# The Object of Representations: Between Wisdom and Certainty[1]

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ABSTRACT. What is that which representations represent? The goal of this article is to discuss the answer to this question provided by two influential contemporary epistemological theories –Piaget's and Wartofsky's. Our discussion is motivated by a practical concern: new technologies of semiotic activity in the classroom (e.g. artefacts, calculators, dynamic software) offer new forms of knowledge representation and text production that are radically different from the time-honoured formats of written traditions based on a sequential organization of sentences. Nevertheless, the use of new forms of knowledge representation (both discursive and artefactual) requires us to better understand how representations in general relate to their conceptual contents. In scrutinizing Piaget's and Wartofsky's epistemologies, it becomes apparent that the answer to the aforementioned question depends not only on our conceptions about how the objects of knowledge become known by the individuals but also on ontological stances that we take about the nature of mathematical objects.

## Introduction

The idea of representation became one of the essential conceptual constructs of 20th Century psychology of learning. It would be virtually impossible to find another so embracing and powerful concept in this field. Perhaps the reason of its spectacular success was rooted in the perplexing situation that we all *think*, although we still do not know exactly how. Capitalizing on the long-lasting venerated opposition

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between body and mind cultivated by Western thought, the idea of representation appeared as a means to inform us about what is occurring in our heads. Thus, traditional cognitive psychology proposed a distinction between external representations (symbols, graphics, language, etc.) and internal representations (mental forms of information processing). However, the point in studying internal representations was not so much to crack the "black box". Cognitive psychology was not so ambitious. Rather the point was to get a glimpse of our internal psychological processes. And this, of course, seduced many of us.

Nevertheless, conceptions of thinking have changed substantially in the past few years. Current discourse in psychology tends to include more and more the role played by social interaction and the use of artefacts in the representations that an individual comes to form. The progressive abandonment of the Cartesian cogitator model and the solitary mind has led us to a point in which we need to reconceptualize the concept of representation and its psychological and epistemological roles. It is, nonetheless, beyond the scope of this paper to offer the elements for such a reconceptualization. My aim is much more modest. I want to focus on *that* which a representation is about, that is, its *object*: the represented as represented by the representation.

I do not take this question as a fancy philosophical one. On the contrary, the advent of sophisticated technologies in past decades has made available new means to *objectify* knowledge (Radford, 2000a, in press-1, in press-2). The means of objectification of new technologies (e.g. dynamic software in geometry) are different from those cast in the time-honoured formats of written tradition, based on the classical linear organization of sentences, and have put on the table pressing questions about the legitimacy of traditional versus new modes of knowledge representation. Of course, the controversies that have arisen from mathematical computer-based proofs are a good example. But there are simpler ones, much closer to school mathematics. For instance, in a recent article, Rodd discusses the role of *figures* in mathematics and asks whether visualization can be an a-linguistic mathematical warrant (Rodd, 2000, p. 237). Questions like this cannot be answered without adopting an ontological and epistemological position. For the answer will depend on what we consider the 'nature' of the object to be (i.e. the objects' ontological status) and on the kind of relationship between the representation (e.g. a diagram or a figure) and its object (i.e. the epistemological status of the representation).

In this article I want to discuss some key conceptualizations that influenced the elaboration of contemporary views on the object of representations. Since this is a very broad problem, to narrow it, I will focus on two of the more influential 20th century theories: Piaget's genetic epistemology and Wartofsky's historical epistemology. In order to understand Piaget's and Wartorfsky's theoretical approaches, I will also consider the problem of the object of representation as it was formulated by Kant, Plato and Aristotle, on the one hand, and by the anthropologists Durkheim and Lévi-Strauss, on the other hand.

## The Limits of Sense and Reason: The Kantian Doubt

The period between 1770 and 1780 is known as the "Silent Decade" in Kant's intellectual life. It was the period in which Kant was struggling with one of the problems that have hunted philosophers and mathematicians of all centuries, namely, the problem of the relationship between representation and the represented object. In a letter sent to his friend and former student Marcus Herz, dated February 21, 1772, Kant said: "What is the ground of the relation of that in us which we call "representation" to the object?"[2]

Kant had at his disposal two solutions, provided by two different theories but he was not satisfied with either of them. The first one was provided by empiricists such as Hume and Locke. The empiricists argued that our representations, concepts and beliefs derive from the information that we receive through our senses. The second one was offered by rationalists like Descartes and Leibniz. According to the second, the human mind has a stock of innate concepts and principles through which we generate and construct the representations of the world. One of the main differences between these views was, as Kant noticed, to be found in the active role with which the mind is endowed in the rationalist account. Indeed, instead of receiving sensuous impressions and being affected by them in a passive way, in the rationalist theory, the mind is conceived of as playing an active role in bringing up the representations.

In his letter to Herz, Kant summarized first the empiricist solution:

> If a representation is only a way in which the subject is affected by the object, then it is easy to see how the representation is in conformity with this object, namely as an effect in accord with its cause, and it is easy to see how this

[2] The letter was published in Zweig, 1970, pp. 32-38.

