# CHAPTER 25

# Historical Conceptual Developments and the Teaching of Mathematics: from Phylogenesis and Ontogenesis Theory to Classroom Practice*

**Fulvia Furinghetti**
*Università di Genova, Genova, Italy*
**Luis Radford**
*Université Laurentienne, Ontario, Canada*

## 1. INTRODUCTION

More than a century ago, Hieronymus Georg Zeuthen wrote a book about the history of mathematics (Zeuthen, 1902). Of course, this was not the first book on the topic, but what made Zeuthen's book different was that it was intended for teachers. Zeuthen proposed that the history of mathematics should be part of teachers' general education. His humanistic orientation fitted well with the work of Cajori, 1894 who, more or less by the same time, saw in the history of mathematics an inspiring source of information for teachers. Since then, mathematics educators have increasingly made use of the history of mathematics in their lesson plans, and the spectrum of its uses has widened. For instance, the history of mathematics has been used as a powerful

*The authors of this chapter are listed in alphabetical order. This paper was made possible by a research program funded by the Social Sciences and Humanities Research Council of Canada, Grant 410-98-1287 and by the Italian national project "Ricerche di matematica e informatica per la didattica" (director F. Arzarello) with a grant of MURST.

tool to counter teachers' and students' widespread perception that mathematical truths and methods have never been disputed. The biographies of several mathematicians have been a source of motivation for students. By stressing how certain mathematical theories flourished in various countries, the diverse contributions of various cultures to contemporary mathematics becomes evident. Specialized study groups have emerged in the past years as a result of the increasing interest in the history of mathematics in educational circles. Two of these are the Commission INTER-IREM Épistemologie et Histoire des Mathématiques in France and the *International Study Group on the Relations between History and Pedagogy of Mathematics*, which is related to International Commission on Mathematical Instruction (ICMI). In addition, regular conferences are organized, such as the European Summer Universities on the History and the Epistemology in Mathematics Education (see Lalande, Jaboeuf, & Nouazé, 1995, and Lagarto, Vieira, & Veloso, 1996, for proceedings). Concomitantly, an important number of books are now available to help teachers use the history of mathematics (Calinger, 1996; Chabert, Barbin, Guillemot, Michel-Pajus, Borowczyk, Djebbar, & Martzloff, 1994; Dhombres, Dahan-Dalmedico, Bkouche, Houzel, & Guillemot, 1987; Fauvel & van Maanen, 2000; Katz, 2000; Reimer & Reimer, 1995; Swetz, Fauvel, Bekken, Johansson, & Katz, 1995).

Instead of offering an overview of the different domains in which the pedagogical use of the history of mathematics is now ramified, we want, in this chapter, to focus on something that Cajori started and in which mathematics educators interested in the history of mathematics are still involved. That is, in considering history not only as a window from where to draw a better knowledge of the nature of mathematics but as a means to transform the teaching itself. The specificity of this pedagogical use of history is that it interweaves our knowledge of past conceptual developments with the design of classroom activities, the goal of which is to enhance the students' development of mathematical thinking.

Cajori's 1894 ideas have led us to developments that he could not have suspected. Indeed, Cajori adopted a positivistic view of the formation of knowledge. He saw knowledge as an objective entity that grows gradually and cumulatively. His reading of the history of mathematics was framed by viewing history as an unfolding process. The direction or completion of the process guaranteed by the idea of progress—an idea underpinning the Enlightenment philosophy and attitudes toward life from which modern thought arose. Nonpositivistic views about the formation of knowledge were later elaborated by philosophers and epistemologists such as Bachelard, Foucault, and Piaget, among others, and by anthropologists such as Durkheim, Levy-Bruhl, and Lévi-Strauss, to mention but a few. Bachelard presented an interpretation of the formation of knowledge in terms of ruptures and discontinuities. Piaget was interested in explaining genetic developments in terms of stages and the intellectual mechanisms allowing the passage from one level to another. Foucault was opposed to the conception of history as a date-labeling practice and investigated the problem of the constitution of knowledge in terms of the conditions of its emergence, which he related to the different spheres of human activity. Bachelard, Foucault, and Piaget had different goals, and thus their projects differed. But what is important for our discussion here is that, contrary to what Cajori and many other positivist thinkers believed, knowledge in general and mathematical knowledge in particular cannot be taken as an unproblematic concept. Behind a concept of knowledge there is an epistemological stance, and this epistemological stance conditions our understanding of the formation of students' mathematical thinking as it conditions the interpretation of historical conceptual developments (Grugnetti & Rogers, 2000; Radford, Boero, & Vasco, 2000). Nevertheless, the study of the development of students' thinking and of the conceptual development of mathematics belong to two different domains—the psychological and the historical, respectively. Each has its specific problems as well as the tools with which to investigate them. Students' conceptualizations can be investigated through classroom observations, interviews, tests, and soforth. The same cannot be done in the historical domain, where historical records are the only available material for study. The difference in methodologies in both domains is, in fact, a token of more profound differences. These cannot be ignored in the context of a pedagogical use of the history of mathematics as a useful tool to enhance the development of students' mathematical thinking. Despite their differences, the psychological and historical domains need to be weighed and articulated in a specific way. One of today's more controversial themes concerns the terms in which such an articulation must be understood. More specifically, the question is how to relate the development of students' mathematical thinking to historical conceptual mathematical developments. Psychological recapitulation, which transposes the biological law of recapitulation, claims that in their intellectual development our students naturally traverse more or less the same stages as mankind once did; it has been taken as a guarantee (sometimes implicitly) to ensure the link between both domains. In its different variants, however, psychological recapitulation has been subject to a deep revision recently, in part because of the emergence of new conceptions about the role of culture in the way we come to know and think.

The purpose of this chapter is to discuss in some detail the basic problems referred to in this introduction. In the next section, we deal with psychological recapitulation and mention some of the current arguments against it. In section 3, we examine key ideas about ontogenesis and phylogenesis as found in the works of Piaget and in the works of Vygotsky. In section 4, we present some paradigmatic examples of mathematicians who commented on phylogenesis and its relation to ontogenesis. Section 5 focuses on a particular interpretation of the recapitulation law that led to the so-called "genetic approach", which had an obvious impact on early mathematics education. In section 6, we discuss some examples of teachers who take into consideration the history of mathematics to improve their teaching; determining how interpretations of the recapitulation law articulate the teachers' goals and actions guides our discussion. Section 7 provides a brief account of a few current approaches in contemporary mathematics education that relate to the history of mathematics regarding either theoretical or practical links between the development of students mathematical thinking and historical conceptual developments. In the last section, we offer a critical assessment of the law of recapitulation and recommend ideas for conceptual and applied research in the 21st century regarding historical and ontogenetic developments in mathematics education.

## 2. FROM BIOLOGICAL TO PSYCHOLOGICAL RECAPITULATION

The way in which people perceived psychological recapitulation at the beginning of the 20th century was linked to the way they perceived themselves in the overall view of the world. As long as humans thought of themselves as essentially different from animals and plants, no relation in terms of ancestry could be advocated. Even in the early 18th century, a common scholarly view to explain the origin of species and to understand the formation of living things was that species came from those beings fortunate enough to survive the deluge, as indicated in the Genesis (see, e.g., Osborn, 1929), by finding refuge on Noah's ark. But with the appearance of the early 19th-century philosophy of nature, humans came to join the greater kingdom of species. In their broader sense, however, recapitulationist ideas data back, to the pre-Socratic thinkers. They did not state them in terms of a telescoping or condensed process of lower life that culminates with humans. Often their reference point was the cosmos. Thus, Empedocles believed that the growth of the embryo echos in a foreshortened way the cosmogonic process: The embryo is submerged into amniotic fluid that evokes the originally fluid earth (de Santillana, 1961, p. 114). During the 18th and early 19th centuries, a vigorous debate separated two opposing schools with regard to the concept of recapitulation. One of them, which became known as preformation theory, stated that ontogenesis was the unfolding or growing of preformed structures, whereas the other school adopted a more dynamic stance, arguing that the embryo was neither the exact miniature of the developed species nor the unfolding of preformed structures, but a being in a state of development. The "causes" originating embryo's the unfolding or the changes were variously interpreted. Charles Bonnet (1720–1793), usually recognized as one of the leaders of the preformationists, saw change as coming from an affectionate God who had ordered the world according to increasing perfection and progress. Whereas in the early-19th century Naturphilosophen attributed development to a "natural" final cause, Lamarck and Darwin envisioned a new theory that replaced the philosophical idea of final cause with an efficient cause—individual development. (For a detailed discussion of preformationist and Naturphilosophen ideas, see Gould, 1977.) Indeed, from the mid-19th century onward, the "causes" were seen in the context of the theory of evolution. "Heredity and adaptation are, in fact, the two constructive physiological functions of living things," wrote Haeckel (1912, p. 6), who, in one of the most famous statements ever made in the realm of anthropogenesis (which he modestly called the fundamental law of biogeny), declared that

> The series of forms through which the individual organism passes during its development from the ovum to the complete bodily structure is a brief, condensed repetition of the long series of forms which the animal ancestors of the said organism, or the ancestral forms of the species, have passed through from the earliest period of organic life down to the present day. (pp. 2–3)

Haeckel's law was more than the simple statement of a condensed repetition of steps. What he was suggesting was that embryos of man and dog, at a certain stage of their development, are almost indistinguishable. Indeed, to take one of Haeckel's favorite examples, "the human gill slits *are* (literally) the adult features of an ancestor" (Gould, 1977, p. 7).

