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CRAFTING AN ALGEBRAIC MIND: INTERSECTION FORM HISTORY AND THE CONTEMPORARY MATHEMATICS CLASSROOM

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Introduction

The working group sessions were organized around three activities. The first one dealt with the history of algebra. A text was distributed to all participants who were then asked to answer the questions appearing at the end of each section. This report reproduces the said text and includes the salient small group comments as well as excerpts from our discussions. These comments will be easily noticeable for they will be in the same font as the present introduction.

For the second activity, Luis Radford summarized a part of his classroom research in which he, along with some other teachers, planned a teaching sequence having in mind some historical considerations. A video showing some of these students at work was then presented. A copy of their productions and a transcript of their discussions were also made available to the workshop participants. These documents are not included in the report. The second part of our report will, nevertheless, summarize our discussion that revolved around the semiotic and conceptual limits and possibilities of two models (the use of the two-plate-scale model and the two-container model) intended as pedagogical artefacts to help novice students deal with linear equations.

The last activity consisted of sharing our views on the use of history in the classroom.

Part 1: Mediaeval and Renaissance algebra

I. THE NUMERICAL TRADITION IN MEDIAEVAL ALGEBRA FIRST DEGREE EQUATIONS

Liber augmentis et diminutionis

The text *Liber augmentis et diminutionis*, a mathematical text of a Hindu origin and dealing with first degree equations, was translated by Abraham ben Ezra during the 11th Century. The author of the text seems to have been Ajjūb al-Basrī, "the first Arab who mastered the Hindu technique of solving equations." (Hughes 1994, p. 31) We find in this text some problems solved using different techniques. Two of them are false position and *regula infusa*.

Regula infusa consists of operations performed on the "positive coefficient" of the unknown and the constant term. Of course, there are no definitions; the *regula infusa* is learnt through examples. The problems involve fractional parts of the unknown and, in the complex variants, sums of linear expressions and fractional parts of them. The main problem consists in bringing all the fractional parts to integer terms.

The problems solved by *regula infusa* arose probably in the context of division of fortunes left by a person to the family (the terms of the division of the fortune being stated through fractional parts). Certainly, the problems in the *Liber augmenti and diminutionis* go beyond the requirement of practical situations. This mathematical text and its *regula infusa* method have to be seen as a pedagogical attempt to teach techniques to solve linear equations whose difficulty resided in the handling of fractional parts.

One of the problems is the following (Libri 1838-1841, p. 321):

English	Français
A treasure is increased by a third [of it]. Then a fourth of this aggregated is added to the first sum. The new sum is 30. How much was the treasure originally?	Un trésor (censo) est augmenté de son tiers. Alors une quatrième partie de cet agrégat est ajoutée à la première addition. La nouvelle addition est 30. Combien le trésor originel était-il ?

In modern notations, the problem is the following:

$$t+\frac{1}{3}t+\frac{1}{4}\left(t+\frac{1}{3}t\right)=30$$

Census (treasure) and *res* (thing) were two different terms used in mediaeval algebra. In later texts the relation between them became standardized. It was that of a number to its square (the square of the *res* was the census). Here, however, their relation is different. We are symbolizing here treasure as t.

The solution is as follows:

> Assume *res* and add its fourth to it and you have a *res* and a fourth of a *res*. How much less will bring *res* and fourth of *res* to a *res*? You will find that what that is, is its fifth. Subtract therefore from thirty its fifth and twenty-four will remain. Then take the second res and add it to its third part and you will have *res* and its third. How much less therefore will bring *res* and third of *res* to a *res*? You will find in truth that how much that is, is its fourth. Therefore subtract from twenty-four its fourth and 18 will remain. (Libri, op. cit.)

As we see, the text starts by dividing the original problem into two sub-problems. By taking $t+\frac{1}{3}t$ as a *res*, that is, in modern notations, by making x = $t+\frac{1}{3}t$, the first sub-problem is:

$$x+\frac{1}{4}x = 30.$$

Then, the left side of the equation $x+\frac{1}{4}x = 30$ gives $\frac{5}{4}x = 30$ and in order to reduce this to one *x*, $\frac{1}{5}$ of $\frac{5}{4}$ of *x* has to be subtracted from each side.
This gives $\frac{5}{4}x-\frac{1}{5}\cdot\frac{5}{4}x = 30-\frac{1}{5}30$, that is, $x = 30 - 6 = 24$.

Now the text deals with the equation $t+\frac{1}{3}t = 24$. The method is the same. The new equation is hence $\frac{4}{3}t = 24$. To get one *t*, we need to subtract $^1/_4$ of $\frac{4}{3}t$ from $\frac{4}{3}t$. Thus, the problem is now: $\frac{4}{3}t-\frac{1}{4}\cdot\frac{4}{3}t = 24-\frac{1}{4}24$, that is: $t = 18$.

