Published in *Mediterranean Journal for Research in Mathematics Education*, 2002, **1** (1), 31-56.

Algebra as *tekhnē*

Artefacts, Symbols and Equations in the classroom[1]

Luis Radford
École des sciences de l'éducation
Université Laurentienne
Ontario, Canada

Abstract: In this article we describe and analyze a teaching sequence whose goal was to introduce students to the operation of the unknown and to the related elementary algebraic techniques to solve linear equations. The design of the activity and the analysis of the students' reasoning were conducted within a semiotic-cultural theoretical framework. A discourse analysis of the students' interaction in the classroom suggests that the overcoming of the well-known difficulties that novice students usually encounter in operating on the unknown, as found in traditional approaches to algebra, was accomplished here through the students' complex co-ordination of different sign systems (gestures, natural language, drawings, etc.) and their successful mastery and articulation of levels of abstraction (going from manual industrious actions to intellectual and sign-based actions).

KEY WORDS: artefacts, symbols, meaning, iconic algebraic thinking, symbolic algebraic thinking, semiotics, semiotic-cultural approach.

1. Introduction

The introduction to symbolic algebra in Junior High School is one of the teaching points that really worries teachers. Indeed, experience shows, again and again, that the learning of algebra represents a key moment in the students' school life and one in which many problems usually arise. If students manage to get a certain sense of algebra, then they will probably be able to go further. For many, however, this is not the case.

One of the didactic problems in the teaching of and learning of algebra is related to the students' understanding of algebraic symbolism and its meaning. A traditional approach to algebra will put the students in front of an endless page of equations to be solved. This activity (as well as the formal manipulation of symbolic expressions) seems rather

[1] This article is part of a research program funded by the Social Sciences and Humanities Research Council of Canada, grant # 410-98-1287 and Laurentian University Research Funds.

esoteric and vain to the students who, in the end, end up making errors that previous research has clearly identified. More than 20 years ago, Matz published an inventory of students' errors. Among them are the following:

(1) Evaluating 4X when X=6 as 46 or as 46X

(2) Simplifying $\frac{AX+BY}{X+Y}$ to $A+B$

Matz (1980) held that these errors are produced as *extrapolation* techniques. That is, the individual interprets the problem in terms of a previous *base* knowledge and applies *rules* that are valid in the *base* knowledge (without necessarily being valid in the extrapolated case, like in the aforementioned examples).

The presence of errors such as those in Matz's examples is a token of the students' difficulties in understanding the intricate system of rules of sign-use in algebra.

One of the didactic problems for a teacher is precisely to look for meaningful situations out of which signs, equations, and symbolic expressions will make sense. This problem is far from trivial (see e.g. Bednarz *et al*. (1996), Sutherland *et al*. (2001)). One of the reasons is that to get more from algebra, symbols are required. But if we start by asking the students to deal with formal manipulations then, as said previously, it is difficult for the students to make sense of algebra. To make matters worse, students' willingness to learn (as broad as it may be) has a limit and it is not surprising that soon the students start asking the reasons for learning such a hieroglyphic-like activity. The students' short-term expectations are difficult to meet despite the teachers' enthusiastic and firm arguments of a future use of algebra in engineering, applied mathematics and other scientific domains.

Several years ago, a group of Junior High School teachers and I found ourselves immersed in this situation. We decided to conceive and participate in a longitudinal research program in order to devise mathematical tasks susceptible of offering the students the occasion to engage in meaningful mathematical activities and, at the same time, allowing us to investigate relevant ontogenetic aspects of the emergence of symbolic algebraic thinking.

The elaboration of mathematical activities to introduce algebra in school led us to make explicit choices that are sustained by theoretical matters lying in the crossroads of epistemological and pedagogical concerns. And underneath these matters rests the long-standing venerable metaphysical question: "*What is algebra?*" Is it a formal game? Is it a set of abstract structures? Is it a language? Is it a tool with which to investigate natural phenomena?

In dealing with this question, our attempt was not really to answer the metaphysical question (whose formulation in the third person singular already reveals an ontological choice) but rather to look for theoretical orientations that could conveniently meet the curricular demands. There are two major topics in the Ontario Curriculum of mathematics in the introduction to algebra. These are equations and patterns. I shall deal here with equations only[2]. The question to be discussed here is hence: how to introduce equations?

Although there is no royal road to algebra, we decided to introduce equations in the context of problem-solving activities –a context in which algebra appears as a *tool*. As with any choice, it has its advantages and disadvantages. Before we discuss the latter, let me specify what is meant here by algebra as a problem-solving tool.

2. The theoretical orientation

2.1 Algebra as a problem-solving tool

In our approach, the conceptualization of algebra as a problem-solving tool is to be understood in the Aristotelian sense of *tekhnē,* that is, a reflective work on certain objects in the course of which intellectual knowledge becomes dialectically interwoven with the industrious actions on the objects[3]. More precisely, our approach to algebra as a problem-solving tool means the development of an analytic technique based on a conceptually complex kind of mathematical thinking relying on the calculation of known and not-yet-known numbers or magnitudes that acquire a meaning as they are handled in the pursuit of the goal of the activity.

[2] Some of our classroom-based research on patterns can be found in Radford 1999a, 1999b, 2000, in press.

[3] For a discussion of *tekhnē* as a relation between conceptual knowledge and concrete labor, see Mondolfo (1954).

This choice, nevertheless, has to overcome two important difficulties that are constantly mentioned in didactic discussions and that Duval (in press) analyses in depth in a recent article:

(1) The first one relates to the "translation" of the statement of the problem into algebraic symbolism.