How, then, was the discussion about the biological growth of humans transferred to the psychological domain? It was Haeckel who, after discussing the nervous system, said "we are enabled, by this story of the evolution of the nervous system, to understand at length *the natural development of the human mind* and its gradual unfolding" (1912, p. 8, italics as in the original). A sharper formulation was the following: "the psychic development of the child is but a brief repetition of the phylogenetic evolution" (Haeckel quoted by Mengal, 1993, p. 94). The adoption of the psychological version of biological recapitulation served as a general framework to conceive the functioning of child psyche as something traveling the same path as his or her ancestors. For instance, the child was seen as behaving as humans in previous stages of the chain of evolution (e.g., such as having, in an early stage of his or her development, an "animist" view of nature, that is, that immaterial forces animate the universe).

Psychological recapitulation endorses a peculiar view of history and development. Concerning development, for Bonnet and the preformists, there was no development, strictly speaking, but only growing or unfolding.[1] Environment cannot alter the preformed structures and their growth. For evolutionary-based recapitulation theories[2], in contrast the environment is supposed to play[3] in the development of species a role. The individual is seen as an organism adapting to his or her environment; in the interplay between individual and environment, some of the biological and psychological functions may develop, whereas others may be lost according to the natural selection.

As for history, in contrast to views that conceived a world that underwent different creations, Bonnet saw the world as created at one time, with its whole history encapsulated within it. History was therefore the unfolding of a predetermined plan. The concept of history is much more problematic for recapitulationists. Indeed, from a theoretical point of view, history and recapitulation become difficult to reconcil because, on one hand, Haeckel's psychological recapitulation supposes that present intellectual developments are to some extent a condensed version of those of the past. On the other hand, natural selection is presented as a function of the environment against which individuals act. For recapitulation to be possible, therefore, such an environment must remain essentially the same, which obviously is not the case. Given that the environment changes, it becomes difficult to maintain that the children's intellectual development will undergo the same process as the one children experienced in the past. The variability that natural selection imposes on the course of events in history conflicts with the idea of recapitulation as condensed repetition of some intellectual aspects registered in past history. Indeed, this point was recognized as a weakness. Werner (1957), for instance, advocated contextual factors and argued that it is impossible to equate a certain intellectual stage of a child in a modern society to the stage an adult could have reached in a ancient society because the respective environments, as well as the genetic processes involved in them, are completely different (see Radford, 1997a). Elias also mentioned the differences that necessarily result as a consequence of variations in cultural settings. Whereas in traditional societies children participate directly in the life of the adults earlier and their learning is done *in situ* (as apprentices), "modern" children are instructed indirectly in mediating institutions, or schools (Elias, 1991, pp. 66–67). Consider memory, an example that is addressed neither by Werner nor Elias but which conveniently clarifies the previous ideas. As many anthropological accounts clearly show (see e.g., Lévy-Bruhl, 1928), memory plays a central role in illiterate societies. In contrast, sign systems related to writing in literate societies dispense with memory to a certain and fundamental extent. They create a different way to handle and distribute knowledge and information between the members of the society and shapes attitudes about how to scrutinize the future (see Lotman, 1990).

The theoretical difficulties encompassing the crude version of psychological recapitulation encouraged new reflections to find more suitable explanations concerning the relations between phylogenesis and ontogenesis. In the next section, we will discuss two different views that have been influential in the use of history in mathematics education.

## 3. PIAGET AND VYGOTSKY ON ONTOGENESIS AND PHYLOGENESIS

Piaget was interested in understanding the process of the formation of knowledge. To do so, he considered knowledge as something that can be described in terms of levels and strived to describe those levels, as well as the passage from one level to a more complex one. He said, "The study of such transformations of knowledge, the progressive adjustment of knowledge, is what I call genetic epistemology" (Piaget cited in Bringuier, 1980, p. 7). As a reaction to the simplistic psychological version of recapitulation and the positivist view of knowledge that we mentioned in the introduction, Piaget and Garcia elaborated the concept of *genetic development*. They envisioned the problem of knowledge in terms of the intellectual instruments and mechanisms allowing its acquisition. According to Piaget and Garcia, the first of those mechanisms is a general process that accounts for the individual's assimilation and integration of what is new on the basis of his or her previous knowledge. In addition to the assimilation mechanism, they identified a second mechanism, a process that leads from the *intraobject*, or analysis of objects, to the *interobject*, or analysis of the transformations and relations of objects, to the *transobject*, or construction of structures. This epistemological viewpoint led them to revisit the parallelism that recapitulationists had emphasized. Therefore, Piaget concluded, "We mustn't exaggerate the parallel between history and the individual development, but in broad outline there certainly are stages that are the same" (Bringuier, p. 48). The two mechanisms were hence considered as invariables, not only in time but also in space. That is, we do not have to specify what they are in a certain geographical space at a particular time because they do not change from place to place or from time to time. They are exactly the same, regardless of the period of history and the place of the individuals.

> In modern mathematics, at the level of algebraic geometry, of quantum mechanics, although it's a much higher level of abstraction, you find the same mechanisms in action—the processes of the development of knowledge or the cognitive system are constructed according to the same kinds of evolutionary laws. (Garcia in Bringuier, 1980, pp. 101–102)

Thus, when Piaget and Garcia investigated the relations between ontogenesis and phylogenesis, they did not seek a parallelism of contents between historical and psychogenetical developments but of the mechanisms of passage from one historical period to the next. They tried to show that those mechanisms are analogous to those of the passage from one psychogenetic stage to the next.

The two mechanisms of passage discussed by Piaget and Garcia have a different theoretical background. The second, that of the intra-, inter- and trans-objectual relations, obeys a structural conception of knowledge and reflects the role that mathematical and scientific thinking played in Piaget's work. As Walkerdine noted, "In the work of Piaget, an evolutionary model was used in which scientific and mathematical reasoning were understood as the pinnacle of an evolutionary process of adaptation" (Walkerdine, 1997, p. 59). The first one, the assimilation mechanism, has its roots in the conception of knowledge as the prolongation of the biological nature of the individuals: "The human mind is a product of biological organization, a refined and superior product, but still a product like another" (Piaget in Bringuier, 1980, p. 108).

Both intellectual mechanisms of knowledge development embody a general conception of rationality that has been contested by some critics who find missing, among other things, a more vivid role of the culture and the social practices in the formation of knowledge. For instance, the epistemologist Wartofsky, who has stressed an intimate link between knowledge and the activities from which knowledge arises and is used, said:

> We are, in effect, the products of our own activity, in this way; we transform our own perceptual and cognitive modes, our ways of seeing and of understanding, by means of the representations we make. . . . Theoretical artifacts, in the sciences, and pictorial or literary artifacts, in the arts constitute the a priori forms of our perception and cognition. But contrary to the ahistorical and essentialist traditional forms of Kantianism, I propose instead that it is we who create and transform these a priori structures. Thus, they are neither the unchanging transcendental structures of the understanding, nor only the biologically evolved a priori structures which emerge in species evolution (as, for example, Piaget and the evolutionary epistemologists suggest). Piaget's dynamic, or genetic structuralism is important here, of course. His dictum, "no genesis without structure, no structure without genesis," suggests the dialectical interplay of the practical emergence and transformation *of* structures with the shaping of our experience and thought *by* structures. But the domain of this genesis I take to be the context of our social, cultural and scientific practice, and not that of biological species-evolution

> alone. . . . In a sense, then, our ways of knowing are themselves artifacts which we ourselves have created and changed, using the raw materials of our biological inheritance. (Wartofsky, 1979, p. xxiii)

Vygotsky, in many writings, dealt with the problem of recapitulation and, like Piaget, believed that the understanding of ontogenesis and phylogenesis had to be based on a deep understanding of our biological nature. (This is clear, for instance, in his book *Speech and Thinking*, as well as in the influence he had on his student Luria and the huge amount of physiological research that the latter conducted.) Instead of posing the problem of the formation of knowledge in terms of universal and atemporal mechanisms functioning beyond culture, however, he saw the cognitive functions allowing the production of knowledge as inevitably overlapping with the context in which individuals act and live. His basic distinction between lower and higher mental functions is reinforced by the idea that the former belong to the sphere of the biological structure, whereas the latter are intrinsically social. Thus, in a passage from *Tool and Symbol in Child Development*, when discussing the problem of the history of the higher psychological functions, Vygotsky and Luria commented:

> *Within* this general process of development two qualitatively original main lines can already be distinguished: the line of biological formation of elementary processes and the line of the socio-cultural formation of the higher psychological functions; the real history of child behaviour is born from the interweaving of these two lines. (Vygotsky & Luria, 1994, p. 148)

The merging of the natural and the sociocultural lines of development in the intellectual development of the child definitely precludes any recapitulation:

> In the development of the child, two types of mental development are represented (not repeated) which we find in an isolated form in phylogenesis: biological and historical, or natural and cultural development of behavior. In ontogenesis both processes have their analogs (not parallels). . . . By this, we do not mean to say that ontogenesis in any form or degree repeats or produces phylogenesis or is its parallel. We have in mind something completely different which only by lazy thinking could be taken to be a return to the reasoning of biogenetic law. (Vygotsky, 1997, p. 19)