Exercise 1:
Solve the following problem using the *regula infusa* method:

> A treasure is increased by a third of itself and four dragmas. Then a fourth of this sum was added to the first sum. The result was forty. (Libri 1838-1841, p. 322).

Question 1:
1.1 What are the key algebraic concepts used in the *regula infusa* method?
1.2 The concept of equality is one of the more important concepts in algebra. What are the properties of the equality that were required in the *regula infusa* method?

Question 2:
Problems such as the first one shown above were also solved by the method of *false position*. To solve the first equation $x + \frac{1}{4}x = 30$, and in order to avoid fractional parts, one may assume that $x = 4$. The act of assuming a value for x was referred to as making position, and since the assumption was known to be false a priori the method was called false position.

If $x = 4$ is substituted in the equation, one gets 5 instead of 30. So, in those cases, the author of *Liber augmentis and diminutionis* asks the same type of question: "Tell me then: by how much must you multiply 5 until you get 30". The answer is obviously by 6. Hence the answer is 6 times the false position, that is $x = 6x4 = 24$.

- Using the method of false position, solve the second equation involved in the first problem, namely, $t + \frac{1}{3}t = 24$

Question 3:
Do you think that the method of false position is "more algebraic" than the *regula infusa* method? Explain!

DISCUSSION

The discussion was mainly related to question 3.

In a certain way, the *regula infusa* method is more restrictive than the method of *false position.* Indeed, from the point of view of the scope of the method, this first method applies to a more specific type of problem, that is, those problems whose equations are of the general form $x + \frac{x}{n} = k$.

However, the method of *false position* seems less algebraic than the other method. The equality is treated differently in these two methods. In the *regula infusa*, the equality is seen more like a statement of the equivalence between two ways of seeing the same quantity. The manipulations used to solve the problem show that k is interpreted as the *n+1 nth* parts of the *unknown*. In the method of *false position*, the "equality" indicates the result of an operation. The result of the calculation $x + \frac{x}{n}$ gives k. So, one tries to calculate the value of this operation by substituting the unknown by a concrete number. If one doesn't get the expected result, one modifies the number, taking into account what happened in the first calculation. The notion of equality used in *regula infusa* then seems closer to the one used in algebra.

Some noticed that, in general, students don't feel at ease with the method of *false position*.

II. THE GEOMETRIC TRADITION IN MEDIAEVAL ALGEBRA

Second degree equations

Abû Bekr's Liber Mensuratonium

One of the interests of Abû Bekr's *Liber Mensuratonium* (ca. 9^{th} Century) for the study of the conceptual development of algebra is that this book contains several problems solved using two different methods. One of the methods is referred to as belonging to the people of *al-gabr* (that is, the people of algebra: to be read, people practicing algebra). The other method does not bear any specific name. Following Høyrup (1990) the method shall be identified as belonging to the "cut-and-paste geometry".

In what follows, one problem from Abû Bekr's *Liber Mensuratonium* is presented. It comes from the edition made by Busard in 1968, an edition based on the translation to Latin by Gerardo de Cremona in the 12th Century.

Problem 25

> "If in truth he will say to you: the area is 48 and you have added two sides and what results was 14, what then is the quantity for each side?

The solution according to the "cut-and-paste geometry" is the following:

> This will be the way to solve it: when you halve fourteen, it will be seven, the same then multiply by itself and what will result, will be 49. Then deduct from it 48 and one will remain, of which obtain the root, which is one; if you will have added to it half of 14, that what will result will be the longer side. And if you will have deducted it from the half of 14 that what will result is the shorter side."

Comments:
Although it is not explicitly stated, the problem deals with a rectangle. The question is to find the length of the sides.

The colloquial style of the text evokes unambiguously the oral setting in which the mathematical discussions were held and makes us sensitive to the characteristics of teaching in oral traditions. In all likelihood, the solution was accompanied by some drawings that were not inserted in the text. The text, indeed, rather has the value of an aide-mémoire, and not that of an autonomous item in the mediaeval market of manuscripts' circulation.

As to the solution, the idea underlying the problem-solving procedure is to start from a square whose side is half of the sum of the sides of the sought after rectangle, that is to start with a square having a side equal to half of 14. This procedure is a kind of false position method in that the problem-solver knows that by taking the side as equal to half of 14 some adjustments will be required later.