> modification of our mind can *represent* something, that is, have an object (Kant in Zweig, 1970, p. 33).

Then, Kant summarized the rationalist solution:

> If that in us which we call "representation" were active with regard to the object, that is, if the object itself were created by the representation […], the conformity of these representations to their objects could be understood (Kant in Zweig, 1970, p. 33).

Nevertheless, Kant contended, each one of these positions has its own problems. As for empiricist theory, the crucial difficulty lies, as Kant remarked, in the assumption that representations are reduced as the effects of objects through our senses. It may be fair to say that

> The passive or sensuous representations have an understandable relationship to objects, and the principles that are derived from the nature of our soul have an understandable validity for all things insofar as those things are supposed to be objects of the senses (Kant in Zweig, 1970, p. 33).

But, Kant asked, how according to this view can we be affected by objects that cannot be given through the senses? How can we be affected by conceptual objects? In this point Kant found the crucial difficulty of empiricism.

As for the rationalist account, Kant said:

> If such intellectual representations depend on our inner activity, whence comes the agreement that they are supposed to have with objects –objects that are nevertheless not possibly produced thereby? (Kant in Zweig, 1970, p. 34)

In other words, the difficulty in rationalism is its impossibility to explain how the products of inner activity can be said to coincide with those of the external world. In his letter, Kant confided to Herz that he was planning to write a work that might have the title "The Limits of Sense and Reason", a book that would lead the difficult problem of representation and the represented object beyond the point in which his *Inaugural Dissertation* of 1770 had left it. Talking about the *Dissertation*, Kant said: "I silently passed over the further question of how a representation that refers to an object without being in any way affected by it can be possible." (Kant in Zweig, 1970, p. 34). "Now", he continued in another passage, "I am in a position to bring out a "Critique of Pure Reason" (*op. cit.* p. 35).

In the *Critique* (published in 1781 and re-published with modifications in 1787) the dilemma was synthesized as follows[3]. Either the object alone makes the

representation possible, or the representation alone makes the object possible. In considering the first case, Kant said that the relation between object and representation would only be empirical. In the second case, Kant observed that the representation cannot produce the objective existence of the object, for what the representation does produce is a *representation* of the object but not the *object itself*. (see B 125, Kant, 1781/1787/1996, p. 147)

To attempt to provide a new solution to this problem, Kant drew on the aforementioned empiricist and rationalist viewpoints and came up with something different. In conformity with the former, he maintained that all knowledge starts *with* experience. But, in agreement with the latter, he added, "that does not mean that all of it arises *from* experience" (A1, Kant, 1781/1787/1996, p. 44). He built a system in which he stressed the human faculty or capacity of receiving or being affected by the particular objects of sensual experience. He termed this faculty *sensibility* (from *sense*). According to Kant, in being affected by objects, we form a kind of representation that he called *intuitions*. The epistemological role of intuitions is a corner stone in Kant's theoretical edifice. Intuitions are the gates through which objects have to enter in order to become known. As he said in the *Critique of Pure Reason*, "no object can be given to us in any other manner than through sensibility" (B33/A19, Kant, 1781/1787/1996, p. 72).

Evidently, sensibility and intuition are tokens of the empiricist influence in Kant's thought. But there were also other less crude sensualistic considerations. Kant observed, as Russell noted, "that the geometers of his day could not prove their theorems by unaided argument, but required an appeal to the figure" (Russell, 1919, p. 145). Kant was hence led to realize that our inferences always require the support of something tangible and came to the conclusion that, even in the case of mathematical objects, we need the support of intuitions, that is, of particular representations of the objects.

But the faculty of sensibility, Kant contended, cannot organize by itself the intuitions of sensual experience. This organization or synthesis is done by what Kant called *understanding*, another faculty of the mind. In his *Anthropology*, one of his later writings, Kant says that the power of intuition is limited to objects in

[3] Passages from the edition of 1781 of the Critique of Pure Reason are usually referred to with the letter A followed by the corresponding pages of the original edition, and those of the amended edition of 1787 with the letter B and its corresponding pages. I will conform to this tradition in Kantian studies but will add the pages according to Pluhar's unified translation (Kant, 1781/1787/1996).

their *singularity*, whereas understanding represents things to us by *concepts* thorough the unification of sensuous intuitions (Kant, 1797/1974, pp. 68-69). The difference between sensibility and understanding is hence that the former does not think, it is rather a receptive faculty. The latter, on the contrary, is the faculty of thinking, that is, of judging in accordance to pure concepts or categories to which the intuitions are presented (see Figure 1).