(2) The second one concerns the symbolic calculations required to solve the equation.

Duval argues that the difficulties in both cases are different. To *translate* the statement of problem from natural language into the symbolic language of algebra requires a semiotic transformation of representations, something that he calls a *conversion*. To *solve* the equation, different transformations from the previous ones are needed. He calls these *processing transformations*.

As for conversion, Duval insists upon the difference between the linguistic discursive procedures and the syntactic ones proper to the symbolic algebraic language. In a collective work (Groupe Math-Français de l'IREM de Strasbourg, 1996), this difference accounts for difficulties encountered by students in tasks where they had to translate word-problems into equations. Conversions are especially difficult for novice students to accomplish in those cases where expressions in natural language cannot be translated directly into symbolic expressions because of differences in the syntax of both systems – something that Duval (1995) calls the problem of *congruence/non-congruence* (of statements) and for which I had proposed the more complicated term of *vectorial direction* (Radford 1992). Probably the best known example of non-congruence translation is the famous problem of students and professors (see Rosnick and Clement 1980).

A viable pedagogical choice to introduce algebra from a problem-solving tool perspective needs to take into account the aforementioned difficulties. In order to make an attempt to reduce the problems arising from the conversion between semiotic systems (i.e. the problems of "translation"), a possibility is to have recourse to word-problems that will not expose the students, unnecessarily from the very beginning, to non-congruent situations and the subtleties of the complex syntax of symbolic algebraic language.

There is still an important point concerning the conceptual development of algebra that can help us refine the previous pedagogical option. Even if in terms of the usual curricular demands our final point is symbolic algebra, this does not entail that, in the first contact with algebra, we need to put the students in contact with letters. Actually, symbolic algebra is a relatively recent invention. It dates back to the 16^{th} Century AD, while mathematical activity with unknown terms goes back to at least the 16^{th} Century BC (see Radford 2001, 1995, 1996). Hence it may be interesting to start with a choice of representational semiotic systems based on soft syntaxes and meanings and elementary word-problems aimed at relieving the problem of conversion (a problem that can be addressed later, once the students have become acquainted with some essential algebraic concepts).

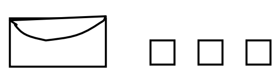
Regardless of the target semiotic system (TSS) to be chosen, the word-problem, as Duval notices, already contains a designation of the unknown quantities in the source semiotic system (i.e. natural language). To convert a word-problem into one or more semiotic expressions in the TSS therefore demands identifying and *re-designating* the unknown quantities and the relations between known and unknown quantities (one of these relations, although not the only, being the equality relation). At this point, a central cognitive problem is the articulation of the semiotic expressions in the TSS and the meaning with which these expressions have to be endowed in the process of their transformation in the solution of the equations. Since the meaning of the signs of the TSS, resulting from the translation phase, cannot be preserved throughout the successive transformations of semiotic expressions, the signs and these semiotic expressions have to acquire a new meaning.

Along this line of thought, I want to suggest that, among the wide variety of signs, from a didactic point of view, icons can be helpful transitional devices to re-designate unknown quantities before a full formal treatment of symbols becomes possible for the students. True, icons have obvious limitations. For one thing, it is not possible to write a compendium of the syntax of icons. But, for our purposes, this may not be necessary, as it was not necessary for the Babylonian scribes who wonderfully managed to solve, by iconic methods, second-degree equations (see Radford 2001; Radford & Guérette 2000). Let us further elaborate these ideas in the next section in light of supplementary semiotic considerations and the idea of algebra as *tekhnē*.

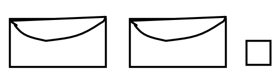
2.2 Iconic algebraic thinking

In the introductory activity to algebra that we elaborated, no letters were required. We opted for an approach based on artefacts, that is, concrete objects out of which the algebraic *tekhnē* and the conceptualization of its theoretical objects arose. Two kinds of artefacts were considered: envelopes and cards. They were taken as *signs* in a Vygotskian sense (see Radford 1998). The cards were icon-signs re-designing known quantities, while the envelopes were icon-signs re-designing unknown quantities. This conceptualization of the unknown is, in fact, inspired from the work of one of the more talented 14th Century Italian algebraists, Antonio de Mazzinghi, who, to explain what the *cosa* is (i.e. the *thing* or the *unknown*) used to say that the *cosa* is "a hidden quantity" (see Radford & Grenier 1996). A statement S_1 such as "one envelope and three hockey cards" or S_2 "two envelopes and one hockey card" can be represented by or converted into iconic statements using material envelopes and cards as follows:

(S_1)



(S_2)



Iconic statements rely on a kind of elementary syntax based on juxtaposition of terms (somewhat similar to the rules of Generative Grammars inspired by Chomsky's works and widely used in the field of Computer Sciences).

We can introduce a new sign into the iconic semiotic system to allow one to express the equality between iconic statements like S_1 and S_2 –the equality being something that Duval (1995, p. 111 ff), following Aristotle, relates to one of the apophantic functions of a language[4]. Such a sign can be any concrete mark (e.g. a piece of paper, or a piece of paper having the traditional equal sign on it). The resulting semiotic system constituted of two kinds of signs ("envelope-sign", "card-sign") and a third sign for the two-place equality predicate along with a set of syntactic rules for the formation of expressions (the 'well-formed' expressions) and deduction rules indicating the passage from a predicate to another, constitute what I want to term an *elementary iconic symbolic language* (EISL).

