do not show any 'specialised' or 'mathematical' symbols to designate the unknowns; secondly, the texts do not display grandiose eloquence concerning the explanation of the problem-solving methods followed to reach the answer of the problem. In fact, concerning this last point, with the very moderate exception of the Textes Mathématiques de Suse, the scribe limits him/herself20 to indicate only the calculation to be carried out to solve the problem (if not, as is the case of many problems, to simply mention the answer) ${ }^{21}$.

The reason of what we see as a «silence» is not the absence of mathematical 'specialised' signs like ' $x$ ', ' $y$ ' (signs which are but merely inconceivable and unnecessary within the realm of Babylonian thought). This «silence» is due neither to an
incapacity to write a mathematical expression - and eventually, in the Old Babylonian period (2000-1600 BC), any algebraic one - but an accepted sociocultural way to transmit and record the information which demarcates the frontiers of what has to be written and how ${ }^{22}$.

To clarify this last point, remember that the tablets bearing some «algebraic» content were, in all likelihood, not produced in professional activities (e.g accounting, book-keeping, surveying) but in the Scribal Tablet-Houses, that is, in the institutions created to train the future scribes. The clay tablets were, without a doubt, privileged tools in the teaching practices. However, they do not merely reflect the content of the teaching but they also mirror the method of teaching and its symbolic forms as well. Concerning the method of teaching, some tablets show that instruction was heavily based on the mastering of cuneiform writing; another, no less important point, was to prepare oral recitations (that were under the supervision of the second person in the hierarchy of the Tablet-House, the sesgal or «elder brother», the assistant of the «father» of the School). In the written component of the scribal training, students had to copy what the teacher said or did. In many cases, the clay tablet shows a sentence (or a passage of a literary work) on one side and, on the other side, with a less confident calligraphy, a copy (visibly the student's copy) of the given sentence (see Lucas, 1979, p. 311 ff .). It is not difficult to imagine the enormous difficulties that young students had to face trying to master the stylus and the rules of cuneiform writing. In a tablet, known as 'In the Prise of the Scribal Art', we can read: 'The scribal art is not (easily) learned, (but) he who has learned it need no longer be anxious about it.' (Sjöberg, 1971-72, p. 127). Another very well known text, 'Examination Text A', that dates back to the Old Babylonian period, deals with the examination of a scribe in the courtyard of the Tablet-House. Besides the precise idea that this text provides us with an examination scene, the text uncovers some accepted teacher-student relationships such as symbolic forms emerging in the dynamic of the scribal school. The examination covers topics such as the translation from Sumerian (by that time, a dead language, as mentioned in footnote 22) to Akkadian and vice-versa, different types of calligraphy, the explanation of the specialised language (or jargon) of several professions, the resolution of mathematical problems relating to the allocation of rations and the division of fields. When the teacher starts asking questions about the techniques employed in playing musical instruments, the candidate gives up the examination. He complains that he was not sufficiently taught. Then the teacher says:
> 'What have you done, what good came of your sitting here? You are already a ripe man and close to being aged! Like an old ass you are not teachable any more. Like withered grain you have passed the season. How long will you play around? But, it is still not late! If you study night and day and work all the time modestly and without arrogance, if you listen to your colleagues and teachers, you still can become a seribe! Then you can share the scribal craft which is good fortune for its owner, a good angel leading you
a bright eye, possessed by you, and it is what the palace needs.' (Quoted by Lucas, 1979, p. 314).

In the previous passage we find mentioned explicitly the intensive work that is expected to be done by the scribe. More importantly still, we also find in the previous quotation clear instances of a symbolic form that emerges as a particular social student-teacher relationship. The symbolic form is that which forces the student to show some specific attitudes: s /he is supposed to be modest and without arrogance as well as a good listener. By the same token, the same symbolic form allows the teacher to say what he said in the text. More eloquent is a passage of another text called 'Schooldays' (Kramer, 1949) in which the scribe tells us that he is caned by his teachers for doing unsatisfactory work.

This symbolic form was supported by the scribes' relatives, who encouraged their sons to follow the teachers' requirements. In a tablet, the father says to his son: «Be humble and show humility before your school monitor. When you make a show of modesty, the monitor will like you.» (Quoted in Lucas, 1979, p. 321).

We do not need to go further into the detail. It suffices to say that the aforementioned scenes clearly suggest that the teaching model relied heavily upon an incontestable imitation model concentrated on the proving of, among other things, the replication of passages from literary texts and procedural and computational mathematical skills. In this sense, the formal, semiotical content of the mathematical texts are but the mirror of the sociocultural web of relations on which schooling and, in general, all Babylonian social activities were based. To expect that the scribe produce a mathematical text containing an analytic algebraic explanation of the procedure is to expect him/her to do something that was out of all the enculturation with which s /he was provided in the Tablet-House. Explanation is, in fact, a sociocultural value, not a transcultural item ${ }^{23}$. Of course, Mesopotamian explanations did exist. Nevertheless, they were not largely based upon deductive analytical principles but on metaphorical ones. A survey of literary and mythological texts is very enlightening in this respect (see for instance Kramer, 1961a, 1961b).