For Vygotsky even the elementary intellectual functions of the individual are intrisically human, acquired through the activities and actions on which are based the intercourse between individuals and between people and objects. One of the central reasons for this is that human activities are mediated by diverse kinds of tools, artifacts, languages, and other systems of signs which, Vygotsky argued, are a constitutive part of our cognitive functions. Most important, these systems of signs, as well as tools and artifacts, are much more than technical aids: They modify our cognitive functioning. The knowledge produced by the individuals hence becomes intimately related to the activities out of which knowledge arises and the conceptual and material "cultural tool kit" (to borrow Bruner's expression, see Bruner, 1990) with which the individuals are equipped. Of course, it does not mean that with every new generation, all knowledge must be constructed anew. As Tulviste (1991) noted, whereas rats are still doing what they did centuries ago, humans have, from one generation to the next, assimilated, produced, and passed on their knowledge. During this process, humans have changed their activities and the way in which they think about the world. In Vygotsky's view, knowledge appears as an individual and social creative reappropriation and coconstruction carried out using conceptual and material tools that culture makes available to its individuals. In turn, in the course of this process, the previous tools and signs may become modified, and new ones may be created. It is in this sense that tools and concepts have embodied the social characteristics from which they arose, and their insertion into other activities allows their transformation and eventually their growth. Because activities, sign use, and attitudes toward the meaning of scientific inquiry do not necessarily remain the same throughout time, changes are effected in phylogenetic lines (and the plural of lines needs to be emphasized here) serving as the historicocultural starting point to new genetic developments. Epistamologica reflexions have then to evidence the relation between cognitive context and action. As Wartofsky pointed out:

> If, in fact, our modes of cognitive practice change with changes in our modes of production, of social organization, of technology and technique, then the connection between cognition and action, between theoretical and applied practice, between consciousness and conduct, has to be shown. (Wartofsky, 1979, p. xxii)

One implication of the previous remarks for the use of the history of mathematics in education is that the study of recapitulation can be advantageously replaced by the contextual study of the social elements in which the historical geneses of concepts are subsumed. This can be accomplished through a careful investigation of the cultural symbolic webs shaping the form and content of scientific inquiry and the ways in which mathematical concepts are semiotically represented (Radford, 1997a, 1998, 1999a, 2000a). We return to this point in section 7.

## 4. INTERPRETATION OF RECAPITULATION LAW BY MATHEMATICIANS

In the period when the treatises of Zeuthen and Cajori appeared, the history of mathematics was growing as a scientific discipline. The first journals dealing exclusively with the history of mathematics were appearing in that period. We have extensive evidence that mathematicians and mathematics educators were both looking at the history of mathematics with great interest. Mathematics educators were creating new areas of work in their field linked to changes in societies. As discussed in Furinghetti (2000) and in Furinghetti and Somaglia (1998), the history of mathematics was considered a suitable means to find efficient ways of teaching in different situations. Among mathematicians, the axiomatization and the foundational works were undertaken. These themes were addressing mathematicians' attention to reflections on the nature of mathematics and on the activity of doing mathematics. The history of mathematics was considered a field that offered inspiration to discuss these kinds of problems. In this context, we consider some interpretations of recapitulation law made by important mathematicians.

In the first issue (1899) of *L'enseignement mathématique*, an important journal devoted to the teaching of mathematics, the eminent mathematician Henri Poincaré clearly stated his position on the relations between conceptual and historical developments:

> Without a doubt, it is difficult for a teacher to teach a reasoning that does not satisfy him completely.... But the teacher's satisfaction is not the sole purpose of teaching... above all one should be concerned with the student's mind and of what we want him to become.
>
> Zoologists claim that the embryonal development of animals summarizes in a very short time all the history of its ancestors of geologic epochs. It seems that the same happens to the mind's development. The educators' task is to make children follow the path that was followed by their fathers, passing quickly through certain stages without eliminating any of them. In this way, the history of sciences has to be our guide. (Poincaré, 1899, p. 159; our translation)

Poincaré gave examples of concepts to be taught at an intuitive stage before presenting them rigorously. Among these examples were fractions, continuity, and area. As far as we know, Poincaré never used his ideas on the efficacy of recapitulation law directly with teachers. This makes Poincaré's position different from that of Felix Klein, another supporter of the use of history in mathematics in teaching. In contrast, Klein applied his ideas in courses for prospective teachers and in related texts that he wrote.

Klein supported the German translation of the famous book *A study of Mathematical Education* by Benchara Branford (1921) in which, according to Fauvel (1991, p. 3), the theory of recapitulation "reached its apogee." This can be considered evidence of Klein's agreement to the recapitulation law (Fauvel, 1991, p. 3). Nevertheless, from what Klein wrote in his articles and books (see Klein, 1924), we understand that the application of the law was not advocated in a literal sense. As in the case of Poincaré, his opinion on the use of history was born of his wish to abolish the use of mathematical logic and the excesses of rigor advocated by some of his colleagues. Klein was interested in the dichotomy of "intuition versus rigor" and, as far as school is concerned, was in favor of intuition. He singled out the history of mathematics as being the suitable context for bringing intuition back into the teaching and learning process:

> I maintain that mathematical intuition ... is always far in advance of logical reasoning and covers a wider field.... I might now introduce a historical excursus, showing that in the development of most of the branches of our science [mathematics], intuition was the starting point, while logical treatment followed. This holds in fact, not only of the origin of the infinitesimal calculus as a whole [this issue was discussed at the beginning of Klein's paper] but also of many subjects that have come into existence only in the present [19th] century. (Klein, 1896, p. 246)

Klein claimed that in school, as well as in research, the phase of formalization must be preceded by a phase of exploration based on intuition.

We find an analogous statement in a secondary school geometry book written by a famous Italian mathematician, Francesco Severi, which clearly refers to school practice:

> We need to take inspiration from the principle that in learning new notions, the mind tends to follow a process analogous to that according to which science has developed. One who is aware of the value of foundation theories [in Italian *critica dei principi*] does not make the dangerous mistake of giving to the elementary teaching a critical and excessively abstract style. It is necessary to know foundation theories for personal intellectual maturity; but in the elementary teaching they are not to be considered as a pedagogical means. (Severi, 1930, p. IX; our translation)

Both Klein and Severi do not clearly state what "intuition" means for them, but both state to what intuition is opposed: rigor, excessive abstraction, and formal logic used at the beginning of the presentation of a mathematical notion. (It may be interesting to note that Severi, famous during the first half of the 20th century, is one of the scholars of the Italian school of algebraic geometry who based his results on intuition to such a degree that these were published without being careful verified by a mathematical proof, as reported by Hanna, 1996).

## 5. THE GENETIC APPROACH

Using the history of mathematics in teaching does not necessarily entail a direct assumption of the recapitulation law; it also may be used in the so-called *genetic approach* to teaching. The term "genetic" is an ambiguous one because it is used with different meanings. In particular, in the foundation literature, the term *genetic method* is used in contrast to *axiomatic method.* David Hilbert probably introduced this term, which was popularized by Edward V. Huntington. Before Hilbert, we find other uses of the word "genetic." Immanuel Kant stated that all mathematical definitions are genetic; after Kant, the term "genetic definition" is present in all major logic treatises.

In addition to its use among mathematicians and philosophers, we find the word "genetic" in other fields of research. Piaget and Garcia used it in their epistemological studies. As to mathematics education, Ed Dubinsky, who dealt with genetic decomposition, used the word.

Here we are concerned with the word "genetic" as it is used in connection with history. In the 1920s the idea of a genetic principle was taking shape, as evidenced by the work of N. A. Izvolsky.[1]

Gusev and Safuanov (2000) report that, according to Izvolsky, nor teachers nor textbooks try to explain the origin of geometrical theorems. He suggested that, when attempts to do this are done, students see geometry in a different way. Moreover sometimes students themselves guess that a given theorem was not originated by a mere wish of the teacher or textbooks' authors, but by questions arisen in previous works. It happens that students try to imagine the origin of a theorem. According to Izvolsky, even if their hypotheses are not correct from the historical point of view, this approach to the teaching of geometry is valuable.

The idea of a genetic approach later took a definite form in a work by Otto Toeplitz that he wrote to describe a method of presenting analysis to university students.[2] The following passage illustrates the ideas underlying the genetic method:

> Regarding all these basic topics in infinitesimal calculus which we teach today as canonical requisites, e.g., mean-value theorem, Taylor series, the concept of convergence, the definite integral, and the differential quotient itself, the question is never raised "Why so?" or "How does one arrive at them?" Yet all these matters must at one time have been goals of an urgent quest, answers to burning questions, at the time, namely, when they were created. If we were to go back to the origins of these ideas, they would lose that dead appearance of cut and dried facts and instead take on fresh and vibrant life again.[3]

Burn explains in this way Toeplitz's ideas:

> The question which Toeplitz was addressing was the question of how to remain rigorous in one's mathematical exposition and the teaching structure while at the same time unravelling a deductive presentation far enough to let a learner meet the ideas in a developmental sequence and not just in a logical sequence. While the genetic method depends on careful historical scholarship it is not itself the study of history. For it is selective in its choice of history, and it uses modern symbolism and terminology (which of course have their own genesis) without restraint. (Burn, 1999, p. 8)

It is not by chance that this alternative approach developed in the domain of teaching calculus. It is in this domain where the notion that learning mathematics takes place in a sequence predetermined by mathematical logic has shown its pedagogical

---

[1]Nikolai Alexandrovich Izvolsky was born in 1870 in Tula, Western Russia. He worked as a teacher at the 2nd Moscow Military School, and from 1922, he was a professor at the 2nd Moscow State University (now Moscow State Pedagogical University). He wrote papers on mathematics education and some textbooks in arithmetic, algebra, and geometry. Izvolsky died in Yaroslavi in 1938. The authors are grateful to Professor Idlar Safuanov from the Pedagogical Institute of Naberezhnye Chelny for the information he kindly provided concerning the life of Izvolsky.