Question 1:
Try to make sense of the solutions using the following reconstructed sequence of drawings:

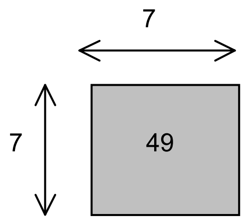


fig. 1

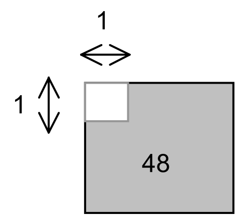


fig. 2

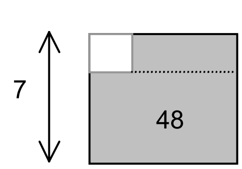


fig. 3

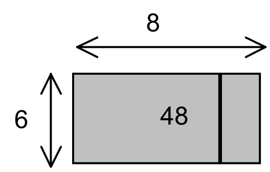


fig. 4

The text presents another method to solve the same problem, which is the method of algebra. As we will see, the algebraic solution is given following a kind of a discursive protocol which seems strange to us. But not to Abû Bekr's students! The text does not use letters as we do now. However an abstract terminology was already in place. Two of the basic terms were the thing (*res* in Latin) and its square –*census*. We have edited the text using numbered lines to make references to the text easier.

Question 2:
Read the algebraic solution (lines 1-7) and find out what is the equation solved in the text. Provide some explanations of the following key terms:

- To confront
- To restore

1. There is another way for it, according to algebra, that is: put one side as one thing (*res*) and the second 14 less thing.
2. Then multiply the thing with 14 less thing and the result will be 14 things less the *census*.
3. Confront then (*oppone ergo*) the area, that is, you restore 14 things with census subtracted and add [the restoration] to 48.
4. It will therefore be census and 48 dragmas that equal 14 things.

5. You will therefore have after the confrontation *censo* and 48 dragmas that equal fourteen things.
6. Then do according to what was given in the fifth question of algebra which is, when you halve 14 things and multiply them by themselves and deduct from it 48 and you will obtain the root of that which remains.
7. Afterwards, if you will have added half of 14, that which will result will be the longer side and if you subtract it, it will be the shorter one.

Question 3:

- Write in modern symbolism the equation in line 5. Then follow the instructions given in line 6.
- Follow the instructions given in line 7 and verify the result. We will come back later to the justification of this algorithm, when we shall read an excerpt of Al-Khwarizmi's work.
- What is the role of number 48 in the problem?

Question 4:
We saw that the unknown was represented by *res* (thing). Did Abû Bekr operate on/with the unknown? Explain!

Question 5:
What are the main differences in both methods, the cut-and-paste geometry and the algebraic one? Explain the differences in terms of the kind of symbolization and meanings on which each solution relies.

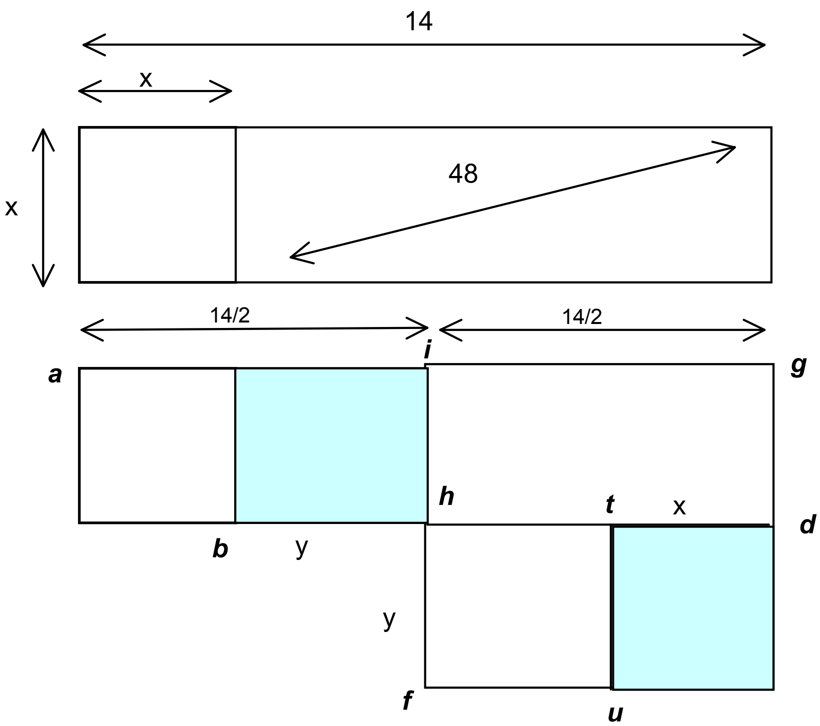
Note: Abû Bekr's problem 25 has a long story. It appears in a different formulation and conceptualization in Diophantus' *Arithmetic* (see Radford 1991/92, 1993, 1996) at the end of Antiquity. But in all likelihood the problem was known and solved by Babylonian scribes around 1600 B.C.