The case of Diophantus was completely different. From the 5th century BC, arguing and explaining were two important social activities that shaped Greek thought. On the other hand, Diophantus had, at his disposal, an alphabetical language and a very well-established socially accepted system of producing and transmitting information out of which books attained an autonomous life ${ }^{24}$.

Concerning algebra, even though Diophantus could use letters to represent the unknown he did not. As it is well established, the use of letters in Diophantus' Ar ithmetica stand for an abbreviation of the word - hence, contrary to our modern use, as merely an economic writing device.

Even though he was not probably the first to do this, as suggested by the papyrus Mich. 620, one of Diophantus' most important semiotic contributions is to be found
in the peculiar use of the expression «arithmos». We previously said that he accomplished a transcendental act by including indeterminate units into the realm of calculations. But this is not to what we are referring now. Here we refer to the use of this expression of the Greek language to designate the algebraic experience that the concept of unknown carries out with itself. In the Arithmetica and in the specific definition of arithmos we found this experience «uttered». While Mesopotamian scribes used the semiotic experience of everyday life and used words like the length or the width of a rectangle to handle their unknowns (using expressions like 'as much as' or 'the contribution of the length' to refer to what we now call the coefficients of a polynomial), Diophantus transformed the word «arithmos» into a more general concept. Because of its generality, this concept could apply to a great variety of situations. The «arithmos» thus became a genuine algebraic symbol ${ }^{25}$. Through this symbol, a numerical assimilating process of (some) geometric algebraic techniques was undertaken, leading to a new formulation of old problems and the rising of new methods in solving new versions of old problems. In turn, the new methods also allowed the Greek calculators to tackle new problems: Diophantus' Arithmetica contains problems which do not have any corresponding version in the algebra embedded in the Naïve Geometry, as is the case of problems concerning square-squares and cubo-square categories. Nonetheless, it is important to note that the great generalising enterprise was supported by the socially committed Greek conception of mathematics (further details in Radford, 1996b).

The «arithmos»-symbolism played a particular role in a different semantisation of problems in the problem-solving phase and led to a more systematic and global treatment of problems. In order to illustrate this ideas more precisely, let us consider problems 27-30 of Book 1 of Diophantus' Arithmetica. There is no doubt that these problems belong to an early tradition. According to Høyrup (1994, pp. 5-9), these problems, formulated, of course, in a geometrical language, go back to the surveyors' Mesopotamian mathematical tradition of the late 3rd millennium BC. These problems constitute part of a stock of problems that played a role in the rise of the Old Babylonian mathematics.

Using Høyrup's notations (modified very little), these problems can be stated as follows:

$$
\begin{aligned}
& \text { 27) } s_{1}+s_{2}=\alpha \\
& s_{1} \cdot s_{2}=\beta \\
& \text { 28) } s_{1}+s_{2}=\alpha \\
& Q_{1}+Q_{2}=\beta \\
& \text { 29) } \begin{array}{l}
s_{1}+s_{2}=\alpha \\
Q_{1}-Q_{2}=\beta
\end{array} \\
& \text { 30) } s_{1}-s_{2}=\alpha \\
& s_{1} \cdot \mathrm{~s}_{2}=\beta
\end{aligned}
$$

In Mesopotamia, problem 27 was solved through the cut-and-paste technique. Diophantus does not use this technique directly. However, as was mentioned above, there are traces of the geometrical ideas in the new numerical algebraic problemsolving procedure (see Høyrup, 1985, p. 103-105; Radford, 1996a). Problem 28 appears in tablet BM 13901, problem No. 8 and in YBC 4714, problem No. 1. There is no direct Mesopotamian evidence of problem 29. (However, according to Høyrup, it is possible that originally problem 9 could be contained in a missing part of Text V of the Textes Mathematiques de Suse).

While the cut-and-paste technique does not apply to all those problems in the same way, the revolutionary concept of the numerical algebraic unknown provides a similar way of tackling all these problems. The idea is to take the half of the sum of the sides added (or subtracted) by a certain number. (As was mentioned above, this is a procedure at the very cross-roads of geometrical and numerical ideas: see Radford, 1995b). Thus, in problems 27, 28 and 29, Diophantus represents the soughtafter numbers as «10+1 'number'» and «10-1 'number'» (in modern notations, it means $10+x$ and $10-x$ ). In problem 30 , the sought-after numbers are chosen as «1 'number' +2 » and «1 'number' -2 ».