[2]A complete study of the genetic method as intended by Toeplitz can be found in Schubring (1978).

[3]This passage, taken from *Jahresbericht der deutschen mathematischen Vereinigung*, XXXVI, 1927, 88–100, is reprinted in 1963 in the English version of Toeplitz's treatise *The calculus, a genetic approach.* The University of Chicago Press, Chicago London, 1963.

limitations. Indeed, when organized around their logical basis, the definitions of the main concepts of calculus (integrals, limits, derivatives) are abstract, and therein lies the burden of formal rules and theorems. Students have difficulty grasping the meaning of that with which they are asked to work. At present there are projects (not based on history) that take into account these difficulties and organize the teaching of calculus according to different patterns. (See, for example, the Harvard project based on giving an informal, operative approach to concepts in Hughes-Hallet et al., 1994).

What Toeplitz proposed is realistic and may be considered a compromise between the two ways of thinking about teaching mathematics, the logical versus developmental sequences. Toeplitz's historically based approach aims to provide a slow process of understanding that the student performs through a sequence of steps. Because Toeplitz's aim is to provide teaching materials that facilitate the learning of calculus, the main concern of the author is not to teach history, but to find learning sequences. Burn (1999) elaborated on these ideas. If we analyze Toeplitz's proposals or the more recent ideas of Burn, we may find an example of the history of mathematics used as a key element in the construction of a teaching sequence (on calculus) from intuition to logical deduction. The role of history is therefore that of providing materials on which to develop intuition. The presentation of the historical materials is not shaped according to recapitulationist principles because it uses modern symbols, verbal expressions, and cultural tools that are different from those of past authors.

An older example of the use of the genetic method (intertwined with a naïve heuristic approach) is in the treatise on geometry by Alexis-Claude Clairaut (1771). The preface of his book is an early example of predidactic literature. Its importance lies in the traces of Clairaut's thought that can be found in works on mathematics education through the 20th century. Clairaut wrote:

> Even if geometry is abstract in itself, we nonetheless must agree that the difficulties suffered by beginners come mostly from the way it is taught in usual treatises. They always start with a great deal of definitions, questions, axioms, and preliminary principles, which only seem to promise dry issues for readers.... To avoid this dry quality that is naturally linked to the study of geometry, some authors put examples after each proposition to show it is possible to do them; but in this way, they only prove the usefulness of geometry without making it any easier to learn. Because each proposition is presented before its use, the mind reaches concrete ideas after having toiled with abstract ideas. Having realized this fact, I intended to find out what may have given birth to geometry and tried to explain principles with the most natural methods, which I suppose were adopted by the first inventors, while trying to avoid the wrong attempts they had necessarily made. (Clairaut, 1771, pp. 2–4; our translation)

According to Glaeser (1983), Clairaut contributed greatly to the introduction of the genetic method. Glaeser commented on Clairaut's work with the following observations: "Giving up the dogmatic exposition, and to follow the true historical development of discovery, this method consists on imagining a road that learned peoples "could have followed"! Thus this is pretense education". (Glaeser, 1983, p. 341, our translation)

In spite of Glaeser's criticism, Clairaut's attempts present interesting features, even more so if we consider that in the period when this author conceived his project, the paradigm of geometrical teaching was based on the hypothetical-deductive Euclidean method. If we compare the passage from Toeplitz's book and Clairaut's passage, we see an extraordinary coincidence of intentions and didactic observations (i.e., the idea of "dryness" that is present in the work of both authors).

Freudenthal (1973) provided an interpretation of the genetic method:

> Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours

> cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now.... It is not the historical footprints of the inventor we should follow but an improved and better *guided* course of history. (Freudenthal, 1973, pp. 101, 103; our italics)

Freundenthal termed this way of using history "guided reinvention." It implies an active and aware participation of the teacher in designing and carrying out teaching experiments with history.

## 6. THE HISTORY OF MATHEMATICS IN THE CLASSROOM FROM THE TEACHER'S POINT OF VIEW

We have argued elsewhere (Furinghetti, 1997) that to study the applications of the history of mathematics in the classroom, we need a systemic net of experiments to analyze. For this reason, one of the authors (F. F.) has constituted a permanent monitor to keep track of the use of history in mathematics teaching in Italy. This means that teachers experimenting with the use of the history of mathematics, or only wishing to do so, are invited to contact the monitor and to discuss their ideas. In this way, it has been possible to create a file containing a range of different situations. The examples that we shall present in what follows come from these data.

First, we report on a workshop of teachers held by Jan van Maanen in Italy to present and discuss the ICMI Study document, "The role of the history of mathematics in the teaching and learning of mathematics," together with Italian researchers in mathematics education and high school teachers. Teachers participating in the workshop were asked if they use history in their classrooms. The answer, in general, was negative because of the constraints of the school system. Nonetheless, all the teachers expressed the strong interest in using it if they were given the opportunity. When asked to explain why they consider the use of history fruitful, the answer was something echoing—usually unintentionally—the recapitulation law. Some of the paradigmatic statements (quoted literally) include the following: "The students' development of concepts follow the historical sequence," "The historical genesis of the concept may help teachers understand the genesis of the concept in students' minds," and "If I present the students with how algebra developed in history, they feel differently about their difficulties in learning it."

Although not necessarily in a conscious or explicit way, the answers exhibit an understanding of the relation between ontogenesis and phylogenesis that is close to Haeckel's psychological version of the law of recapitulation. The following three examples illustrate, in a more detailed way, some teachers' positions about recapitulation.

We will see that in these cases the initial stimulus to consider the history of mathematics in their teaching is the vague idea that some parallelism between child development and mathematical development exists. Nonetheless, the kind and amount of adaptations that result from changes due to differences in historical periods and their cultural contexts are so significant that it is not possible to talk about some form of genuine recapitulation.

### 6.1. First Example

The first teacher is a mathematics instructor in a middle school (students aged 11 to 13), who studied biological sciences in college (and hence does not have a substantially deep understanding of mathematics) but is fond of mathematics and of teaching. She confesses her difficulties in teaching because of students' lack of motivation and her personal incapacity to interpret their difficulties. She has never carried out experiments in the classroom encompassing the use of history in mathematics teaching; nonetheless, she wrote (see also Gallo, 1999):

> I feel that my mathematical preparation lacks a historical perspective. I think I could find in history some answer to my teaching problems.
>
> In my opinion, to follow the evolution of the mathematical thinking could help the teacher understand how learning mathematics develops in children and preadolescents.
>
> As an example, I mention the use of fractions by the Egyptians: It is closer to the intuitive concept held by a primary pupil. I gave my 10-year-old daughter an Egyptian problem of dividing loaves among men taken from a seventh-grade mathematics textbook. She solved the problem in the way that the Papyrus Rhind solves it.
>
> I think it could be interesting to show students other issues taken from history: the geometrical representations of numbers, the geometrical representations of algebraic situations offered by Euclid. I think that the latter are more illuminating than the usual modern presentations.

The division problem the teacher used is the following problem in the Rhind Papyrus (ca. 1650 BCE): *"Example of reckoning out 100 loaves for 10 men, a sailor, a foreman and a watchman with double"* (see Peet, 1923, p. 109). Here we have an example of a teacher who does not have historical preparation; she only has some scattered ideas taken from notes in books and articles. She never carried out experiments using history in the classroom. Her experience is based on anecdotal facts. We interpret what she writes about history as being representative of the ideas that teachers in similar situations have about the use of history in teaching: There is a parallel between history and the way students learn.

## 6.2. Second Example

Other examples of the relationship of teachers with history that are more precise focus on experiments performed in the classroom. In these cases, the ideas expressed by the teacher are not mere intuition but are based on fact. The first case concerns a class of twenty-one 15-year-old high school students. We only briefly report on this experiment. (For a wider account, see Paola, 1998.) The teacher has a extensive experience in instruction and research in mathematics education. In the experiment, he acted as a teacher and as an observer. His purpose was to work with students on the concept of probability, which they had already encountered in previous school years. He chose to work with history to return to the concept of probability using a different (historical) approach. The work in the classroom was centered on a problem that is treated in many books of arithmetic from the Middle Ages "How can the stake be divided in a game where the two players are of the same value (in modern terms, have the same probability of winning) if the game is interrupted before one of the two players has realized the winning score?" This problem is known as "the problem of partition." Luca Pacioli gave his solution (based on proportionality) to this problem in his famous treatise *Summa de arithmetica geometria proportioni et proportionalità* (Printed in 1494). The classroom activity was developed through discussion of the problem between students divided into groups. The teacher not only orchestrated the discussion but also acted as an observer and reported all that happened in the classroom. Initially all students agreed that the best way to solve the problem would be to divide the stake in parts that were proportional to the scores earned by each player. The teacher easily refuted this solution by proposing that one of the two players had a score of zero when the game was interrupted. After a discussion on this particular case, another group of students proposed other ways of solving it that did not satisfy their classmates. At this point, the teacher read Pacioli's solution, which is similar to that of the students, allowing them to see that an important historical personage followed the same process they did. The students seemed ready to approach the concept of fair division of the stake. Additional classes were dedicated to discussing this concept, but the students did not arrive at effective results on their own (i.e., they were not able to grasp the concept of probability). The teacher expounded Pascal's the solution to the problem, as reported in (Pascal, 1954), and thus introduced students to the concept of probability.