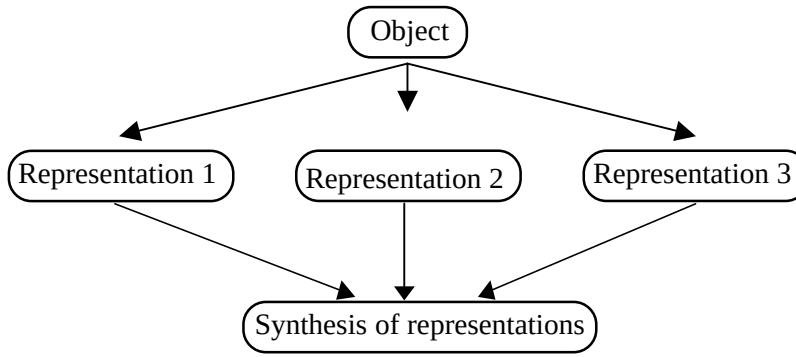


Figure 1. The diverse intuitions (representations) of the object are made possible by the faculty of Sensibility. The concepts of the faculty of Understanding ensure the synthesis of intuitions.

Evidently, the pure concepts and the faculty of understanding are a token of the rationalist influence in Kant's thought. Pure concepts are, according to Kant, prior to experience; they are part of the architecture of the mind and serve to guide experience. But neither the empiricist idea of sensibility nor the rationalist idea of understanding can lead, when separated from each other, to knowledge. Kant insisted that sensibility and understanding need to work together. Without sensibility, the understanding would not be able to proceed to the synthesis of particular representations and to proceed to the unity necessary to accomplish knowledge. This is why, Kant said, sensibility is more necessary and indispensable (Kant, 1797/1974, p. 69). Within this context Kant was able to formulate the two conditions for knowledge to be possible:

> The first condition is *intuition* [i.e. sensible representation –L.R.]; through it the object is given, though only as appearance. The second condition is the *concept*; through it an object is thought that corresponds to this intuition. (A93/ B125, Kant, 1781/1787/1996, p. 147).

What was, then, Kant's final answer to the question of the adequacy between object and representation? How did he resolve the problem that he mentioned to

Herz in his letter? He drew a clear border limiting what can humanly be known. The object of representations is not 'the object-in-itself', that is, what he also termed the 'object = *x*' or 'transcendental object', but the object as it appears in the phenomena and is constructed through sensual experiences by the acting individual. He gave a clear example using rain drops. Suppose, he said, that we take something empirical like a rain drop. This empirical intuition is a mere appearance in which nothing whatever belonging to some thing in itself is to be found. However deeply we explore the objects, we deal with nothing else but appearances:

> Not only are these drops mere appearances; rather, even their round shape, and indeed even the space in which they fall, are nothing in themselves. They are, rather, mere modifications, or foundations, of our sensible intuition. The transcendental object, however, remains unknown to us (Kant A45-46, Kant, 1781/1787/1996, p. 97).

To sum up, for Kant the object of representation is not the thing-in-itself but the thing as it becomes known through our faculty of sensibility, through the intuitions that we form of it and the way these are synthesized in a coherent unit by the faculty of understanding. In accordance to the rationalist trend, the representation, he maintained, determines the object, but, in accordance to the empiricist trend, he added, the representation determines the object in the sense that only through the sensual representation can the object become knowable. As Wolff noted, prior to Kant

> the object was assumed to exist, the problem being to explain how it could be known. […] Now, however, Kant proposed to reverse this order, making the realm of existing objects dependent upon the subjective conditions of knowledge. The *a priori* representations determine what can and cannot be known as an object, and hence what can and cannot be considered to exist. The realms of being and knowledge are co-terminous, and even more significantly, the latter defines the former. Wolff, 1963, pp. 97-98).

In drawing the border between the thing in itself and its appearance, Kant accomplished something really new, a revolution that, short of modesty, he compared to the one Copernicus accomplished in Sciences.

## Plato and Aristotle

According to Kant, Plato was misled on two counts. First, since, according to Plato, senses confuse our ideas, the sensual realm was seen by him as something to

be distrusted. Here Plato committed an *excess of prudence*. Second, Plato thought that the real objects were intelligible. Here, in Kant's eyes, Plato committed an *extravagance*. Plato showed a lack of prudence concerning the limits of the intelligibility of objects. Plato's epistemological presuppositions lead obviously to a different perspective on representations. It is true that being first of all a moralist, Plato did not deal with representations in a systematic way. However, it is clear that he assumed a kind of *resemblance* between representations and their objects. In a very often quoted passage of *The Republic*, Plato says:

> You know too that they [the geometers] make use of and argue about visible figures, through they are not really thinking about them, but about the originals which they resemble. (*The Republic,* Book 6, 510 d).

Aristotle saw the problem differently. While the link between object and representation appears in Plato' ontology as a kind of assumed *iconicity*, Aristotle envisaged it as an act performed by the individuals in their commerce with things. At the centre of this act lies the human possibility of *separating* in thought, the physical and the conceptual. He wrote:

> The mathematician is able to study surfaces, volumes, lengths, and points in isolation from their physical instantiations because (in some way which needs to be explained) he is able to separate the two in thought. (Aristotle, *Physics* B2, 193b33 ff.)