EISLs, as conceived here, are good enough to deal with some elementary word-problems through algebra. Although an explicit list of rules of sign-use and deduction could be provided, it is of no interest here. As said previously, the interest of an EISL is the friendly environment it offers for non-expert users in symbolic languages, which is the case of novice students stepping into the realm of algebra[5].

As we will see in the next section, because of the signifying nature of icons, transformation and deduction of expressions (that is, *inferences* in Peirce's sense) allowed the students to elaborate, in a non complicated way, a mathematical technique operating on known and unknown quantities and relying on perceptual actions –a technique underpinned by what can consequently be called *iconic algebraic thinking*.

3. Stepping into Algebra

In this section, we shall discuss some aspects of the first two days in which two classes of Grade 8 students were introduced to algebra. The goal of the first day was to familiarize the students with the principles of the algebraic techniques required to solve equations without using letters. The second day, the students were asked to start using letters to write and solve equations.

[4] Statements S_1 and S_2 have a referential nature in that they tell us something about the objects of the universe of discourse without making any judgment. To go beyond mere reference, a language has to add new possibilities accounting for acts of expressions that have a logic value (i.e. trueness), an epistemic value (certainty) or a social value (see Duval, *op. cit*. p. 112). Equality, I want to submit, is not restricted to a logic value. Depending on the specificities of the context, equality may have a logical, a social or an epistemic function and even more than one of these at the same time.

[5] Formal languages can also have an iconic component. Actually, Peirce considered that mathematical formulas were icons: "an algebraic formula is an icon, rendered such by the rules of commutation, association, and distribution of the symbols." (Peirce, 1955, p. 105).

3.1 The first day

The first day, the lesson was divided into three parts. In the first one, the teacher discussed a word-problem and its solution with the class. This problem (as well as those given to the students in the second part of the lesson) dealt with hockey cards and envelopes. The teacher had envelopes and hockey cards and, according to the information given in the statement of the problem, placed them on the blackboard. He divided the blackboard into two parts and referred to each part as one of the sides of an equation. In the very beginning, to justify the actions on artefacts to solve the problem, the teacher stretched out his arms and made a quick reference to a two-plate scale. His explanation, however, focused on the comparison of the two sides of the equation. In the second part, the students were provided with a kit of envelopes and hockey cards and worked on a series of 6 problems according to a small-group structure (the small-groups were usually comprised of 3 students). Then, the teacher conducted a general discussion to compare the students' solutions and enhance mathematical understanding. In what follows, we shall present excerpts from one of the small-groups. The students will be identified as Paul, Carole and Madeleine (their names have been changed for deontological reasons). The spatial disposition of our group of students is shown in Figure 1. They worked on two desks put together.

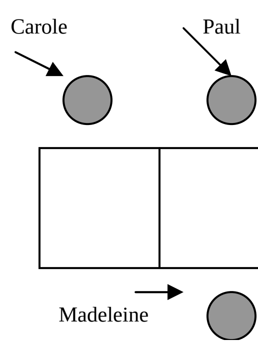


Figure 1. Students' spatial distribution

The mathematical activity required them to work on concrete objects, to write the answers on a sheet of paper and, in some cases, to write and explain their methods. For each problem, a different student was in charge of recording the group's answer.

Here is the first problem of the students' small-group activity:

Problem 1:

> Paulette's and Richard's mother decides to give her children a gift. She will give them envelopes containing hockey cards. In order for the envelopes to be identical, she puts the same number of hockey cards in each envelope.
> Paulette already had 7 cards and her mother gives her 1 envelope.
> Richard already had 2 cards and his mother gives him 2 envelopes.
> Now, the 2 children have <u>the same number</u> of hockey cards.
> How many cards are in each envelope?

Following the strategy showed by the teacher, the students formed an equation with the concrete material. In this part, Paul and Madeleine were in charge of using the concrete material and Carole was in charge of writing on the activity sheet. They thought of Paul's desk as a plane surface divided into two parts. On one part, they placed 7 cards and 1 envelope, which corresponded to Paulette's cards and envelopes. On the other part, they placed 2 cards and 2 envelopes, which corresponded to Richard's share (see Figures 2 and 3)[6]. Here is an excerpt of their dialogue:

1. Paul: Okay. … What are the data, please. Oh. (*He looks up at the blackboard and sees the problem.) (See Figure 1 for the students' spatial distribution.)*
2. Carole: (*looks at the paper and without raising her eyes answers Paul's question by saying)* 7 cards … 1 envelope.
3. Paul: (*begins to construct Paulette's situation. He arranges 7 cards on the left side of his desk all the while counting out loud*) 1, 2, 3, 4, 5, 6, 7 … Then 1 envelope? (*he places 1 envelope along with the 7 cards towards one of the corners of the desk*; see *Figure 2*)

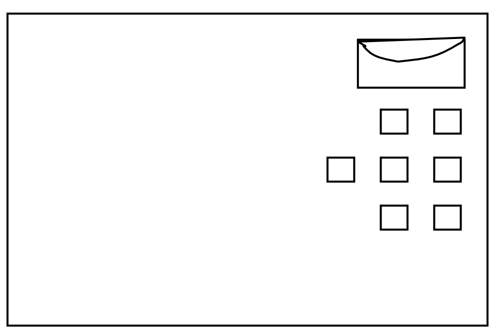


Figure 2. Paul's desk

4. Carole: 2 and 2 …(*meaning 2 cards and 2 envelopes to represent Richard's situation*)
5. Madeleine: After this, this one has 2 (*she puts her hand over the spot on Paul's desk where they are supposed to depict Richard's situation.)* Right?
6. Paul: (*counts aloud and places 2 cards*) 2 …(*then, addressing himself to Carole, says*) 2 envelopes, please. (*Madeleine quickly takes 2 envelopes from among those on Carole's desk and places them beside the 2 cards representing Richard's situation*.) […] (see Figure 3)

[6] Of course, in symbolic terms, the problem would correspond to the equation: $x+7=2x+2$.