On the other hand, symbolism shifts the thinking from the figures themselves and makes it possible to carry out operations that do not have any corresponding sense with the initial statement of the problem. This is not the case of the cut-and-paste-technique, where it is possible to distinguish the sequence of geometrical transformations and its link with the original configuration.

The introduction of the «arithmean» symbolic language provides an autonomous way of thinking -autonomous with regards to the context of the problem. In contrast, it requires a new semantisation that has its own difficulties. This is to what Diophantus probably refers when he says, at the beginning of Book 1: 'Perhaps the subject will appear rather difficult, inasmuch as it is not yet familiar (beginners are, as a rule, too ready to despair of success)' (Heath, 1910, p. 129). Indeed, many of Diophantus' Arithmetica Scholia or comments (cf. Allard (ed.), 1983) deal with detailed explanations about the elementary symbolic treatment of the problems. Some of them use a geometric context to give a sense to the solving procedure. This is the case of a scholium of problem 26 from book 1, which explains the solution of the equation $25 \mathrm{x}^{2}=200 \mathrm{x}$ in terms of two rectangles of the same width. (Allard (ed.), 1983, p. 727).

## SOME REMARKS FOR TEACHING

The historical itinerary that we have followed in this work provides some information about past trends in the historical construction of very early algebraic thinking. These trends can help us to better understand the deep and different sociocultural and cognitive meanings of algebraic thinking and provide teachers
with new paths to teach algebra in the classroom. In particular, our historical epistemological itinerary can shed some light on the didactic problem of how to introduce algebra in school. Of course, as we have pointed out in previous works, we do not claim that we must follow the historical path. History cannot be normative for teaching. There are social and cultural aspects in the development of algebra that we cannot reproduce in the classroom. Furthermore, these aspects may not be necessary for our purposes. There are other aspects which could be more interesting, such as the following:
(1) The epistemological meaning of algebra, i.e., that of mathematical knowledge developed around a problem-solving oriented activity, can provide some insights about the way of introducing and structuring algebra in the school; and this us to re-think, within a new perspective, the role of problems in teaching algebra.
(2) However, our study of Mesopotamian and Greek algebra clearly suggests that the specific form in which each algebra was conceived was deeply rooted in and shaped by the corresponding sociocultural settings. This point raises the question of the explicitness and the controlling of the social meanings that we inevitably convey in the classroom through our discursive practices.
(3) Our epistemological analysis suggests that algebraic language emerged as a tool or technique and later evolved socio-culturally to a level in which it was considered as a mathematical object. Usually, in the modern curriculum, algebraic language appears from the beginning as a mathematical object per se. Taking into account this result, it is possible to make some changes in the way of introducing algebraic language in the classroom.
Following some insights of our historical studies of the development of algebra (see also Radford, 1995c), we elaborated a teaching sequence whose goal was to introduce students to algebraic methods based on different semiotic levels which culminates with the progressive introduction of symbolletters (Radford and Grenier, 1996).
(4) A fourth point that we can consider, from a teaching point of view, could be the historical movement of arithmetisation of geometric algebra (which occurs in a recursive way through the history of algebra until the pre-modern epoch). Until now, the algebra embedded in cut-and-paste Naïve Geometry does not belong to the modern curriculum of mathematics; inspired by this historical research we were able to successfully develop a teaching sequence in the classroom that, through cut-and-paste geometry, has been allowing High-School students to re-discover the formula for second degree equations (Radford and Guérette, 1996).
(5) Another aspect to consider could be the link between proportional thinking and algebraic thinking. Ratio and proportions are not presented, in modern school curricula, as being linked to algebraic thinking in the way that history suggests it happened. It seems to me that the historical metaphorical link between proportions and algebra is another interesting element to be explored in the teaching of mathematics.

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## NOTES

1 This research was supported by a grant from FCAR 95ER0716 (Québec) and a grant from the Research Funds of Laurentian University (Ontario).
2 A collection of problems from Tell Harmal shows how to calculate the total price of some current items used in commercial business using silver as the 'monetary' unit of goods. Such items included sesame, dates and lard; see Goetze, 1951, p. 153.
${ }^{3}$ Some such 'non-practical' problems will be discussed in this paper. One of them is found at the end of this section.
4 As pointed out by Damerow, numbers in Mesopotamian mathematics are not abstract entities; they are attached to specific contexts (e.g. weight of objects, grain production). A relative detachment from the context is suggested, however, by the tables of reciprocals in the early Old Babylonian period (ca. 2000 BC) (see Damerow, 1996, particularly pp. 242-246).
5 In what follows, we will represent the numbers in base 60 ; for instance a number represented by $2^{*} 3^{\circ}$
$5^{\circ} 6^{\prime} 11^{\prime}$ means