As we said previously, the teacher acted as an observer, and he accurately reported the activity in the classroom (Paola, 1998). Even if some elements of probability had been taught to these students in the previous school years, it is clear from the chronicle of the classroom activity that their strategies were based on proportionality, as Pacioli's were. The teacher believes this experiment shows that students follow the path of history:

> The voice of history is again evoked by the teacher to give dignity to the students' solutions which actually follow the path hinted by mathematicians before Pascal and Fermat. (Paola 1998, p. 34)

There are many passages suggesting that the teacher is concerned with the mistakes in the ancient attempts of solving Pacioli's problem. For example: "The incursion into history had the goal of giving dignity to the mistake made by students: it was not a trivial mistake if a mathematician made it" (p. 33).

The teacher showed interest in the parallels between the strategies his pupils and Pacioli used, but he did not draw general theoretical conclusions concerning the recapitulation law. From his conclusions, we see only that he has a certain confidence in the validity of following the stages of the historical development for didactic purposes:

> With another session I could have read and commented on the Pascal–Fermat letters in the classroom and thus I would have stressed the role of history [in helping students to bypass some obstacles in constructing concepts of probability theory] (Paola, p. 35).

## 6.3. Third Example

The last case we present concerns a high school mathematics and physics teacher who works with students ranging in age from 16 to 19 years. The teacher has researched the history of mathematics. She is interested in proof and tries to develop students' abilities on this subject using historical examples. To this end, she uses the method of analysis and synthesis, found in the Pappus's *Collectiones Mathematicae*. We describe this method with the following passage taken from Hintikka and Remes (1974):

> Now analysis is the way from what is sought—as if it were admitted—through its concomitants [the usual translation reads consequences] in order to something admitted in synthesis. For in analysis we suppose that which is sought to be already done, and we inquire from what it results, and again what is the antecedent of the latter, until we on our backward way light upon something already known and being first in order. And we call such a method analysis, as being a solution backwards. In synthesis, on the other hand, we suppose that which was reached last in analysis to be already done, and arranging in their natural order as consequents the former antecedents and linking them one with another, we in the end arrive at the construction of the thing sought. And this we call synthesis. (p. 8)

The method of analysis is described in a manual for teachers (Smith, 1911) as follows:

> I can prove this proposition if I can prove this thing; I can prove this thing if I can prove that; I can prove that if I can prove a third thing," and so the reasoning runs until the pupil comes to the point where he is able to add, "but I can prove that." This does not prove the proposition, but it enables him to reverse the process, beginning with the thing he can prove and going back, step by step, to the thing that he is to prove. Analysis is, therefore, his method of discovery of the way in which he may arrange his synthetic proof. (Smith, 1911, pp. 161–162)

Historically this method originated in the field of geometry, but it has since been used in other branches of mathematics. For example, the method of analysis is at the heart of algebra: The introduction of symbols made by Viète in the 16th century did not arise spontaneously but was a consequence of having adopted the method of analysis for solving algebraic problems (Charbonneau, 1996). The method of analysis also is not specific to mathematics; for example, in Marchi (1980), it is applied to chemistry. The method of analysis represents a link between history and education. In their chapter on proof, Alibert and Thomas (1991) proposed a method of proving that is similar to the method of analysis, probably without considering the history of mathematics.

The teaching experiment with this method that the teacher in this example carried out lasted for many years. We report on only briefly this experiment; for a lengthier account, see Somaglia (1998). At the beginning of the lesson, the teacher presents her students with the method of analysis in the field of Euclidean geometry. Students experience the application of this method in different problems until the method is mastered and recognized as a tool for attacking geometrical problems. Afterward, the teacher has the students apply the method to other parts of mathematics (algebra and calculus) so that they become aware of the transversality of the method (i.e., that the method is not linked to a particular domain of knowledge but can be generalized). Students are then ready to attack problems in physics and in chemistry using this method (see Clavarino & Somaglia, 2001).

In the description of her work, the teacher never mentioned any parallel between the strategies of her students and those of past mathematicians, nor the persistence of errors. In our experience, this fact is unusual among teachers dealing with mathematics history. There are two developments in the work of this high school teacher, the historical and the educational, that interact, and her way of looking at these processes is very positive. The teacher looks for what can give students the means to realize the condensation of concepts (see Sfard, 1991). This teacher has an excellent knowledge of mathematics history, and moreover it is quite natural for her to work with original sources. Thus, history is an integral part in her mathematics teaching. Her contact with the past is not that of someone who looks at the past with the eyes of the present but one who sees the concepts of the past as real and important content—as foundations in an architectonic sense—upon which our modern concepts and methods are based. She puts in action Gadamer's way of looking at the past, that is, as "a dialogical process in which two horizons (the past and the present) are fused together" (Radford, 1997a, p. 27).

## 7. THE RECOURSE TO HISTORY IN CONTEMPORARY MATHEMATICS EDUCATION

In the previous sections, we discussed some interpretations of recapitulation law made by past mathematicians and teachers. Let us now examine a few examples of contemporary mathematics educators, confining our discussion to two specific cases. The first emphasizes (mainly although not exclusively) a theoretical interest. The second appears closer to specific contexts arising from the needs to enhance teaching and learning processes in mathematics instruction. In the first case, the history of mathematics appears as a theoretical tool to understand developmental aspects of mathematical thinking. The purpose of the second case is to facilitate, through explicit pedagogical interventions, students' learning of mathematics by attempting to relate the development of students' mathematical thinking to historical conceptual developments.

### 7.1. The Interface Between History and Developmental Aspects of Mathematical Thinking

The work of Sfard (1995) provides a clear example of contemporary views on the relation between history and the developmental aspects of mathematical thinking. She analyzed the development of algebra by blending historical and psychological perspectives. At the beginning of her article, she claimed that

> there are good reasons to expect that, when scrutinized, the phylogeny and ontogeny of mathematics will reveal more than marginal similarities. At least, this is what follows from the constructivist view according to which learning consists in the reconstruction of knowledge. (p. 15)

The similarities between the phylogenetic and ontogenetic domains result in this account from "inherent properties of knowledge." For Sfard, who follows a Piagetian epistemological perspective, knowledge can be theoretically described in terms of genetic structural levels, and it is precisely the nature of the relationship between the different levels that accounts for the similarity of phenomena appearing in the historical and in the individual's construction of knowledge. As she noted, "difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those that once challenged generations of mathematicians" (Sfard, 1995, pp. 15–16). A large part of the text is devoted to the discussion of the development of algebraic language. Indeed, using Nesselmann's (1842) distinction between rhetorical, syncopated, and symbolic algebra, Sfard endeavored to locate those "constants" (more precisely, those "developmental invariants") that ensure the passage from rhetorical and syncopated algebra to symbolic algebra. Rhetorical algebra refers to the reliance on nonsymbolic, verbal expressions to state and solve a problem, as it appears, for instance, in Arabic, Hindu, and Italian Medieval texts. Syncopated algebra is seen as a more elaborate algebra in that, although still relying heavily on verbal expressions, it introduces some symbols, the work of Diophantus being the canonical example. Viète's systematic introduction of letters epitomizes symbolic algebra. After confronting experimental classroom results with the traditional view of the historical development of algebra, Sfard concluded that one of the development invariants underpinning the passage from rhetorical and syncopated algebra to symbolic (Vietan) algebra is the precedence of operational over structural thinking. Operational thinking, in this context, means a way of thinking about algebraic objects in terms of computational operations. Structural thinking is related to more abstract objects conceived structurally on a higher level.

As we can see, the use of history in Sfard's approach is an attempt to corroborate parallelisms between ontogenetic and phylogenetic developments. As she said, "history will be used here only to the extent which is necessary to substantiate the claims about historical and psychological parallels" (Sfard, 1995, p. 17). Although she stressed the importance for teachers to be aware of the historical development of mathematics, the intention is not that of creating an historically inspired classroom activity. This is the goal of another perspective in contemporary mathematics education, discussed in section 7.2. For the time being, we want to mention a sociocultural approach that shares Sfard's use of history for epistemological reasons but, in contrast, emphasizes the crucial link between cognition and the practical human activity in which cognition is embedded. This approach (see Radford, 1997a; Radford et al., 2000), inspired by key ideas of the Vygotskian and cultural perspectives alluded to in section 3 of this chapter, is driven by a conception of knowledge that differs from Piagetian genetic structuralism, particularly in that knowledge and the individuals' intellectual means to produce it are seen as intimately and contextually related to their cultural setting. Knowledge, in fact, is conceived as the product of a *mediated cognitive reflexive praxis* (see Radford, 2000b). The mediated character of knowledge refers to the role played by artifacts, tools, sign systems, and other means to achieve and objectify the cognitive praxis. The reflexive nature of knowledge is to be understood in Ilyenkov's sense, that is, as the distinctive component that makes cognition an intellectual reflection of the external world in the forms of the individual's activity (Ilyenkov, 1977, p. 252). Knowledge as the result of a cognitive praxis (*praxis cogitans*) emphasizes the fact that what we know and the way we come to know it is framed by ontological stances and by cultural meaning-making processes that shape a certain kind of rationality out of which specific kinds of mathematical questions and problems are posed.