Al-Khwarizmi and the geometric proof of canonical equations

Al-Khwarizmi gave, in an explicit manner, a geometrical proof of the procedure solving second degree equations. He distinguished three kinds of objects: numbers, the unknown (*Jidhr, i.e. root,* which the abacus masters will translate as *thing*) and *Mal* (the square of the unknown). One of the six types of equations which he studied was: squares and numbers equal to roots. Al-Khwarizmi says:

For instance, "a square and twenty-one in numbers are equal to ten roots of the same square." That is to say, what must be the amount of the square, which when twenty-one dirhems are added to it, becomes equal to the equivalent of ten roots of that square.

This type of equation corresponds to the equation for which Abû Bekr gave the steps leading to the solution in the problem seen above. Al-Khwarizmi provided a geometrical proof for the particular equation stated above ($x^2 + 21 = 10x$) (see Rosen 1831, p. 16 ff). We will give it here for the equation of problem 25 of Abû Bekr. (Details in Radford 1995a), that is, the equation: $x^2 + 48 = 14x$.



Let *i* be the middle of ag. Construct hf such that bh=hf. Let td be equal to hi. The rectangles bi and ud are equal. Hence, square fg minus rectangle bg is equal to square ft.
That is: area square ft=(14/2)(14/2) - 48.
Its side is obtained by taking the square root of the area and x is found by subtracting the square root from (14/2). This is the small side in Abu Bekr problem. The big side is obtained by subtracting the small side to 14.

III. A PROBLEM FROM FIBONACCI'S *LIBER ABACI*

Leonardo Pisano or Fibonacci was instrumental in the introduction of Arabic algebra into the West. His *Liber Abaci* (1202) was a sort of encyclopedia containing an exposition of several methods to solve a variety of commercial and non-commercial problems. In the *Liber abaci* there is no algebraic symbolism yet. And Fibonacci and the mathematicians of the 13^{th}, 14^{th} and 15^{th} centuries were still using only one unknown to solve problems —even if most of the time in the statement of the problems the question was to find several numbers (see Radford 1997, in press). This is the case in the following problem.

Divide 10 into two parts, add together their squares, and that makes $62\frac{1}{2}$.

Exercise 1: Becoming rhetoric!
Using the mediaeval algebraic technique, find the second-degree equation corresponding to this problem. (If needed, make transformations so that the "coefficient" of censo be one). As the mathematicians of that period, you will use one unknown only (which you

will represent by ‘thing’ and its square by ‘censo’). In the problem-solving process indicate the passages where ‘restoration’ was needed.

Exercise 2:
Instead of solving the equation by the usual modern formula, solve it using the geometric argument explained in the previous section.

Exercise 3:
Compare your solution to Fibonacci’s:

> “Let the first part be one thing, and this multiplied by itself makes a censo.
> In the same way, multiply the second part, which is 10 minus one thing, by itself; for the multiplication you do this: 10 times 10 equals 100; a subtracted thing multiplied by a subtracted thing makes a censo to add. And twice 10 multiplied by a subtracted thing makes 20 subtracted things. And so for 10 minus 1 thing multiplied by itself makes 100 and a censo diminished by 20 things. Adding this to the square of the first part, that is, to the censo, there will be 100 and two censi minus twenty things, and this equals $62\frac{1}{2}$ denarii. Add therefore, twenty things to each part, there will be 100 and two censi equal to 20 things and $62\frac{1}{2}$ denarii. Take away, therefore, $62\frac{1}{2}$ from each part, there will remain two censi and $37\frac{1}{2}$ denarii that equal 20 roots; this investigation has thus been brought to the third rule of mixed cases, that is, censi and numbers are equal to roots.
> In order to introduce the rule, divide the numbers and roots by the number of censi, that is by 2, and it will make one censo and $18\frac{3}{4}$ denarii equal to 10 roots. Therefore …” (According to Boncompagni’s edition of *Liber Abaci*; 1857, p. 411. The problem is fully discussed in Radford 1995a).

DISCUSSION

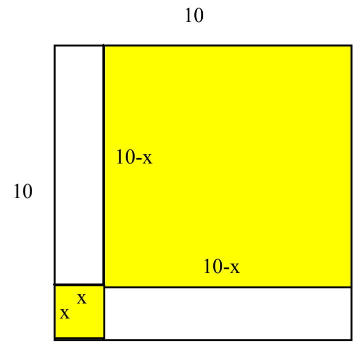
At the beginning, the discussions focused on the nature of numbers and their relation with geometry. But quickly, it evolved toward ways of solving Fibonacci’s problem. Frédéric Gourdeau presented his own solution.

Frédéric’s solution:

The square has an area of 100. The two squares (grey) have a total area of $62^1/_2$.

Therefore, the two remaining equal rectangles are left with a total area of 100 - $62^1/_2$, that is $^{75}/_2$, and thus one of those rectangles has an area of $^{75}/_4$.

The problem is then to find a rectangle of which half of the perimeter is 10 and the area is $^{75}/_4$.
This is the type of problem solved by Abû Bekr.