Aristotle's distinction between bodily and conceptual objects led him to an interesting point: conceptual objects can be managed in ways that their bodily counterparts cannot. Thus, in a conceptual rectangle we can imagine diagonals that were *not* in the concrete object (e.g. in a bronze rectangle). With the help of representations we can also separate the conceptual rectangle into two triangles, make visible new properties, and so on. For Aristotle, the link between object and representation appears hence ensured by the fact that arithmetic and geometry were conceived of as a conservative extension of physical theory (see Lear, 1982, p. 188). And, according to him, that which makes the extension possible is *abstraction*. Piaget also insisted on the role of abstraction in the link between object and representation. However, for Piaget, abstraction does not occur by retaining some features of the object and mentally omitting other features of it. The ground of abstraction for Piaget is the *actions of the individual*. Naturally, the theoretical shift from the Aristotelian emphasis on material objects to the Piagetian focus on actions leads to a different idea of conceptual objects. In the next section we will

look at representations and the nature of their conceptual object in Piaget's genetic epistemology.

## Piaget's dilemma

Piaget's epistemology was certainly influenced by Kant's work. But he had to modify some of Kant's basic assumptions. Piaget was dissatisfied in particular with Kant's rationalist ideas of "a priori categories" (e.g. causality) and "pure concepts", that is, concepts prior to *all* experience. He said:

> For my part, I consider myself to be profoundly Kantian, but of a Kantianism that is not static, that is, the categories are not there at the outset ; it is rather a Kantianism that is dynamic, that is, with each category raising new possibilities, which is something completely different (Piaget in Piattelli-Palmarini, 1980, p. 150)

Piaget also had to deal with a problem that Kant left beyond discussion, namely, the problem of the *origin* of knowledge. The Copernican revolution accomplished by Kant is, as Derrida noted (Derrida in Husserl, 1978, pp. 39-40), a "revelation" for the first geometer as opposed to a "construction". "The [Kantian] a priori nature of [the geometrical] concept within which we operate precludes all historical investigation whatever about its subject matter." (*op. cit*, p. 41). Piaget, hence, had to theoretically thematize the abstraction of the actions in such a way that the origin and development of knowledge could become clear. And he did so against the background of two influential theories of knowledge development of his time, namely, empiricism and preformation theory[4]. In the famous debate with Chomsky, held in France in 1975, after considering these theories, Piaget concluded that "an epistemology conforming to the data of psychogenesis could be neither empiricist nor preformationist, but could consist only of a constructivism, with a continual elaboration of new operations and structures." (Piaget in Piattelli-Palmarini, 1980, p. 23)

What can be said, within this context, of the conceptual object in Piaget's epistemology? What is the relation between representations and their objects? The conceptual object, in Piaget's epistemology, arises neither from the impressions

[4] Preformation theory, which was widely known in the 18th Century, contended that human development was the unfolding or growing of *preformed* structures. It re-emerged in the 20th Century under a biological form of innatism. For a more detailed discussion see Furinghetti and Radford 2002.

that we receive from a material object nor from what we discover by separating in thought the physical and the mental. They arise from the actions that the individual carry out in his/her physical experience with the material object:

> The acquisition of knowledge … results from an abstraction, which we must consider as starting from these actions, since the properties discovered in the objects are the very ones which the actions have introduced to begin with. (Beth and Piaget, 1966, p. 232)

Piaget contended that, through abstraction, the actions become transformed into operations. These operations, he added, "can sooner or later be carried out symbolically without any further attention being paid to the objects which were in any case 'any whatever' from the start." (Beth and Piaget, 1966, p. 238). The relationship between representation and the represented object appears thereby linked in a strong manner to the operations to which the abstracted concrete actions lead.

There is, of course, a generous dose of Kantianism in the manner in which the conceptual object is conceived in Piaget's epistemology. But to what extent Piaget followed Kant? In particular, what can we say about the ontological status of the conceptual *object* in Piaget's epistemology?

The question of the ontological status of the conceptual object in Piaget's epistemology was brought up by René Thom and Norbert Bischof during the debate Piaget/Chomsky as a question concerning the difference between construction and reality. Referring to Piaget's approach, Thom said that either the space exists as a universal framework prior and exterior to the individuals, as a space where all reality would be subsumed or, in contrast, the space is a Kantian projection of an internal (a priori) structure of the individuals. The problem for Piaget was, as Thom put it, to reconcile a genetic approach of subjective constructions, where what the object *is* depends on the individual's action, with a realist ontology where what exists is beyond the will and actions of the individual. Thom was hence urging Piaget to explain how the individuals' *construction* of space can coincide with the reality of space. (Thom in Piattelli-Palmarini, 1979, p. 503).

Piaget had in fact dealt with this question in a book written more than ten years before the debate with Chomsky. In his book, co-authored with the logician Beth, Piaget himself asked: "in what sense can we then speak of a pre-established harmony between deduction and experience to explain the accordance of mathematics with reality?" (Beth & Piaget, 1966, p. 284). The answer was given in

terms of "a common origin" to be found in "the laws of organic co-ordination" and "the physico-chemical environment". (*op. cit*). In addressing Thom's question, Piaget did not present any opposition to Thom's suggestion that "Piaget seems indeed to adopt the realist thesis of an external existence of space" (Thom in Piattelli-Palmarini, 1980, p. 362). Instead, drawing on the idea of a "common origin" elaborated the previous years, Piaget stated that the space constructed by the individual is in accordance with the exterior one: "both exist without conflict and converge without merging." (Piaget in Piattelli-Palmarini, 1980, p. 369). And that what ensures their convergence is the fact the individual is a physicochemical and spatial object, the individual "starts from neurological and biological sources whose laws are those of reality" (*op. cit*. p. 369).