Theoretically, however, this does not mean that the study of knowledge is determined by social, economical, and political factors because these are also historically produced. Certainly, the link between culture and cognition is more subtle than the distinction between the "internal" and "external" realms employed in many historiographic approaches that see the external as mere stimulus for conceptual changes and development. Methodologically, this means that the study of the historical development of mathematics cannot be reduced to the sociology of knowledge. This also means that such a study cannot be done through the analysis of texts only. The "archive" (to borrow Foucault's expression), as a historical repository of previous experiences and conceptualizations, bears the sediments of social, economic, and symbolic human activities. Therefore, understanding the rationality within which a mathematical text was produced requires relocating the text with in its own context. The goal of this kind of epistemological reflection is not to find a parallel between phylogenetic and ontogenetic developments. In the sociocultural approach that we advocate, mathematical texts from other cultures are investigated while taking into account the cultures in which they were embedded. This allows the researcher to scrutinize the way mathematical concepts, notations, and meanings were produced.

Through an oblique contrast with the notations and concepts taught in contemporary curricula, we seek to gain insights about the intellectual requirements that learning mathematics demands of our students. We also seek to broaden the scope of our interpretations of classroom activities. In designing classroom activities, we aim at eventually adapting conceptualizations embedded in history to facilitate students' understanding of mathematics. Our work on Babylonian algebra and the teaching of second-degree equations (Radford & Guérette, 2000) is an example of the latter. Our classroom research on the strategies students use to deal with the algebraic generalization of patterns and the way they conceive relations between the concrete and the abstract (see Radford, 1999b, 2000c)—research based on our investigation of pre- and Euclidean forms to convey generality (Radford, 1995a)—is an example of oblique contrast between past developments and contemporary students' conceptualizing processes.

Our classroom research on the introduction of algebraic symbolism also benefited from our epistemological inquiries based on editions of original texts from Medieval and Renaissance Italian mathematics (Radford, 1995b, 1997b). Space constraints do not allow us to go further, but this anthropological approach to the epistemology of mathematics offers a new view of the rise of symbolic algebra in the 16th century. The difference from traditional views stressing the passage from syncopated to abstract algebra in terms of abstractive processes is that, in our account, changes in development are related to changes in societal practices and the way in which mathematical conceptualizations are subsumed in them. Briefly, what we find in our analysis is that there were two main mathematical practices in the early Renaissance, that used by merchants and abacus mathematicians and that used by humanists and court mathematicians. While the latter were busy with the restoration of Greek texts, the former were applying Arabic algebraic techniques to practical as well as nonpractical problems (e.g., problems about numbers). Symbolic algebra was a timeconsuming effort made by Italian humanist and engineer mathematicians, such as the priest Francesco Maurolico, who eradicated all commercial content in his *Demonstratio Algebrae*, which was completed October 7, 1569 and edited by Napoli in the 19th Century (Napoli, 1876). Another example is the engineer Rafael Bombelli, who, after having learned that the first books of Diophantus' *Arithmetic* were on the shelves of a Roman library, studied them and ended up eliminating the commercial problems in his *Algebra*. Bombelli provided a final version of it that conformed much more to the humanist understanding of Greek mathematics. In France, a similar effort was made by the humanists Jacques Peletier and Guillaume Gosselin (although in this case, the promotion of French as a scientific language was an important drive; Cifoletti, 1992). The underlying reason for the effort to introduce a specific symbolism in algebra was not due to the limitations of vernacular language. Mathematicians working within the possibilities offered by rhetorical algebra produced many difficult problems involving several unknowns, as can be seen in Fibonacci's *Il Flos* (Picutti, 1983). These problems could not be simplified by the introduction of letters because what was symbolized in the emergence of symbolic algebra did not include all of the unknowns mentioned in a problem but only one of them. (See, for instance Bombelli's symbolism or the neogeometrical example in Piero della Francesca's *Trattato d'abaco*, edited by Arrighi, 1970.) It was only later that some in Germany began using letters for several unknowns (see Radford, 1997b). In our approach, the emergence of algebraic symbolism appears to be related to the effort made by humanists and court-related mathematicians to render the merchant's algebra noble and Court worthy (details in Radford, 2000b). This was accomplished by the lawyer and mathematician François Viète, at the French court, who followed the prestigious Greek traditions typified by Diophantus' *Arithmetic* rather than the multitude 15th- and 16th-century of abacus treatises.

We now discuss a second reference to the use of history in contemporary mathematics education, that which aims at enhancing, through explicit pedagogical interventions the students' learning of mathematics.

### 7.2. Enhancing Students' Mathematical Thinking Through Historically Based Pedagogical Actions

Boero and collaborators (see Boero, Pedemonte, & Robotti, 1997; Boero, Pedemonte, Robotti, & Chiappini, 1998) made use of the mathematics history to investigate the nature of theoretical knowledge and the conditions by which it emerges. Their historico-epistemological analysis aims at looking for elements considered typical of mathematical thinking, such as organization, coherence, and systematic character. They have investigated the role played by definitions and proofs, as well as by the type of theoretical discourse. The framework draws from Bakhtin's theory of discourse, mainly from the theoretical construct of "voice" (Bachtin, 1968, Wertsch, 1991) and from Vygostky's distinction between scientific and everyday concepts (Vygotsky, 1962). The historico-epistemological inquiry is subsequently invested in the design and implementation of teaching settings based on a careful selection of primary sources of which the main objective is to allow the students to echo the voice of past mathematicians. In the students' echoing process, the students bring their individual subjective and cultural backgrounds to build from it a "voices and echoes game," which proves to be fruitful for the acquisition of theoretical knowledge. The voices from the past are not listened to passively but actively appropriated through an effort of interpretation. Usually the students' echoes may take various forms. Boero and his team have provided a categorization of some of the ways in which the students enter the dialogical game. For instance, a "mechanical echo" consists in precise paraphrasing of a verbal voice, whereas an "assimilation echo" refers to the transfer of the content and method conveyed by a voice to other problem situations. A "resonance" is a student's appropriation of a voice as a way of reconsidering and representing his or her experience.

Among the concrete instance of theoretical knowledge examined by the authors are the theories of the falling bodies of Galileo and Newton, Mendel's probabilistic model of the transmission of hereditary traits, and theories of mathematical proof and algebraic language, all of which feature aspects of a counterintuitive character.

Another example of the contemporary use of history in the classroom is the research of Sierpinska and collaborators. One of the goals of this research is to provide an alternative, based on the use of the Cabri-Géomètre software, to the traditional axiomatic approach to the teaching linear algebra in undergraduate courses. A problem examined in this research, which underlies important aspects of the learning of basic linear algebra, is that of understanding key differences in the representations of mathematical objects. In this line of thought, Sierpinska has emphasized the distinction between a "numerical" and "geometrical" space. The objects of the arithmetic spaces are sets of n-tuples of real numbers defined by conditions (in the form of equations, inequalities, etc.) on the terms of the n-tuples belonging to the sets. It stresses the fact that these objects can be represented by geometric figures (e.g., lines, surfaces). Geometric objects, in contrast, are defined as a locus of points verifying some conditions (e.g., the "geometric circle" means the locus of points equidistant from a given point). The geometric objects can be represented by sets of n-tuples defined by conditions on their terms (e.g., by equations). Thus, in the case of arithmetical spaces, the geometrical aspect is derived from the numerical one; in the case of geometrical spaces, the numerical aspect results from the geometrical one. A suitable understanding of elementary linear algebra requires the students to establish a convenient relation between the geometrical and the numerical views of the objects of linear algebra and to grasp that the roles of objects and representations are reversed.

The difference between geometrical and numerical space is clear in the history of linear algebra. Sierpinska, Defence, Khatcherian, and Saldanha (1997) identified three modes of reasoning, which they labeled "synthetic-geometric," "analytic-arithmetic," and "analytic-structural." As they noted (a more detailed report is in Bartolini Bussi & Sierpinska, 2000), the concepts of linear algebra do not all have the same meaning and, in the classroom, they are not equally accessible to beginning students. The design of the teaching activities as well as the understanding of students' answers took into account the modes of reasoning as determined in the historico-epistemological analysis. (An extended account of the teaching activities can be found in Sierpinska, Trgalová, Hillel, & Dreyfus, 1999a and Sierpinska, Dreyfus, & Hillel, 1999b.)

## 8. SYNTHESIS AND CONCLUSION

In this chapter, we dealt with one of the many uses of the history of mathematics in mathematics education, namely, a use that can be characterized as an attempt to investigate historical conceptual developments to deepen our understanding of mathematical thinking and to enhance the students' conceptual achievement. In the first part of the article, we saw how psychological recapitulation was imported from biological recapitulation and gave rise to a discourse that framed much of the discussions about child development since the beginning of the 20th century. Psychological recapitulation was adopted by some eminent mathematicians who, in one form or another, supported the idea that in developing their mathematical thinking, children would traverse similar steps as those followed by humans. Within this conception, children will supposedly find during their development some similar problems, difficulties, or obstacles as those encountered by past mathematicians. Recapitulationism, we argued, served the cause of some mathematicians as a means to counter the teaching orientation based on commitments to rigor and logical structures arising in the flow of the research on the foundations of mathematics at the turn of the 20th century.

Nonetheless, one of the problems with the recapitulationist approach is that conceptual developments are seen as chronologically self-explanatory, and psychological evolution is taken for granted. Furthermore, knowledge is conceived as having little (if any) bond to its context, and the idea of history is reduced to a linear sequence of events judged from the vantage point of the modern observer. In all likelihood, the extremely low number of studies that attempt to check the validity of recapitulation law is evidence of the impossibility of reproducing the conditions in which ideas developed in the past. As Dorier and Rogers noted, "'naive recapitulationism' has persisted in many forms and now we accept that the relation between ontogenesis and phylogenesis is universally recognized to be much more complex than was originally believed" (Dorier & Rogers, 2000, p. 168).