We also noticed that writing an expression such as $x^2 - 2x + 100$ is misleading. This expression is the area of the square of sides (10-*x*). In the context of Fibonacci's epoch, it would be impossible to have a negative number, even a virtual one, since this expression represents an area which is necessarily positive. In the expression $x^2 - 2x + 100$, it may happen that *x* be such that $x^2 - 2x$ becomes negative. So, we should write instead $x^2 + 100 - 2x$. This shows the danger of using algebraic symbolism while studying the early history of algebra or while using concrete material such as algebraic tiles.

The fact that, at that time, memory was of paramount importance while solving mathematical problems was also discussed. In the Middle Ages, writing was a lot less widely spread than today. The high cost of paper limited the availability of support for writing. Also, students, and society in general, developed many ways of using memory. With the widespread use of writing and other cultural means to keep track of things and events, those abilities have been transformed and required to a lesser extent.

IV. OPERATING WITH AND ON THE UNKNOWN

In Section I we encountered a first type of operation with the unknown: addition of the unknown and fractional parts of it (e.g. $x + \frac{1}{4}x = \frac{5}{4}x$). In Section II, we encountered a different type of operation with the unknown. Indeed, in this case, the unknown (or its square) appeared as a subtractive term (i.e. a term that is provisionally lacking from another term. More specifically x^2 was seen as being absent from 14 in the left expression of the equation $14x - x^2 = 48$). So, the expression $14x - x^2$ was *repaired* or *restored* by adding x^2 to both sides of the equality. In this section we will see another type of operation with the unknown by referring to a problem from R. Canacci's *Ragionamenti d'algebra*. This book belongs to the tradition of algebra books of the Renaissance. With the increase of economic activity in the 13th Century onwards, exchange of merchandise and money were the main means to acquire goods. But cash money was not always available. Sometimes, to do business required more money than merchants could afford. So small companies developed and with this a systematic study of mathematical techniques to calculate gains. In teaching settings, problems were ideated in such a way that merchants and their sons were trained to cope with calculations and problem-solving needs —merchants' daughters were not usually involved in commercial activity (a contextual analysis of commercial mathematics and its relation to the humanistic thinking of the Quattrocento can be found in Radford 2000).

The following problem is a kind of pedagogical effort to provide training to use algebra in a non-realistic setting. The difficulty resides in finding the equation and handling it.

> Two men have a certain amount of money. The first says to the second: if you give me 5 denarii, I will have 7 times what you have

left. The second says to the first: if you give me 7 denarii, I will have 5 times what you have left. How much money do they each have?

Exercise 1:
Using one unknown only, write an equation for this problem. Then solve it justifying the algebraic actions.

Compare your solution to Canacci's.

Canacci's solution	Comments
The first man has 7 things minus 5; the second man has one thing and 5 D[enarii]. The second [gives] to the first 5D. He is left with a thing. The first will have 7 things. Therefore, the first has 7 things minus 5D. He gives 7 to the second who has one thing and 5D, for which he asked, and the first will have 7 things minus 12D. This is equal to 5 times the [amount] of the first. Therefore, multiply the amount of the first by 5 and that gives 5 times 7 things minus 12, that which gives 35 things minus 60D. This is equal to one thing plus 12. Even up the parts by adding to each 60D and subtracting a thing from each part this will give 34 things equal to 72D. Divide the things, as the rule says, and the thing is $2\frac{2}{19}$ [the text is incorrect: the division gives $2\frac{2}{17}$ —L.R.] Therefore, since the thing is $2\frac{2}{19}$, come back to the beginning of the problem. The first man had 7 things minus 5D, the second man had a thing and 5D. Therefore, the first had …	First man= 7*x*-5 Second man= *x*+5. After the second person gives the 5 denarii, the amount are 7*x* and *x*, respectively. After the first person gives 7 denarii, the amounts are 7*x*-12 and *x*+12, respectively. The equation is: $x+12=5(7x-12)$ $x+12=35x-60$: operating *x* leads to: $34x=72$ and $x=2\ 2/17$.

(This problem is fully discussed in Radford 1995a)

DISCUSSION:

Two original solutions have been given to Canacci's problem.

Darren's solution

While playing with real pieces of money, cents and obviously not *dinarii*, Darren Stanley found the following solution. It seems that working with concrete money played an important role in the making of Darren's solution. We may speculate, and it is pure speculation, that Canacci's solution may have been found the same way.

> Let's say, at the beginning of the discussion, that the second man has *x*+5 denarii. But, since *if you* (the second man) *give me* (the first man*) 5 denarii, I will have 7 times what you have left*, after this exchange, the second man has *x* denarii, since he gave 5 denarii to the first man, and the first has 7*x* dinarii, 7 times what the second has. Thus, the first man having gotten 5 denarii in this exchange, before it, at the beginning, he had 7*x*-5 denarii.