$$
2 \times 60^{2}+3 \times 60+5+\frac{6}{60}+\frac{11}{60^{2}}
$$

${ }^{6}$ It is important to note this nuance: Thureau-Dangin was very careful with the philological aspects of the translations: in contrast, the interpretation of such translations very often had recourse to modern algebra (see Høyrup, 1996. 7-9). For instance, when discussing one of the problems of the tablet VAT 8389, Thureau-Dangin refers to the equation $40^{\circ} x-30^{\circ} y=8^{\prime} 20$ and says: 'Le scribe ne formule pas cette equation, mais il l'a certainement en vue'. (The scribe does not formulate this equation, however, he certainly has it in view). (Thureau-Dangin, 1938b, p. xx).
${ }^{7}$ As we suggested in the previous section, false quantities were generated as metaphors of true quantities. Here, a new metaphor would be used by the scribes to generate a new concept -that of algebraic unknown.

* We will use modern algebraic notations in some passages of our paper in order to have an idea of the problems and the methods of solution under consideration. Modern notations are not used as structural artefacts in our interpretation of ancient mathematics.
* Stated, of course, in a Babylonian 'natural' context (e.g. a stone and its weight).
${ }^{10}$ The Textes Mathématiques de Suse were translated by Bruins and Rutten (1961). In these Textes, there are two problems called problems $A$ and $B$ of text VII, related to the width and the length of a rectangle. In a recent re-interpretation made by Høyrup (1993b), Problem A of Text VII concerns the equation that, translated into modern notations, reads as follows: $\frac{1}{7}\left(x+\frac{1}{4} y\right) \cdot 10=x+y$, where x represents the length and y the width of a rectangle; nevertheless, our modern translation does not
distinguish some of the different conceptualisations between ancient numerical operations and the modern ones. Keeping this in mind, some of the steps of the translating solution include the following calculations:
$1 / 7[(4-1) x+(x+y)] 10=4 \cdot(x+y), 3 x \cdot 10+(x+y) \cdot 10=28 \cdot(x+y), 3 x \cdot 10=18 \cdot(x+y), x \cdot 10=6 \cdot(x+y)$ Then, the scribe chooses $x=6$ and $10=x+y$ and he arrives at $y=4$. (For a complete translation see Høyrup, 1993b). One of the points to be stressed here is the fact that the calculations showed in the previous sequence are based on an (implicit) analytical procedure: the scribe's calculations comprise the unknown quantities $x, y$ (as seen in their own mathematical conceptualisation); the unknown quantities are considered and handled as known numbers, even though their numerical values are not discovered until the end of the process.
"J 'J'ai mangé les deux tiers du tiers de ma provende: le reste est 7. Qu'était la (quantité) originaire de ma provende?' (Thureau-Dangin, 1938b, p. 209).
12 The problem of the transmission of algebraic knowledge and the sources of Greek (numerical and geometrical) algebra has been studied by J. Høyrup in terms of sub-scientific mathematical traditions. (see, e.g., Høyrup, 1990a).
13 The problem of whether a conceptual organisation is scientific or not is evidently a cultural decision. In the case of the Alexandrian algebra of the 3rd century B. C., it is hardly possible to ascribe to Diophantus the whole merit of building such a theory (Klein, 1968, p. 147). Nevertheless, we can say that, in all likelihood, his contribution was conclusive to this enterprise.
Freeman, 1956, fragment 4, p. 74.
When reading this quotation we have to keep in mind that Heath's translation is tainted by a modern outlook. Diophantus never spoke about 'negative terms'. Diophantus spoke rather of leipsis, i.e. of deficiencies in the sense of missing objects; this is why we might remember that a leipsis does not have an existence per se but was always related to another bigger term of which it is the missing part.

It would be teleologically erroneous to think that the non-alphabetical cuneiform language of the Old Babylonian period was a delaying factor to the emergence of algebraic symbols in the Ancient Near East. The cuneiform language was a marvelous tool to crystallise the experiences, the meanings and conceptualisations of the people that spoke Sumerian and later Akkadian. Alphabetic languages correspond to new ways to see, describe and construct the word. One language is not stricto sensu better than the other: they are just different (For a critique of the alphabetical ethnocentric point of view, see Larsen 1986, pp. 7-9).
25 In the light of this discussion, it is easy to realise that it is an anachronism to see the development of algebra in terms of Nesselmann's three well-known stages: rhetorical, syncopated and symbolic (Nesselmann, 1842, pp. 301-306); further details in Radford, 1997


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