This statement corresponds well with recent nonpositivist epistemological and anthropological trends. Indeed, in emphasizing the relation between knowledge and social practices, these trends have raised some criticisms to the acultural stance conveyed by the general and universal character of the recapitulation law, thereby opening new ways to reconceptualize the relations between historical conceptual developments and the teaching of mathematics.

In the course of our discussion, we mentioned two different and critical stances toward the relation between ontogenesis and phylogenesis as elaborated by Piaget and Garcia on one hand and by Vygotsky and his collaborators on the other. The way Piagetian and Vygotskian epistemologies have inspired current work on contemporary mathematics education was made clear in the brief presentations of specific traits in the works of Sfard, Radford, Boero, and Sierpinska, works that attempt to contrast (with different purposes and in different senses) ontogenetic and phylogenetic developments to shed light on the nature of mathematical knowing as well as on the teaching and learning of mathematics.

Regarding recommendations for future research, it can be suggested, in light of the previous discussion, that a pedagogical use of the history of mathematics committed to enhance students' conceptual achievements requires a critical reflexion about the conceptions of ontogenesis and phylogenesis and, of course, of knowledge itself. But to be fruitful in practical terms, such a critical reflexion must be clear about its classroom implications. In particular, efforts to include teachers in the reflexive enterprise must be made. The work of Furinghetti suggests that to reach effectiveness in using history, teachers' willingness is not enough. To use history productively, teachers need to gain an appropriate understanding of differences between ontogenetic and phylogenetic developments and to bear a critical stance toward recapitulation views. As the sophisticated methodology of Boero's approach suggests, this requires teachers to be amply comfortable in handling cognitive and historical aspects. Let us make three suggestions concerning actions for research.

1. On a theoretical level, discussions about recapitulation and its different meanings should be promoted among historians, epistemologists, psychologists, anthropologists, and mathematics educators.
2. On a practical level, models of contrasts and conceptualizations between ontogenetic and phylogenetic developments also should be considered further. Models

of contrast may help us to better grasp specific traits of mathematical thinking, its relation to the cultural settings, and the mathematical concepts thus produced. This can lead to a better understanding of the kind of practical pedagogical interventions that can be envisioned.

3. Theoretical reconceptualizations of recapitulation and contrasts and comparisons between ontogenetic and phylogenetic domains should be explicit as to how they can frame the engineering of material and teaching sequences.

We consider these related research topics as being interactively fed by theoretical enquiries, historical studies, and also classroom observations.

The course of the three aforementioned actions for future research will ultimately depend on the very conception of mathematical knowledge to be adopted. At this point, two main contrasting trends seem to be emerging. In the first trend, what makes the specificity of mathematical knowledge is its systemic, objective, and logical nature (see Fujimura, 1998). In the second trend, which is much more anthropologically driven, knowledge is conceived as a kind of culturally framed activity enabling individuals to enquire about their world and themselves. Here "systematicity" and "logicality" are seen as circumscribed characteristics of knowledge that can be different from culture to culture (see Radford, 1999c). Between them, of course, many possibilities can be envisaged. To theoretically elaborate on some of those possibilities, to build practical and conceptual reflexions about historical and contemporary "developments," and to deepen our understanding of mathematics and facilitate the way students learn is a challenge for the years to come.

## ACKNOWLEDGMENT

We wish to thank Rossella Dorigo for her help in the preparation of this chapter.

## REFERENCES

Alibert, D., & Thomas, M. (1991). Research on mathematical proof. In D. O. Tall (Ed.), *Advanced mathematical thinking* (pp. 215–230). Dordrecht, The Netherlands: Kluwer.

Arrighi, G. (Ed.). (1970). *Piero Della Francesca: Trattato de Abaco*. Pisa, Italy: Domus Galileana.

Bachtin, M. (1968). *Dostoevskij, poetica e stilistica*. Torino: Einaudi.

Bartolini Bussi, M., & Sierpinska, A. (2000). The relevance of the historical studies in designing and analysing classroom activities. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education. The ICMI Study* (pp. 154–161). Dordrecht, The Netherlands: Kluwer.

Boero, P., Pedemonte, B., & Robotti, E. (1997). Approaching theoretical knowledge through voices and echoes: A Vygotskian perspective. *Proceedings of the 21st Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 81–88), University of Helsinki, Finland.

Boero, P., Pedemonte, B., Robotti, E., & Chiappini, G. (1998). The "voices and echoes game" and the interiorization of crucial aspects of theoretical knowledge in a Vygotskian perspective: Ongoing research. *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 2, pp. 120–127), University of Stellenbosch, South Africa.

Branford, B. (1921). *A study of mathematical education*. Oxford, U.K.: Clarendon Press. (Original work published in 1908)

Bringuier, J-C. (1980). *Conversations with Jean Piaget*. Chicago: Chicago University Press.

Bruner, J. (1990). *Acts of meaning*. Cambridge, Massachusetts/London, England: Harvard University Press.

Burn, B. (1999). Integration, a genetic approach. *Nordisk Matematikk Didaktikk*, 7(1), 7–27.

Cajori, F. (1894). *A history of mathematics*. New York: Macmillan.

Calinger, R. (1996). *Vita mathematica: Historical research and integration with teaching*. Washington, DC: Mathematical Association of America.

Chabert, J-L., Barbin, É., Guillemot, M., Michel-Pajus, A., Borowczyk, J., Djebbar, A., & Martzloff, J.-C. (1994). *Histoire d'algorithmes. Du caillou à la puce*. Paris: Belin.

Charbonneau, L. (1996). From Euclid to Descartes: Algebra and its relation to geometry. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra. Perspectives for research and teaching* (pp. 15–37). Dordrecht: Kluwer.

Cifoletti, G. (1992). *Mathematics and rhetoric. Peletier and Gosselin and the making of the French algebraic tradition.* Unpublished doctoral dissertation, Princeton University, NJ.

Clairaut, A.-C. (1771). *Elementi di geometria* (2nd ed.). Rome: V. Monaldini. (Original work published) *Elements of geometry*, London: C. Kegan Paul, 1881; English translation: 1741)

Clavarino, F., & Somaglia, A. (2001). Il procedimento di analisi-sintesi e la sua visualizzazione: Una guida attraverso la matematica, le scienze, l'arte. *L'insegnamento della Matematica e delle Scienze Integrate, 24B,* 257–278.

de Santillana, G. (1961). *The origins of scientific thought.* Chicago: University of Chicago Press.

Dhombres, J., Dahan-Dalmedico, A., Bkouche, R., Houzel, C., & Guillemot, M. (1987). *Mathématiques au fil des âges.* Paris: Gautier-Villars.

Dorier, J.-L., & Rogers, L. (2000). The historical formation of mathematical thinking and the contemporary student understanding of mathematics. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education. The ICMI Study* (pp. 168–170). Dordrecht, The Netherlands: Kluwer.

Elias, N. (1991). *La société des individus.* Paris: Fayard.

Fauvel, J. (1991). Using history in mathematics education. *For the Learning of Mathematics, 11(2),* 3–6.

Fauvel, J., & Van Maaren, J. (Eds.). (2000). *History in mathematics education. The ICMI Study.* Dordrecht, The Netherlands: Kluwer.

Freudenthal, H. (1973). *Mathematics as an educational task.* Dordrecht, The Netherlands: D. Reidel.

Fujimura, J. (1998). Authorizing knowledge in science and anthropology. *American Anthropologist, 100,* 347–360.

Furinghetti, F. (1997). History of mathematics, mathematics education, school practice: Case studies linking different domains. *For the Learning of Mathematics, 17,* 55–61.

Furinghetti, F. (2000). The history of mathematics as a coupling link between secondary and university teaching [Special issue of the selected papers of the ICMI Study in Singapore]. *International Journal of Mathematical Education in Science and Technology, 31,* 43–51.

Furinghetti, F., & Somaglia, A. M. (1998). History of mathematics in school across disciplines [Special Issue]. *Mathematics in School, 27(4),* 48–51.

Gallo, M. T. (1999). Anche nella scuola dell'obbligo. *Lettera PRISTEM, 32,* 60–61.

Glaeser, G. (1983). À propos de la pédagogie de Clairaut. Vers une nouvelle orientation dans l'histoire de l'éducation. *Recherches en Didactique des Mathématiques, 4,* 332–344.

Gould, S. J. (1977). *Ontogeny and phylogeny.* Cambridge, MA: Belknap Press of Harvard University Press.

Grugnetti, L., & Rogers, L. (2000). Philosophical, multicultural and interdisciplinary issues. In J. Fauvel & J. Maanen (Eds.), *History in mathematics education. The ICMI Study* (pp. 39–62). Dordrecht, The Netherlands: Kluwer.

Gusev, V. A., & Safuanov, I. S. (2000). Some theoretical problems of the development of mathematical thinking. In T. Nakahara & M. Koyama (Eds.). *Proceedings of the 24th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, 17–24). University of Hiroshima, Japan.

Haeckel, E. (1912). *The evolution of man.* London: Watts & Co. (Original work published 1874)

Hanna, G. (1996). The ongoing value of proof. In A. Gutierrez & L. Puig (Eds.), *Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, 21–34). University of Valencia, Spain.