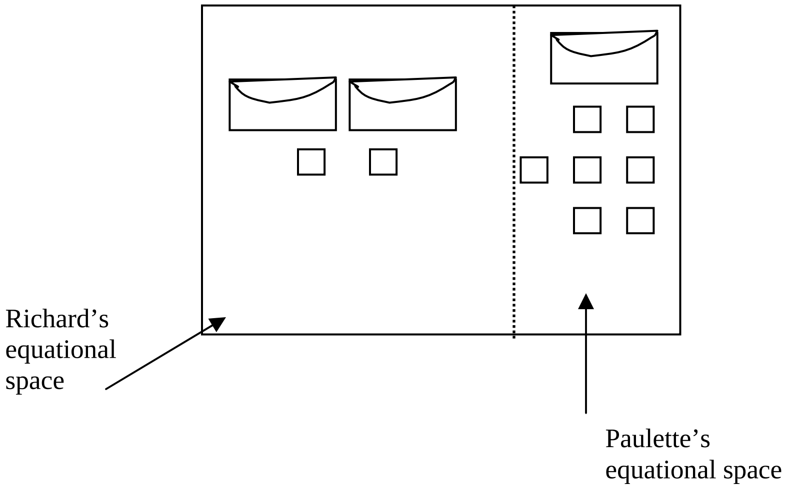


Figure 3. Students' indicate the equality of the iconic expressions through a space between the equational spaces.

So far, the students succeeded in the process of conversion between semiotic systems. In this case, and due to didactic choices concerning the congruence between the respective statements, the iconic predicate in Figure 3 is almost a mirror of the statement of the word-problem. In itself, it may seem as an unimportant step in the students' undergoing mathematical actions. Nevertheless, as Lacan noticed, mirroring marks the beginning of symbolic activity.

7. Paul: Okay (*acting very quickly, while Madeleine moves the concrete material to separate even further the equational space between Richard and Paulette that we have indicated by a dotted line in Figure 3*). So, this (*he takes one of Paulette's cards*), this (*takes one of Richard's cards*) comes out (*he holds up the 2 cards for a moment, one in each hand, so that the other students can see, then he moves them onto Carole's desk)*. So, … (*inaudible*) this one.
8. Madeleine: (*While Paul is still speaking, takes one card from Paulette and one card from Richard and says*) This, this …
9. Paul: (*begins to act when Madeleine is still taking the cards and says*) This, this comes out …
10. Carole: No. … (*then almost simultaneously shows her understanding of the action of her classmates regarding the subtraction of cards*)
11. Paul: (*takes the two cards that Madeleine had taken and still has in her hands, joins them to those that he had taken in Line 7, places them on Carole's desk and concludes, looking down at the desk*) So, there are 5 [cards] in each envelope. (see Figure 4)

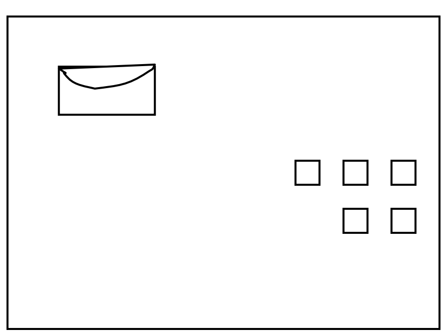


Figure 4. Final solution.

As we can see in this episode (that lasted 29 seconds) the students did not find any difficulty in solving the problem. They operated on the unknown (the envelope) and on the constant terms (the cards).

Successive transformations of the initial iconic equation found their meaning in a *perceptual semiosis* (Radford, submitted), ensured here by the iconic nature of the object-signs. This is manifested in a clear predominance of visualization and of the concrete actions to remove objects over what the students *say*. That is, glances and gestures become the primordial regulatory principle of the activity. Speech functions as a background regulatory principle (in fact, the students utter relatively few word in lines 1 to 11. This amount increases dramatically in the next episodes).

The second and third problems were similar to the previous one. Their goal was to familiarize the students with the algebraic problem-solving technique. They did not find any difficulty in solving them. In problem 4, the students were asked to explain the steps that have to be followed to solve an equation. The didactic reason for including this problem is related to our theoretical framework and its conceptualization of language as a means of objectification (Radford, in press). In fact, we consider that there is an important difference between doing something and saying how to do it. While doing something relies on a kind of social understanding in which coordinated perception plays a fundamental role (the students may use deictic terms such as "this", "that", etc. to direct the others' attention to something; they may also coordinate glances or point to something with a finger), in saying how to do it we enter into a different conceptual realm. Glances and gestures are no longer enough. The written words of language become a new sensation organ, so to speak, that, in contrast to ephemeral pointing and uttered words, endow objects with a permanent status.

Problem 4:

1. Carole: (*reads the question*) Explain in your own words the steps that must be followed to solve an equation.
2. Madeleine: (reads the *question*) […] Well, you have to take them away equally. Like, when you take away 1 card, you have to take away the other. …
3. Carole: Yes, but … but, we have to say it at the beginning …
4. Paul: What you do on one side, you do on the other.
5. Carole: Yes. You always do to one side what you do to the other.