But even assuming an ontologically structured reality, a common origin of actions and the psycho-chemical environment seems not good enough to ensure the convergence of the individual's construction of space with the space of reality. We may still be confusing two different ontological layers, as Bischof suggested in his intervention during the debate. Bischof indeed introduced a distinction between *critical realism* and *naïve realism*. Critical realism presupposes an independent reality, "the so-called "objective world," the structure of which remains whether or not there are organisms who perceive it correctly, or perceive it at all." (Bischof in Piattelli-Palmarini, 1980, p. 233). Furthermore, critical realism distinguishes between the aforementioned objective world and the "phenomenal world", that is, the world in which the individuals live, make their experiences and produce their conceptual constructs. There may be a corresponding relation between them, but they are ontologically different. Naïve realism confounds both. Bischof's major worry was that the products of our thought (Piaget's conceptual objects) are manifested in the course of experiences and that these products may present to us as the "irresistible evidence" of a solution (in the phenomenal world) that, in the end, presents itself as "a good form" only. Thus, given this distinction between phenomenal and objective world, how to ensure, in the Piagetian account, the convergence between knowledge and reality?

Piaget came back to this point in his text *Afterthoughts* and commented again on the relation between knowledge and reality, which now he expressed in terms of divergence. He said:

> As for critical realism, which Bischof analyzes so well, and the characteristics of optimal (but never complete) adaptation of knowledge, and so on, I think that I generally agree with Bischof's considerations, with the exception of

> perhaps one important point. Bischof, if I understand him correctly, thinks as I do that the object is never completely attained; it remains a limit, and he speaks in this regard of an asymptotic progression. But the problem is to know whether this limit allows for convergence or divergence. I believe that the progression is divergent in the sense that the object is transformed as knowledge closes in. Now, this makes me differ even more from the "naïve realism to which Bischof fears that the utilization of mechanisms of equilibrium might lead me. (Piaget in Piattelli-Palmarini, 1980, pp. 283-284)

We see then that in his answer to Thom, Piaget talked about convergence. Now, in his comments on Bischof he talks about divergence. I do not take these answers as contradictory, but as an additional precision that Piaget was bringing about his epistemology. A constructivist approach which adopts a realist ontology cannot say anything about how close we are from reality. The answer to Thom reflects, I think, the fact that in such a framework one *expects* the individual's constructions to be close to reality. The answer to Bischof expresses the awareness that the *abyss* between reality and the individual's conceptual constructions made in the phenomenal world is essentially impossible to fill. Knowledge and reality, as Kant would have said, remain separated.

But Piaget's convergent/divergent stance towards reality is also an indication of the tension that his epistemology born as a result of adopting a realist ontology. All in all, Piaget was confronted with the following dilemma:

- either to follow Kant's ontology and to do what Radical Constructivism did a decade later, that is, to work within the boundaries of a Kantian nominalism and consistently claim that the individuals' constructions do not inform us about the things in themselves
- or to hold a kind of realistic ontology and to try to explain the gap between the subjective construction of reality and objective reality[5].

Piaget preferred the latter. He did so because he valued logical necessity and because of his commitment to rational thought as modeled by sciences and mathematics. The first option is *more philosophico*. The second one is *more mathematico*. To prefer the first option would amount to embrace *wisdom* only. To choose the second option is to stick to *certainty*. Although Piaget said that he turned

[5] Piaget even went on to say that constructivism "in fact, only differs from Platonism in that it does not speak of the universe of possibles as if it were already achieved or "existing". But constructivism retains the Plationist belief that this universe is accessible, through the procedure common to all schools of thought: that of effective construction" (Beth & Piaget, p.303).

away from Kant's a priori categories, he indeed also departed from Kant's ontology. Instead of the "wisdom" of philosophy, he opted for the "certainty" of sciences[6].

## Representations and their objects as cultural constructs

Piaget's genetic epistemology never denied the role of culture in his accounts of intellectual development. The problem, he said, is to determine whether the intervention of inter-individual factors is necessary to explain the development of thinking (by which he meant *logical* thinking) (Piaget, 1967, p. 155; Piaget 1968, p. 9). Piaget conceived of inter-individuality (and social co-operation at large) as a *system of actions*, and attempted to show that inter-individuality was also governed by the same formal structures as the logical one. Thus, he concluded that social cannot explain the intellectual and vice-versa, because both are framed by the same dynamics of evolving structures, namely, by the laws of equilibration:

> Social co-operation is also a system of action, inter-individual actions and not simply individuals, but actions nevertheless, and consequently under the effect of all the laws that characterize them. Hence, one can say that the social actions that result in the co-operation are also governed by the laws of equilibrium. (Études Sociologiques, p. 158, my trans.)[7]