	First man	Second Man
At the beginning	7*x*-5	*x*+5

> Let's now analyze the second exchange. *The second says to the first: if you give me 7 denarii, I will have 5 times what you have left.* Then, after the exchange, the first man has (7*x* – 5) –7 dinarii, and the second has 5 times this amount, that is 5(7x-12) dinarii. Then, before this exchange, the second man had 5(7x-12) –7 dinarii, since he got 7 dinarii from the first man during the exchange.

	First man	Second Man
At the beginning	$7x$-5	$5(7x - 12) - 7$

Therefore, the amount the second man had at the beginning may be expressed in two different ways, x+5 and 5(7x-12) –7. Thus,

$$x+5 = 5(7x-12) - 7.$$

So $x = 2\ ^2/_{17}$ denarii. Then, the first has $9\ ^{14}/_{17}$ denarii, and the second one, $7\ ^2/_{17}$ denarii.

Ralph Mason proposed another solution.

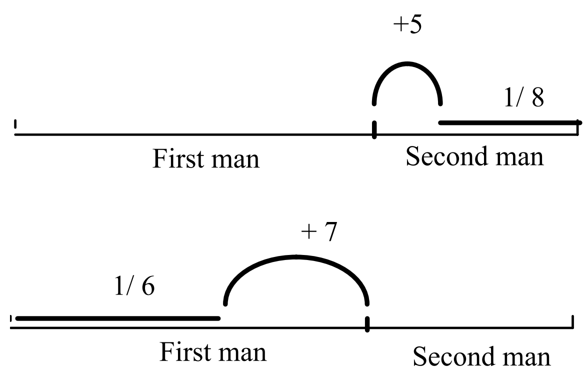
Ralph's solution

Let's resolve this problem using ratios, with a graphical representation.

With the first exchange, the first man has seven times more money than the second one. Thus, the first man has $^7/_8$ of all the money. At the beginning, the first man had then seven eighth of the total, less 5.

With the second exchange, the second man has five times more money than the first one. Thus, the first man has $^1/_6$ of all the money. At the beginning, the first man had then one sixth of the total, plus 7.

Therefore, if one says that the *res* is the total amount of money, and we look at what the first man had at the beginning, one has: $^7/_8\ res - 5 = {}^1/_6\ res + 7$.



From there, it is easy to find the actual amounts each man has.

Part 2: A teaching sequence

The second part of the working group was devoted to the analysis of a video. This video showed two groups of three Grade 8 students each working with manipulatives trying to solve some problems related to linear equations. The members of the working group had the verbatim discussion of one of the groups of three students who participated in an experiment described in Radford and Grenier's papers (Radford, L. & Grenier, M. (1996). Entre les idées, les choses et les symboles — Une séquence d'enseignement d'introduction à l'algèbre. *Revue des sciences de l'éducation, **22**, 253-276*. / Radford, L. & Grenier, M. (1996). On the dialectical relationships between symbols and algebraic ideas. In Puig, L. & Gutiérrez, A. (Eds.), *Proceedings of the 20th international conference for the psychology of mathematics education*, Vol. 4, (pp. 179-186). Universidad de Valencia, Spain). The part we saw focused on the way the students dealt with the question of the equality between the two parts of what could be an equation. The stage at which the students were, they hadn't yet seen any symbolism. The problems on which they had worked were given in terms of hockey cards and envelopes containing an unknown number of hockey cards, the question being how many hockey cards does each envelope contain. One of the ideas was to have the students consider the unknown as an "occult" quantity, as Mazzinghi wrote during the second half of the 14th Century. The students had actual envelopes and hockey cards in their hands.

In the working group, the discussion rapidly focused on the analogy between an equation and a two-plate-scale model. Is this idea of a scale really helping students to understand how to manipulate and transform an equation? Three main objections were raised.

First, it was noted that two–plate scales are not common in our students' environment. Usually, the students don't have any experience with such scales. Can we then say that an analogy between such an instrument and a symbolic equation may help a student to know what to do to manipulate an equation?

Second, the common situation in which the two-plate-scale model is used in the introduction to algebra refers to discrete quantities only, i.e., hockey cards.

Third, the equality corresponds to the fact that there is the same number of cards on "both sides" of the equation. However, in a two-plate-scale, it is not the *number* of objects in each plate that is equal, but the *weight* of those objects.

In light of the previous objections, another different approach was proposed by one of the workshop participants. This approach is based on the use of two identical containers and was illustrated with reference to the following problem. In the first container, there is already 1 dl of liquid, and in the other, there are 4 dl. There is a third, smaller container of an unknown capacity, called a "cup".