6. Madeleine: Uh huh. (*she begins to write*)
7. Carole: If you take away 1, you take away 1. If you take away 2, you take away 2 … Then, what you …
8. Madeleine: (*saying what she writes*) Do the same thing on both sides.
9. Carole: On the 2 sides … On each …
10. Madeleine: Then, if … (*inaudible*) on one side …
11. Paul: You take away the cards on the other side.
12. Madeleine: Okay. So, after that, what's left is …
13. Paul: How many cards …
14. Carole: (*interrupts*) And, when you can't take away any more …
15. Madeleine: (*begins to write again*) That's your answer … (*saying what she is writing*) when you can't … anymore …
16. Carole: Sometimes you have to divide … And then after that, you have to say that.
17. Paul: Or multiply, or add, or …
18. Carole: (*interrupts*) No. Just divide the number of cards by the number of envelopes.
19. Paul: I don't think you'd add, though. (*Carole and Paul examine Madeleine who is busy writing on the paper*.)
20. Madeleine: That's your answer? (*Carole gestures 'yes' with her head and Madeleine begins writing again*.) […] The amount in each … the amount in each envelope (*Carole gestures 'yes' with her head again*) … or the answer. (*Madeleine continues to write on the sheet of paper*.)
21. Carole: Okay. Then, there, if you have many cards and envelopes, you have to divide. (*Madeleine finishes writing*.)

The students' final explanation is as follows:

> You always have to do the same thing on both sides. For example, if you take away one card on (one) side you have to take away one card from the other side too. When you can no longer take away any cards or envelopes that's the answer. If you have many cards and envelopes, you have to divide.

The dialogue and the students' text reveal the dialectical relationship between conceptual knowledge and the reflective work on objects. Thus, lines 16 and 17 make provisions for different possible situations; line 18 is a firm conclusion sustained by the students' previous experiences that leads, in line 19, to a reassessment of line 17. The text includes general statements that are exemplified with concrete cases and hypothetical situations and has the following structure:

General statement. Example. Indicative statement. Hypothetical situation.

In problem 5, the students were asked to invent a word-problem having as the answer: in one envelope there are three cards. Paul was in charge of writing.

Here is an excerpt of the students' dialogue:

1. Carole: Okay. Well, we could start like this. (*She takes one envelope and says*) 1 envelope, 3 cards are left. (*She places 1 envelope and 3 cards in front of her*. *See Figure 5*). Okay. So, there we can add like … (*adds 1 envelope on each side*: *See Figure 6*).

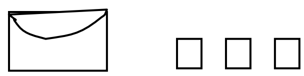


Figure 5

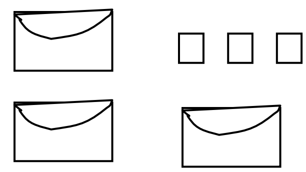


Figure 6

2. Paul: (*interrupting*) 2 envelopes like this.
3. Carole: No, more. (S*he re-arranges the objects and adds 1 more envelope on each side: see Figure 7*. *Then she adds 3 cards on one the sides of the emerging equation and says*) 1, 2, 3. (*Then places 3 cards on the other side of the equation containing the 3 initial cards and says*) 3 have to be left over (*see Figure 8*). We'll put lots on both sides.
4. Madeleine: Yes, okay.

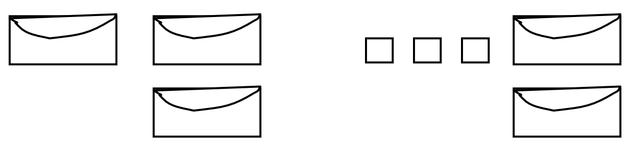


Figure 7

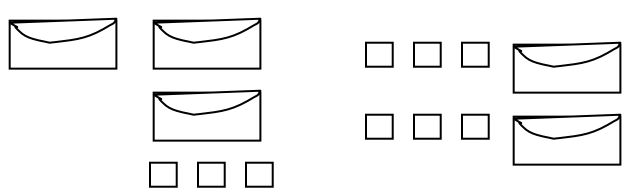


Figure 8

5. Carole: Really lots. (*She adds 3 more cards on both sides, counting aloud*) 1, 2, 3 … 4, 5, 6.

6. Carole: … (*continues*) where do we write our problem? … Well, how did they write it there? (*She turns the page to the preceding problems*) Like, we'll start the same way.

Although the final iconic statement is still under construction, at this point, the students will start a conversion process: they will translate the iconic statement into a statement in natural language:

7. Paul: The mother of …
8. Carole: The mother of [Carole] and … Madeleine …
9. Paul: … decides to give a gift to her children. (*He continues to write*)
10. Madeleine: How many are there? (*She begins to count the cards on one side*) 1, 2, 3, 4, 5, …, 6, 7, …
11. Carole: (*interrupting*) There are 3 left.
12. Madeleine: … (*continues*) 8, 9. There are 9.
13. Carole: (*counts the cards from the second package*) 6 [cards]. (She *gathers 3 and 2 envelopes and says*) 3, 2. (*The students have the concrete equation shown in Figure 9*)