The cultural tradition that emerged from the work of the anthropologists of the early 20th century offered a different perspective. Thus, for Durkheim and the sociological tradition inspired from his work, the objects of our representations are related to collective cultural experiences. The concepts with which we think, he claimed, are those consigned in the vocabulary of a culture. This vocabulary expresses collective experiences beyond the always limited space of any individual's activity. He said:

> There are scarcely any words among those which we usually employ whose meaning does not pass, to a greater or less extent, the limits of our personal experience. Very frequently a term expresses things which we have never

[6] The opposition between wisdom and certainty was elaborated by Piaget in his book *Insights and Illusions of Philosophy*. In this book he said that "philosophy ends in a wisdom and not [in] a mode of knowledge" (Piaget, 1971, p. 116). For knowledge demands that verification (in the sense of the science) be possible. "What is important" he wrote in the postscript to the second edition of his book, "is the trilogy reflection x deduction x experiment, the first term representing the heuristic function and the other two cognitive verification, which is alone constitutive of 'truth'." (*op. cit*. p. 232) (see also Bkouche, 1997, pp. 36-39).

[7] A more detailed analysis of Piaget's conception of inter-individuality as actions can be found in Radford, 2000b.

> perceived or experiences which we have never had or of which we have never been the witnesses. Even when we know some of the objects which it concerns, it is only as particular examples that they serve to illustrate the idea which they would never have been able to form by themselves. Thus there is a great deal of knowledge condensed in the word which I never collected, and which is not individual. (Durkheim, 1915, p. 483).

For Durkheim the object of representations was a collective object. Of course, within this context, the Kantian problem of the relationship between the represented object and its representation cannot have a solution in terms of the standards of objectivity that started with Galileo and that became adopted and expanded in the 18th Century philosophy of Enlightenment. It has a solution, nevertheless, but it requires conceiving science in a different manner. It requires seeing science as a one of the various modes of comprehension of the world, as a kind of "more perfect form of religious thought" (Durkheim, 1915, p. 477). The relationship between representation and its objects is hence to be found in the kind of *ideality* that societies construct, for, as Durkheim noted, "a society can neither create itself nor recreate itself without at the same time creating an ideal" (op. cit. p. 470). Lévi-Strauss held a similar position, although he did not agree with the 'vulgar thesis' according to which magical or religious thinking would be a lesser stage in the evolution of scientific thinking. Instead, he saw both as lying on their own structures and suggested that they should be placed in parallel (Lévi-Strauss, 1962). We may very well understand Piaget's refusal to accept Lévi-Strauss' position according to which all cultural groups have a similar logic –a logic based not on structural features of operations but on complex and subtle oppositions with which we organize our world (Lévi-Strauss, 1962). Piaget maintained that the so-called primitive cultures have reached the stage that he identified as concrete operational thinking (Piaget, 1967, pp. 147-148) and that the disagreement with Lévi-Strauss could only be resolved experimentally. Thus, in an interview, Piaget said:

> I spend a lot of time during every discussion I have with Lévi-Strauss repeating that neither he nor I can decide the matter deductively. It is not the ethnographer's enquiries which will decide the issue for us. What is need are studies in the field by psychologists used to our methods of enquiry who will question adults. (Grinevald 1983a, p. 75)

To which Lévi-Strauss answered:

> What I do ask, and I formulate this question rather naively in ethnological terms, is whether Piaget's research techniques aren't rather artificial in

> character. His experiments are set up in advance, prefabricated, which does not seem to me to be the best way to understand the mind in all its spontaneity. (Grinevald 1983b, p. 84)

In an important sense, Lévi-Strauss' plea was an attempt to invite epistemology to consider in a decisive manner cognition and representations as consubstantial of context and culture. As an anthropologist having been in contact with numerous tribes, he held the conviction that it is impossible to circumscribe all of them in a same Universal History where Reason would unfold stage after stage. Of course, in such a perspective, not only the object of the representation but also the problem of the relationship between representations and the represented objects change. But how can this relation be theoretically thematized? Lévi-Strauss did not offer a straight answer. In the next section I will sketch the historic epistemology of Marx Wartofsky, which constructs the idea of representation in an intimate relation to its cultural context.