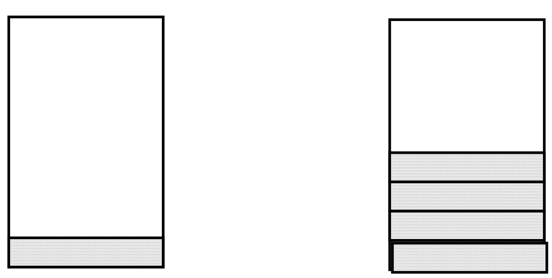


Figure 1

The containers are such that if we add the liquid of two "cups" to the first container and then add one "cup" to the second container, the liquid in both containers will be at the same level. In such a situation, the equality is **visible**. She argued that this situation is more natural than the one based on a two-plate-scale model and that a research process is probably more likely to be engaged by the children themselves and that when one uses a real object to represent the unknown, the actions to be executed are more natural.

The presentation of the two-container model was followed by a discussion. Some objections were made, among them the following.

First, it is not clear that the two-container model is more *visible* than the two-plate-scale model. Indeed, while it is true that you can see that both of the containers have the same amount of liquid, what you see is the *total* of the liquid. Once the water has been poured, it is impossible to visually discern what you had before and what was added. In a real setting, what one sees is not a diagram like the one in Figure 1. The dividing lines are not there. The students lose track of the different components of which the total is made up and can easily forget how many dl were in each container at the beginning –something that can unnecessarily complicate the problem-solving procedure. This problem with the liquid, it was argued, is a common problem of objects with a fuzzy referent. (Actually this is why, in natural language, the plural of objects such as liquid and sand functions differently from the plural of discrete objects: if you add water to water you still have water – and not water*s*). In contrast to what happens in the two-container model, the discrete objects of the two-plate-scale model allow the students to clearly keep track of what they add or remove from the scale (cards and envelopes are *visually* different).

Second, the above-mentioned intrinsic difficulty with the two-container model renders it very unlikely to make it appear more natural than the two-plate-scale model. Furthermore, as to the familiarity of students with two-plate-scale artefacts, it was noticed that contemporary students in primary school learn about the relations " X is lighter than Y", "X is heavier than Y", "X is as heavy as Y", etc., through the use of plastic two-plate scales.

Third, the two-container model requires that the students add and remove amounts of liquid that need to be measured. Since here we are dealing with liquid, measures will always be approximative. Unavoidable errors in measures complicate the students' hands-on procedure and filter unnecessary sources of problems into the learning process.

Fourth, from the concrete demands of the teaching setting, it has to be pointed out that as a result of playing with liquid, the chances are that the students will end up making a mess in the classroom.

Fifth, in the end, the unknown in the two-container model is the amount of liquid held by the "cup". You add (or remove) one, two, three cups. Upon closer examination, as in the case of the two-plate-scale model, you are handling whole quantities. Hence, there is not much to be gained from this point of view.

During the comparison of the two models, it was noticed that both of them bear the risk of not making clear for the student the distinction between the signifier and the signified. In the two-plate-scale model, the unknown is the number of cards in the envelope. The students, however, tend to identify the unknown with the envelopes. In the two-container model, the confusion arises by taking the cup as the unknown instead of the amount of liquid *in* the cup.

These considerations led some members of the group to ask if in using too much concrete material we are not led to a situation where, like in the Middle Ages, negative numbers would not be accepted.

Luis Radford then expressed the view that it is important to distinguish hard uses of the two-plate-scale model (based on the *weight* of objects) from metaphorical uses of it (based on a descriptive general idea of quantities that are equal) and explained that, in his research, the metaphorical model is a way to help students make sense of the comparison of quantities required in algebra (his students are not provided with any scale). In his approach, this kind of comparison —that past mathematicians referred to as “confrontation”— is possible by a division of the space in which the concrete actions occur. The desk becomes the space on which actions unfold. As it was possible to see in different parts of the classroom episodes presented in the video, the desk is divided by the students (in general, mentally) into two parts, each one containing the cards and envelopes according to the problem. The algebraic concept of unknown becomes conceptually related to the concrete actions that the students perform on the objects and that underlie the algebraic techniques. A metaphorical approach referring obliquely to a two-plate scale is intended to invite students to participate in a kind of *language game* (in Wittgenstein’s sense) that, as experimental data shows, they easily join and that gives them an opportunity to conceive the idea of “confrontation” in a concrete way. Radford emphasised the fact that the originality in his approach is to be found not only in the metaphorical recourse to a two-plate scale but, over all, in the recourse to three distinct (but related) semiotic layers in which students have the chance to produce meanings for symbols. The differentiated semiotic layers constitute an important difference to other hands-on approaches (e.g. in terms of the role of symbols and the source of their meaning)