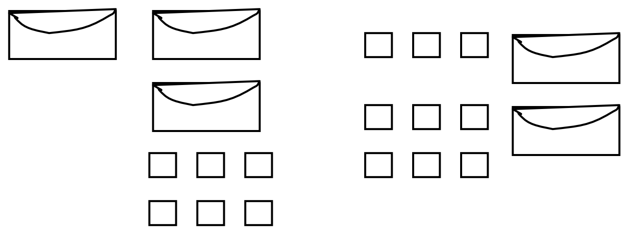


Figure 9. The concrete equation elaborated by the students

14. Madeleine: Well, no, she gives, um …
15. Paul: (*turns the sheet and shows a previous problem*) Here, it says, "She gives them envelopes containing Hockey cards."
16. Carole: Okay. … envelopes containing …(*Paul continues writing then examines the problem on the previous page*)
17. Madeleine: Okay. For Carole … she gives … […] Yes. Carole already had … (*examines the cards and envelopes in front of Carole*) […] 9 [cards], then her mother gives her (*shows the envelopes in one package*) 2 envelopes. (*Carole takes the package and examines the problem*.)
18. Paul: Yes, then she gave …
19. Carole: No. Yes. Well, how many envelopes does my mother give me?
20. Madeleine: 2.
21. Carole: 2 envelopes. So, you have to write, eh, (*shows a place on the sheet*). Carole already had … 9 cards and her mother gave her 2 envelopes.
22. Madeleine: Yes. (*Carole gives the stack of pages back to Paul*.)
23. Paul: Okay. (*He begins to write*)
24. Madeleine: Then, after this, eh, Madeleine already had 6 cards and her mother gives her 3 envelopes. (*Paul continues to write*.)
25. Paul: Then, after this, (*examines the previous page*) we'll write, Madeleine already had 6 cards … Wait a minute! Here (*points to a place on the sheet*) How many are there …
26. Madeleine: … cards in each envelope. (*Paul continues to write*.)
27. Carole: What about, now the two children have the same number?
28. Madeleine: You don't need to put that.

29. Carole: Yes. … That's how they're going to find it out because they have exactly the same thing (*looks at Paul who is writing*) No! Paul, Paul! You have to write "Now, the 2 children have the same number" (*Paul is inaudible*.)
30. Carole: Because it's important to know this for your problem. You have to have it.
31. Paul: So, I write this before this. (*shows a place on the sheet of paper*)
32. Carole: Yes. So, you just have to choose this. *(Paul scoffs at something and continues to write*.)
33. Paul: Okay.
34. Carole: Now, …[…] The 2 children … the 2 children … have the same number of cards … (*inaudible*) have the same number of cards. […]
35. Carole: In each envelope. (*Paul continues to write*.)
36. Madeleine: That works. Right?
37. Carole: Yes. (*Madeleine begins to remove the cards and the envelopes from each side to verify if the problem will give the right solution. Carole helps her do this*. *Madeleine counts the cards that she is taking away and says: "Yes, this works"*.)

The final text reads as follows:

> Madeleine's and Carole's mother decides to give her children a gift. She will give them envelopes containing hockey cards. Carole already had 9 cards and her mother gives her 2 envelopes. Madeleine already had 6 cards and her mother gives her 3 envelopes. ~~How many cards are there~~
> Now the 2 children have <u>the same number of cards</u>. How many cards are in each envelopes (*sic*!)?

Lines 27 to 30 show the students' evolving understanding of one of the more important aspects of the algebraic concept of equality. If the statement of the problem does not say that the children have the same amount of hockey cards, then, Carole argues, it will be impossible to solve the problem. In the beginning, Madeleine thinks that this is not important. However, as a result of the discussion, she realizes that Carole is right.

Although the first-day mathematical activity involved two more problems than those discussed previously, we will not discuss them here. Let us only mention that Problems 6 and 7 were aimed at bringing the students to tackle one problem with an infinity of solutions and one problem with no solution, respectively.[7] The solution to these two problems led the students to become aware of something that they had left implicit in the previous discussions, namely, that the envelopes have to contain the *same* number of hockey-cards.

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[7] In symbolic terms, the problems can be translated as: $2x+2=2x+2$ and $2x+2=2x+3$, respectively. I am analyzing the students' problem-solving methods and conceptualizations in a manuscript in progress.

3.2 The second day: Symbolic algebraic thinking

On the second day, the students were asked to solve the same problems that they had encountered the previous day. This time, however, instead of using artefacts, they were required to use letters and numbers. The teacher solved Problem 1. We will analyze here the students' solution to Problem 2.

Here is the statement of the problem:

> Mario's and Chantal's mother decides to give her children a gift. She will give them envelopes containing hockey cards. In order for the envelopes to be identical, she puts the same number of hockey cards in each envelope.
> Mario already had 12 cards and his mother gives him 1 envelope.
> Chantal already had 3 cards and her mother gives her 4 envelopes.
> Chantal has <u>the same number of hockey cards</u> as Mario.
> How many cards are in each envelope?