## Wartofsky on Representations

Wartofsky started with a question that underpinned Vygotsky and Luria's work: what is it which makes human cognition distinctive? Wartofsky's answer was: "the ability to make representations". Representations for him included not only tables, drawings, formulas but also artefacts, objects and ideas. In their broader sense, representations are culturally and historically constituted modes of action; they are mediating agents in human activity, concrete and theoretical artifacts of our forms of perception and cognition:

> Theoretical artifacts, in the sciences, and pictorial or literary artifacts, in the arts, constitute the *a priori* forms of our perception and cognition. But contrary to the ahistorical and essentialist traditional forms of Kantianism, I propose instead that it is we who create and transform these *a priori* structures. Thus, they are neither the unchanging transcendental structures of the understanding, nor only the biologically evolved *a priori* structures which emerge in species evolution (as, for example, Piaget and the evolutionary epistemologists suggest). Piaget's dynamic, or genetic structuralism is important here, of course. His dictum, "no genesis without structure, no structure without genesis", suggests the dialectical interplay of the practical emergence and transformation *of* structures with the shaping of our experience and thought *by* structures. But the domain of this genesis I take to be the context of our social, cultural and scientific practice, and not that of biological species-evolution alone (…) In a sense, then, our ways of knowing are themselves artifacts which we ourselves

> have created and changed, using the raw materials of our biological inheritance. (*Op. cit*, p. xxiii)

Piaget, as we saw in the previous sections, considered objects and artefacts as accessorial epistemological elements. He shifted the Aristotelian emphasis from artefacts to actions. Wartofsky subsumed the actions into their modes of social praxis and, in addition, he put the artefact back into the cognitive scene and stressed its epistemological role.

The genesis of representations, he argued, has to be examined in the two fundamental forms of human activity: in our making of things (i.e., is to say, in modes of production) and in our interaction with individuals (i.e., in modes of social relations):

> Tools and language (…) become the basic artifacts by means of which the human species differentiates itself from its animal forebears; and it is therefore in an analysis of these basic artifacts that a theory of the genesis of representation needs to be developed. (Wartofsky, 1979, p. xvi)

Of course, these forms of human activity are not to be conceived as functioning separately. Neither should one of them be considered the functional complement of the other. There is a strong (although not necessarily visible) relationship between them. An artefact can be an element of (material or intellectual) production. But it bears the very form of the individuals' activity:

> Thus, spears and axes are not only made for the sake of hunting and cutting, but at the same time represent both the method of their manufacture and the activities of hunting animals or chopping wood. (*op. cit*, p. xiii-xiv)

Reciprocally, individuals' activities are generally organized in terms of artifacts. Thus, we will organize our actions differently when building a rhombus using a pencil-and-paper technology or when using computer software.

In Wartofsky's epistemology the Kantian question of "how is representation possible?" is replaced by the new question "by what means, and in the course of what activity does [representation] take place?" (see *op. cit.*, xvi). How, however, can the problem of the objectivity of knowledge be dealt with here?

> What I am proposing here, only programatically and sketchily, is what I would characterize as an historical-materialist theory of the genesis of theory, or of theoretical cognitive praxis. The apparent instrumentalism of truth in such a

> view, as I set forth here, is then replaced by a realist emphasis: As a representation in some symbolic form of a mode of action or practice, the theoretical formation has its truth-value in the adequacy of the representation: i.e. in its practical conformity to a successful mode of activity, which it institutionalizes, so to speak, in the representation. It is *not*, therefore, an unmediated representation of some external state of affairs, but rather one which is mediated by the practice or mode of action which it represents. The sheer externality of a state of affairs become 'objective' for us, then, only as it is mediated by our practice. What we can know is therefore always conditioned by the way that we come to know it. In a sense, our knowledge of the 'external' world is a knowledge of what this externality is amendable to, in our incursion upon it and intervention in it. (Wartofsky, *op. cit*, p. 136)

We see then that, in Wartofsky's epistemology, the individuals are seen as placed in an independently existing environment. Nevertheless, this environment appears to the individuals as mediated by representations[8]. The object of representations is thereby clearly distinguished from the thing-in-itself, the 'real' object that hits our senses. Our forms of perception –or our faculty of sensibility, as Kant would have put the matter– create an unbridgeable gap between the thing-in-itself and what we know of it. Certainly, in taking this theoretical position, the historical epistemology becomes close to Kant's epistemology and the one adopted by Radical Constructivism and moves away from Piaget's genetic epistemology. However, Wartofsky's epistemology departs from Kant's, in that the faculty of sensibility was taken by Kant as a rigid part of the architecture of the mind. For Wartofsky, on the contrary, the human faculty of sensibility (i.e. our modes of perception) is historically and culturally constituted. With our physical organs we produce representations which come to be integrated in our forms of perception and cognition to the extent that what we perceive is already a world tainted with the historical color of human needs and intentions.

Thus, instead of a "sentimentalist relativism" –to use Goody's splendid term (Goody, 2000, p. 137), or instead of a Radical Relativism, where 'everything goes' (Putnam, 1981), objective knowledge and truth are seen as embedded in culture, in the modes of production and in the social forms of interaction. The 'truth' and 'objectivity' proposed by Wartofsky's epistemology are not transcendental entities.

[8] As he said in another passage, "the 'objects of perception' are taken to be independent of perception, though they are mediated by the activity of perception, in that they are perceived *by means of our representations of them*." (op. cit. p. 193; emphasis as in the original).