Thus, one of the fundamental differences between the two models previously discussed was located in the *source of meaning* for the concept of unknown and the correlated actions carried out on it (addition, subtraction, etc., of known and unknown terms of the “confronted” quantities). In the metaphorical two-plate-scale model, the meaning of concepts arises from the hands-on step, based on the students’ gestures and concrete actions on concrete objects. In this model, there is an important step (systematically placed after the hands-on step and before the symbolic one) in which the students make drawings. No more manipulatives are required. Reasoning on the drawings, an *iconic* type of algebraic thinking is generated. In this stage, meaning is produced from actions on indirect objects –icons of concrete objects. This step is followed by a symbolic one in which symbols are introduced as abbreviations of those gestures and actions on icons. Spatially, the *equation* mimes the iconic objects. In this step, the actions are characterised by new symbols on the page which then becomes a semiotic space endowed with the meaning of the actions undertaken on the iconic layer. The three-step methodology developed by Radford and Grenier is such that each semiotic layer (concrete, iconic and symbolic) functions at different times. In the two-container model, in contrast, because of the difficulty involved in keeping track of the amount of water added or removed, the actions have to be written down or referred to the equation (let’s suppose a student who removes, say, 3 dl and a cup from each container and puts the removed liquid into another recipient, which may already contain some liquid, and then forgets the actions previously undertaken. It will be almost impossible for him or her, by merely looking into the recipient, to identify what was removed from each container). This is why, to keep track of actions (something required in any heuristic process), these have to be written down somewhere. As a result, two semiotic layers enter simultaneously into the scene and it may be unclear for the students where to focus their attention –the containers or the written actions? Furthermore, the necessity of keeping track of actions may lead the teachers to prematurely introduce the symbolic equation, which thereby becomes a focus of attention and the source of meaning.

Despite the theoretical and semiotic principles underpinning the two models, both allow students to construct (although probably not with the same intensity), little by little, complex symbolic representations. For instance, in one of the classroom episodes, discussed in the working group, it was possible to see how the group of students, after

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,solving some word-problems about hockey cards through the two-plate-scale model, succeed in symbolising a word-problem about pizzas into an equation as follows:

$$3P - 6 = 1P - 2 + 18.$$

The problem dealt with pizzas having two missing slices. The number of slices in one of those pizzas was represented by the students as '1P-2'. Thus, the left side of the equation indicates the number of slices in three of those pizzas. The sense of the equation and the meaning of the actions were such that they added 6 slices on the left side to complete the pizzas and, to maintain the 'confrontation', the students added 6 slices on the right side of the equation. Two of the 6 slices were used to complete the pizza on the right side of the equation. Symbolically, these actions were written as follows:

$$\begin{array}{rcl} 3P - 6 & = & 1P - 2 \;+ 18, \\ +\;6 & & +2 \;+\; 4. \end{array}$$

The resulting equation was then 3P = 1P +22; then they proceeded to remove 1P from both sides, etc.

This is reminiscent of what they had done in a previous stage when they were using concrete material.

The longitudinal nature of the research ensured a close follow-up allowing one to see how the concrete actions progressively lose their contextual root and become more and more autonomous vis-à-vis the specific situation of the problems.

The working group participants seemed to agree that, despite the differences in the discussed models, both of them are based on cultural artefacts (containers and scales) that as any artefact, are not “good” or “bad” in themselves. What is important, from a pedagogical point of view, is that, suitably used in the classroom, they can serve as a means to promote mathematical understanding.

Part 3: Using history in the classroom

At the beginning of this last part of our work, we saw an extract of a video done under the supervision of Leigh Wood (*Balancing the equation - The concepts of algebra. University of Technology*. Sydney and the Open Training Education Network.). This video was aimed at students being introduced to algebra. The extract showed a discussion between two characters involved in the resolution of a traditional problem from India and another one from China.

This led us to a discussion about the main reasons for which the members of the working groups used history in their mathematics classroom. Different categories of reasons surfaced. The most often mentioned one was the desire to show to students that mathematics is a human activity. To do that, one may focus on multicultural aspects of the history of mathematics, choosing for example to study Islamic patterns. Another way is to focus on the fact that the history of mathematics is full of controversies and discussions. Opinions played an important role in the evolution of mathematics. It is why the history of mathematics is a rich source of problems for problem solving activities. Having students solve problems and, at the same time, be informed of historical discussions aroused by this same problem may be a very enriching experience for students. A last way of giving students the feeling of mathematics as a human activity is to relate the evolution of mathematics to the fulfilment of human needs, practical needs, related to the economy, the daily life, the military, etc., as will as other needs related to aesthetic or intellectual pursuits.

History is often fascinating for students. Using history often allows teachers to get their attention. History gives then a way to motivate students, even if just for a short period of time. Anecdotal extracts from the life of mathematicians are often used in that way.

It has been noticed, on the other hand, that putting too much emphasis on the life and works of great mathematicians may give the students the impression that mathematics is only accessible to a small group of very bright persons. It is therefore essential to have a healthy diversity of historical intrusions in the mathematical classroom.

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