They proceeded as follows:

1. Madeleine: Okay. The number 2? (*she begins to write on the paper*)
2. Carole: Yes. … (*she looks at the paper in front of her*) 12 cards, 1 envelope.
3. Madeleine: Okay. So, … (*she begins writing again*)
4. Carole: 12 … 12 plus 1 e.
5. Paul: 12 plus … How many envelopes?
6. Madeleine: 12 cards then how many envelopes?
7. Carole: One.
8. Paul: 12 plus 1*e*.
9. Carole: Equals … 3 plus 4e. (*They arrive at the following equation: 12+1 e = 3 + 4 e*)

The students wrote the equation, following the statement of the problem. In doing so, they capitalized on their previous mathematical experience. For instance, the juxtaposition syntax of the EISL is transformed into something new: juxtaposition gives rise to new signs. Thus, the addition sign "+", which was not part of the EISL language, appears in the symbolic expression. Furthermore, the empty space in front of the equational space of the compared iconic expressions (see Figure 3) now becomes the traditional equal sign "=". Last, but not least, the letter "e" functions as an abbreviation of the object "envelope". In a certain sense, in the re-designation of unknown quantities, the letter is still an icon. The difference is that now the letter is endowed with the perceptual meanings that were originally derived from the EISL (problems 1 to 3) and were later objectified in speech (problem 4). We have, indeed, the interplay of different semiotic systems where a reference to a same object is made (although obviously the mode of denotation is *not* the same):

a. The written language where the word *envelope* appears (hence envelope_1)
b. The concrete sign-object ‘envelope’ (hence envelope_2)
c. The uttered word envelope (hence envelope_3)
d. The letter “e”.

It is in the interplay of these semiotic systems that the embryonic symbolic meaning will be expanded further in the following lines, where the students will tackle the symbolic equation that has now reached a *linear* form. Here is the dialogue:

10. Madeleine: Okay. So, it’s … how many cards did we take away?
11. Carole: 3.
12. Madeleine: Right (*she goes back to writing*) minus 3 (*she writes at the same time as she says* 12^{-3})
13. Paul: minus 3.
14. Carole: (*at the same time*) minus 3.
15. Madeleine: Okay. So, this, this gives …
16. Carole: 9.
17. Madeleine: 9 …
18. Carole: plus one *e*.
19. Madeleine: Equals …
20. Carole: 4e. (*They arrive at the following equation*: $9+1e=4e$)

Even though the emerging abstract thinking is a semiotic expression of iconic algebraic thinking, in the new symbolic layer, the students do not reproduce *each* of the concrete actions. We saw that the students took one card after the other (one card at a time on each side of the concrete equation). Here they subtract *three* cards *at the same time*. Symbolic thinking leads to a condensation and a re-organization of previously temporal sequenced actions. There is a fundamental difference between the narrative of concrete iconic actions and the symbolic ones.

At this point of the problem-solving process, the symbolic equation has been simplified. Now the students will operate on the unknown. They will proceed to subtract unknown terms on both sides of the equation:

21. Carole: After this, take away …
22. Madeleine: (*interrupting*) 1 envelope.
23. Carole: … (*continues*) 1. (*Madeleine continues writing on the paper. She writes :* $9+1e^{-1e}=4e^{-1e}$)
24. Paul: Equals 9.
25. Madeleine: … 9.
26. Paul: Equals … 3.

27. Madeleine: (*at the same time*) 3. (*They arrive at the following equation*: 9=3e)
28. Paul: So, it's like 3 cards in 1 envelope.
29. Madeleine: Oh, yes. So there …
30. Paul: You divide.
31. Carole: Yes, but is it correct?
32. Paul: So, you go 9 divided by 3.
33. Madeleine: But, is it correct like this, eh? (*They experience difficulty in symbolically expressing the action of dividing; they write:* $9 \div 3e =$)
34. Carole: Yes.
35. Paul: Yes, but, you have to divide there.
36. Carole: (*shakes her head*) No.
37. Paul: Are you sure? (*Finally, they write:* 3=1e)

Their written work was the following:

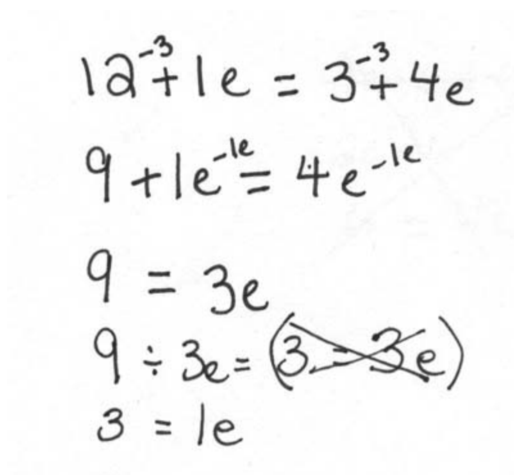

Figure 10. The students' symbolic equation and its solution

As can be seen, subtractive actions were symbolized with small characters in a quasi-exponential format adjacent to the sign on which the action was carried out. These actions were accompanied by speech. For instance, lines 21 and 22 show the regulatory function of speech and its result in line 23. Although, as mentioned previously, the students chose to use a letter that was abbreviating the word that names the object (in this case, the letter "***e***" was an abbreviation of the word "***e***nvelope" that designates the concrete object with which the students were acting), the letter, bit by bit, started losing its contextual reference and by the end of the two-days activity the students were already successfully dealing with direct equations.

It does not mean, however, that algebra was always easy for them. As a matter of fact, coefficients were seen as '*nombres nombrants'*, that is, as numbers that number the amount of unknown objects (*one* envelope, *three* envelopes, etc.)[8]. If our model affords this conceptualization of coefficients, something different is needed in order to provide meaning to, say, decimal or irrational coefficients.