They find their support in the adequacy of representations and in the conformity and success of the latter in terms of institutionalized activities[9]. The verification to which knowledge and its representation have to be submitted finds now its normativity in the cultural institutions serving as the background of the individuals' activities. The normativity is no longer one of the laws of biological equilibrium and of constructive conceptual autoregulations leading to deductive necessity. Normativity has to be understood as a flexible, situated and historical construct. Unfortunately, Wartofsky did not develop this point further. And it is evident that we need to reflect much more on it. In particular, we need to better understand how cultures dynamically produce normative elements and how, in the space opened by the normative elements, scientific and mathematical discourses become constituted. At any rate, in front of the same dilemma to which Piaget was confronted, Wartofsky had also to make a choice. His was not certainty. It was wisdom.

## Concluding Remarks

In this article I dealt with different conceptualisations about the object of the representations. The question was motivated by a practical concern: the spreading of new forms of knowledge representation associated with the increasing use of new technologies of semiotic activity. The time-honoured format of sentence-based mathematical text of written traditions is now becoming part of larger processes of objectification of mathematical knowledge (Radford, 2000, in press-1, in press-2) where other kinds of knowledge representation are being made possible (e.g. discursive and artefactual). Nevertheless, changes in modes of knowledge representation do not go without raising a delicate problem, namely the problem of *legitimacy*. We need to understand it as a problem having two different albeit related aspects, a *political* one and an *epistemological* one.

By a political aspect (from the Greek *polis*) I mean here an aspect related to social actions conveying a distinction between what is good and what is not. Thus, the technology of the written was condemned by the Pythagoreans, who favoured the technology of the word. In a similar vain, not long time ago the question of the pertinence of a calculator in the classroom was still a polemical issue; now the

[9] Naturally, this point is a highly developed form of one of the succinct remarks made by Marx in his *Theses on Feuerbach*: "The question whether objective truth can be attributed to human thinking is not a question of theory but is a *practical* question […] The dispute over the reality or non-reality of thinking which is isolated from practice is a purely *scholastic* question".

same question is no longer under discussion[10]. The written in Plato's time and the calculator in ours were highly controversial in terms of their legitimacy to represent and objectify knowledge.

By an epistemological aspect I mean the manner in which the objects of knowledge become known. But, as Kant noticed, how an object becomes known depends on the manner in which the object is *sensed* or *presented* to us, how it affects us. To express this affectation of our sensibility by the object he used the word *aesthetics*, not as relating to the beautiful but in its etymological sense, *aisthetá*, that is, as pertaining of sensuous perception. We can see, then, that in dealing with the problem of knowledge representation it is not possible to dissociate the political from the epistemological. But what this article wanted to explore was something that underlies the problem of knowledge representation and that often remains implicit, namely, that which we assume to be the nature of the mathematical objects. In other words, their *ontological* status. We cannot deal with the problem of the representation of knowledge without looking into the ontological dimension.

I began with Kant because it is impossible to understand Piaget and Wartofsky without understanding the manner in which Kant posed and tried to solve the problem of knowledge. Of course, Piaget's *épistémologie génétique* has had a remarkable influence in mathematics education. It is one of the best achieved contemporary epistemological accounts[11]. In the course of the article we saw how Piaget drew on, and departed from, Kant. The interpretation that I have offered of Piaget's ontology is different from von Glasersfeld's, who said that "Piaget's position is a bit ambiguous. Despite the important contributions that he has made to constructivism, he has always a tendency for metaphysical realism." (1988, p. 27; my trans.). I think that Piaget was showing much more than an unfortunate or inopportune penchant for realism.

Wartofksy drew from Kant and from Piaget but he departed from both. In thematizing Piaget's basic notion of *action* from a historic and cultural viewpoint, and in resituating the artefact at the center of human activity, Wartofsky's epistemology offers a rich avenue to theorize the problem of knowledge

[10] Thus, The Ministry of Education and Training of Ontario (Canada), for instance, has provided High School students with access to graphic calculators.

[11] I have to say, nevertheless, that to narrow my discussion I had to resist the temptation of dealing with another interesting ontological problem with which Piaget was confronted –that of logical necessity.

representation. To carry out this theorization however involves a change of perspective concerning the ontological status of mathematical and conceptual objects in general. And, as the section on Durkheim and Lévi-Strauss intimated, this change of perspective is not easy in that, in the end, it implies changes that are rooted in basic assumptions that are taken for granted. In this respect, Piaget's commitment to scientific method, which for him delimitated a clear border between philosophy and sciences and a distinction between wisdom and certainty, is most revealing. I am not contending, however, that to find the answer to the question with which we started this article, namely, "What is that which representations represent?", we need to avoid any recourse to basic assumptions. Thinking and understanding are contingent upon assumptions[12]. This is why there is not a unique answer to the aforementioned question for the answer will depend on ontological matters which are in turn supported by basic assumptions. A critical stance towards the object of representations does not require us to necessarily remove assumptions; rather it requires of us an awareness of the assumptions that we are making.

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[12] As Adorno (2001, p. 13) noted "[i]f you refuse to make any assumptions … then you will understand nothing." To further become aware of the role of assumptions in the ontology of mathematical objects one only needs to read hot debates between e.g. Hilbert and Brower or Poincaré and Couturat surrounding the foundation of mathematics.

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