Hintikka, J., & Remes, U. (1974). *The method of analysis.* Dordrecht, The Netherlands: Reidel.

Hughes-Hallett, D. et al. (1994). *Calculus.* New York: John Wiley.

Ilyenkov, E. V. (1977). *Dialectical Logic.,* Moscow: Progress Publishers.

Katz, V. (Ed.). (2000). *Using history to teach mathematics. An international perspective.* Washington, DC: Mathematical Association of America.

Klein, F. (1896). The arithmetizing of mathematics. An address delivered at the public meeting of the Royal Academy of Sciences of Göttingen, November 2, 1895 (I. Maddison, Trans.). *Bulletin of the American mathematical society,* series II, 2, 241–249.

Klein, F. (1924). *Elementarmathematik vom höheren Standpunkte aus, 1.* Berlin: Springer. (English translation: *Elementary mathematics from an advanced standpoint.* New York: Dover, 1945)

Lagarto, M., Vieira, A., & Veloso, E. (Eds.). (1996). Proceedings of the Quadrennial Meeting of the International Study Group on the Relations Between History and Pedagogy of Mathematics and Deuxième Université d'été Européenne sur l'Histoire et l'Épistémologie dans l'éducation Mathématique. Universidade do Minho, Braga, Portugal.

Lalande, F., Jaboeuf, F., & Nouazé, Y. (Eds.). (1995). *Actes de la Première Université d'Été Européenne. Histoire et Épistémologie dans l'Éducation Mathématique.* Montpellier, France: IREM de Montpellier.

Lévy-Bruhl, L. (1928). *Les fonctions mentales dans les sociétés inferieures* (9e éd.). Paris: Librairie Félix Alcan. (Original work published 1910)

Lotman, Y. (1990). *Universe of the mind. A semiotic theory of culture.* London (UK): Bloomington and Idianapolis, Indiana University Press, Taurus.

Marchi, P. (1980). The method of analysis in mathematics. In T. Nickles (Ed.), *Scientific discovery, logic and rationality* (pp. 159–172). Dordrect: Reidel.

Mengal, P. (1993). Psychologie et loi de récapitulation. In P. Mengal (Ed.), *Histoire du concept de récapitulation* (pp. 93–109). Paris/Milan/Barcelone/Bonn: Masson.

Napoli, F. (1876). Intorno alla vita ed ai labori di Francesco Maurolico. *Bullettino di Bibliografia e di Storia delle Scienze Mathematiche e Fisiche, 9*, 1–22, 23–121.

Nesselmann, G. H. F. (1842). *Die Algebra der Griechen*. Berlin. (Reprinted Frankfurt: Minerva GMBH, 1969)

Osborn, H. F. (1929). *From the Greeks to Darwin*. New York/London: Scribner.

Pacioli, L. (1494). *Summa de arithmetica geometria proportioni et proportionalità*. Venice, Italy.

Paola, D. (1998). Il problema delle parti: Prassi didattica e storia della matematica. *Didattica delle Scienze e Informatica nella Scuola, XXXIV (198)*, 31–36.

Pascal, B. (1954). *Oeuvres complètes, par J. Chevallier*. Paris: Gallimard.

Peet, T. E. (1923). *The Rhind mathematical papyrus*. London: University Press of Liverpool—Hodder & Stoughton.

Picutti, E. (1983). Il "Flos" di Leonardo Pisano. *Physis, 25*, 293–387.

Poincaré, H. (1899). La logique et l'intuition dans la science mathématique et dans l'enseignement. *L'enseignement Mathématique, 1*, 157–162.

Radford, L. (1995a). La transformación de una teoría matemática: El caso de los números poligonales, *Mathesis, 11*(3), 217–250.

Radford, L. (1995b). Before the other unknowns were invented: Didactic inquiries on the methods and problems of mediaeval Italian algebra. *For the Learning of Mathematics, 15*(3), 28–38.

Radford, L. (1997a). On psychology, historical epistemology and the teaching of mathematics: Towards a socio-cultural history of mathematics. *For the Learning of Mathematics, 17*(1), 26–33.

Radford, L. (1997b). L'invention d'une idée mathématique: La deuxième inconnue en algèbre. *Repères (Revue des instituts de Recherche sur l'enseignement des Mathématiques—IREMs), 28*, 81–96.

Radford, L. (1998). On signs and representations. A cultural account. *Scientia Paedagogica Experimentalis, 35*, 277–302.

Radford, L. (1999a). *On cultural and mind, a post-Vygotskian semiotic perspective, with an example from Greek mathematical thought*, preprint series n°2/1999, École des sciences de l'éducation, Université Laurentienne, Ontario, Canada. (Available at: http://www.laurentian.ca/educ/lradfordl)

Radford, L. (1999b). The rhetoric of generalization, *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 4, 89–96). Technion-Israel Institute of Technology, Haifa, Israel.

Radford, L. (1999c). La razón desnaturalizada. Ensayo de epistemología antropológica. *Revista Latinoamericana de Investigación en Matemática Educativa, 3*, 47–68.

Radford, L. (2000a). The historical development of mathematical thinking and the contemporary student understanding of mathematics. Introduction. In J. Fauvel & J. Maanen (Eds.), *History in mathematics education. The ICMI Study* (pp. 143–148). Dordrecht, The Netherlands: Kluwer.

Radford, L. (2000b, May). *Savoir mathématique, processus de signification et pratiques sociales: le cas de l'algèbre de la Renaissance*. Paper presented at the 68th Congrès de l'Association Canadienne-Française l'Avancement des Sciences (ACFAS), May 15–19, 2000, Université de Montréal, Québec.

Radford, L. (2000c). Signs and meanings in students' emergent algebraic thinking: A semiotic analysis. *Educational Studies in Mathematics, 42*, 237–268.

Radford, L., Boero, P., & Vasco, C. (2000). Epistemological assumptions framing interpretations of students understanding of mathematics. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education. The ICMI Study* (pp. 162–167). Dordrecht, The Netherlands: Kluwer.

Radford, L., & Guérette, G. (2000). Second degree equations in the classroom: A Babylonian approach. In V. Katz (Ed.), *Using history to teach mathematics. An international perspective* (pp. 69–75). Washington, DC: Mathematical Association of America.

Reimer, L., & Reimer, W. (1995). *Mathematicians are people too*. Palo Alto, CA: Dale Seymour.

Schubring, G. (1978). *Das genetische Prinzip in der Mathematik-Didaktik*. Stuttgart: Klett-Cotta.

Severi, F. (1930). *Elementi di geometria*. Firenze (Italy): Vallecchi.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics, 22*, 1–36.

Sfard, A. (1995). The development of algebra: Confronting historical and psychological perspectives. *Journal of Mathematical Behavior, 14*, 15–39.

Sierpinska, A., Defence, A., Khatcherian, T., & Saldanha, L. (1997). À propos de trois modes de raisonnement en algèbre linéaire. In J.-L. Dorier (Ed.). *L'Enseignement de L'Algèbre Linéaire en Question* (pp. 249–268). Grenoble: La pensée sauvage éditions.

Sierpinska, A., Tragalová, J., Hillel, J., & Dreyfus, T. (1999a). Teaching and learning linear algebra with Cabri. In O. Zaslavski (Ed.), *Proceedings of 23rd Conference of the International Group for the Psychology of Mathematics Education* (Vol. 1, 119–134). Technion-Israel Institute of Technology, Haifa, Israel.

Sierpinska, A., Dreyfus, T., & Hillel, J. (1999b). Evaluation of a teaching design in linear algebra: The case of linear transformations. *Recherches en Didactique des Mathématiques, 19*, 7–41.

Smith, D. E. (1911). *The teaching of geometry*. Boston: Ginn and company.

Somaglia, A. (1998). Analisi-sintesi: Un resoconto di esperienze didattiche. In E. Gallo, L. Giacardi, & C. S. Roero (Eds.), *Conferenze e seminari 1997–1998 della Mathesis Subalpina*, Torino (Italy): Dipartimento di motamatica dell'Universitàà (pp. 16–27).

Swetz, F., Fauvel, J., Bekken, O., Johansson, B., & Katz, V. (1995). *Learn from the masters*. Washington, DC: Mathematical Association of America.
Tulviste, P. (1991). *The cultural-historical development of verbal thinking*. New York: Nova Sciences.
Vygotsky, L. S. (1962), *Thought and language*, Cambridge, MA: MIT Press.
Vygotsky, L. S. (1997). *Collected works, Volume 4* (R. Rieber, Ed.). New York: Plenum Press.
Vygotsky, L. S., & Luria, A. (1994). Tool and symbol in child development. In R. van der Veer & J. Valsiner (Eds.), *The Vygotsky reader* (pp. 99–174). Oxford, U.K.: Blackwell.
Walkerdine, V. (1997). Redefining the subject in situated cognition theory. In D. Kirshner & J. A. Whitson (Eds.), *Situated cognition* (pp. 57–70). Mahwah, NJ: Lawrence Erlbaum Associates.
Wartofsky, M. (1979). *Models, representation and the scientific understanding*. Dordrecht: Reidel.
Werner, H. (1957). *Comparative psychology of mental development*. New York: International Universities Press.
Wertsch, J. V. (1991). *Voices of the mind*. Cambridge, MA: Harvard University Press.
Zeuthen, H.-G. (1902). *Histoire des mathématiques dans l'antiquité et le Moyen Âge*. Paris: Gauthier-Villars. (Original work published in 1892)