4. Synthesis and Conclusion

In this paper, we presented an analysis of a group of Grade 8 students in their first contact with algebraic equations. This first contact occurred in a mathematical activity in which algebra was considered in the Aristotelian sense of *tekhnē*. Paramount to this idea is that algebraic thinking arises in the stream of the students' reflective work on certain objects as a technique that oscillates between manual and intellectual endeavour where each one of these poles sheds light on the other. The mathematical activity allowed the students to progressively enter into the realm of algebraic symbolism. Instead of starting from the highly complex symbolic language of contemporary algebra and to consider equations as objects *per se*, we proposed a more friendly environment: a concrete elementary iconic symbolic system (EISL) having an elementary perceptual syntax. This syntax is not empowered with all the capabilities of an "analytic" or a formal syntax. Hence, a statement comprised of a card and an envelope is the *same*, regardless if the card is above or below the envelope. EISL syntax is less rigid than the formal ones. This is precisely one of its potentials, from a didactic point of view[9].

EISL soft syntax allowed the students to accomplish the translation of elementary word-problems into an iconic statement and to suitably transform these statements in order to reach the solution. These iconic statements form an iconic text that, in the end, appears as

[8] See, for instance, the last line in Figure 10. The symbolic expression "1e" still resounds the echo of the utterance "one envelope" of line 28.

[9] In order to make more tangible the contribution of our approach, it is worthwhile to mention, at this point of our discussion, that manipulatives have been used for many years in the teaching of algebra (for example, in terms of concrete or representational two-plate-scale models). We do not claim any originality in using them. What is new is the cultural-semiotic analysis that we are doing of the students' actions and the conceptual orientation of manipulatives in our teaching lessons resulting from our theoretical framework. If, as any cultural artefact, two-plate-scales are neither good nor bad *in themselves*, our results, nevertheless, point to a relatively low pertinence of them for the learning of algebra. Thus, the students' dialogues show that scales are actually of little (if any) use in their reasoning. What is important is the *equational space* (something similar to the physical counting-board of Chinese mathematics), which becomes the arena of concrete actions. These remarks then converge to Pirie and Martin's observations concerning the irrelevance of scale models (Pirie and Martin, 1997).

a didactic device reducing the gap between the statement of the problem in natural language and a formal symbolic treatment of equations. The didactic goal was not to remove the gap (which is, I believe, an impossible task). The goal was to provide the students with an intermediary semiotic system from where to derive certain meanings to be used later in the semiotic system of symbolic algebra.

EISLs rely on a perceptual semiosis in which algebraic meanings arise in direct contact with signifying features of the human body and its cultural context. In perceptual semiosis the individual's senses are oriented to the inspection of the objects and meanings arise in the contact between senses and objects. As Merleau-Ponty (1964, p. 67) says,

> We must […] recognize that what is designated by the terms "glance", "hand", and in general "body" is a system of systems devoted to the inspection of a world capable of leaping over distances, piercing the perceptual feature, and outlining hollows and reliefs, distances and deviations –a meaning—in the inconceivable flatness of being.

To inspect the world, however, the senses of the human body are not enough. Between body and object, culture interposes a wide range of artefacts that anthropologists and cultural psychologists call *prostheses*, that is, artificial objects that make the body (and, along with it, the mind) go beyond the skin –Bateson's example of the blind man and his cane is probably the best known example of this kind (Bateson, 1973; for a more detailed discussion within cultural psychology see Cole 1996; for a discussion from the perspective of the learning of algebra see Radford 1999b). Envelopes and cards, in our teaching sequence, were artefacts that functioned as prostheses to touch the noumenal mathematical objects through actions driven by perceptual meaning.

Perceptual meaning, whose texture is made from the threads of voices, glances, and touches, later became transformed into a new meaning that relied more heavily on *writing* and out of which symbolic expressions and equations arose.

The passage from perceptual to abstract meaning was ensured by the interplay of different semiotic systems. As the lines of the dialogue shown in Section 3.2 suggest, the meaning of what they were progressively writing in the semiotic system of symbolic algebra, in fact, relied upon what they were *saying*, *seeing*, and the previous experience of the EISL (see Figure 11).

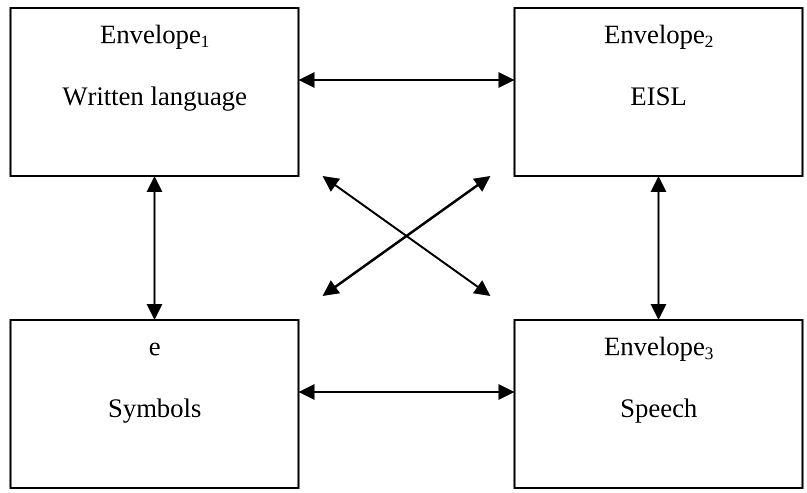

Figure 11. Interplay between different semiotic systems in the production and solution of an equation

By and by, as the students entered further into the realm of symbolic algebra, speech became less and less present. Sometimes slowly, sometimes quickly, the pencil moved across the paper, line after line, up to the point where the solutions were found. The signs, that the solitary hand holding the pencil produced, entered into a kind of mute territory, occasionally interrupted by a whisper –the vestige of the previous tumultuous social exchange that is now crystallized in the silent letter.

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