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# Gestures, Speech, and the Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization

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> All instruction is either about things or about signs; but things are learnt by means of signs.
>
> Augustine, *De doctrina Christiana,* I. 2.

To improve our understanding of novice students' production of symbolic algebraic expressions, this article contrasts students' presymbolic and symbolic procedures in generalizing activities. Although a significant amount of previous research on the learning of algebra has dealt with students' errors in the mastering of the algebraic syntax, the semiotic cultural theoretical approach presented here focuses on the role that body, discourse, and signs play when students' refer to mathematical objects. Three types of generalizations are identified: factual, contextual, and symbolic. The results suggest that the passage from presymbolic to symbolic generalizations requires a specific kind of rupture with the ostensive gestures and contextually based key linguistic terms underpinning presymbolic generalizations. This rupture means a disembodiment of the students' previous spatial temporal embodied mathematical experience.

At first glance, nothing could be easier than continuing a sequence of figures. You simply look at the sequence, grasp the rule, and then draw the figures that would follow. In the classroom, however, things may not be so obvious. Therefore, when a Grade 2 student was given the first three terms of a sequence, she continued as shown in Figure 1.

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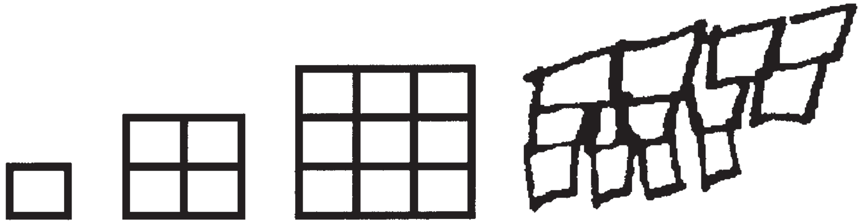

FIGURE 1 A Grade 2 student continuing the pattern.

At the beginning, her answer seemed nonsensical to us but then she explained her procedure. It was clear that she considered the first two terms together. In this case you have $1 \times 4 + 1$ small squares and because, according to her, this rule applies to Term 3 (where you have $2 \times 4 + 1$), then in the next term you will have $3 \times 4 + 1$ small squares, in the following term you will have $4 \times 4 + 1$ and so on.

The pursuit of a generalizing task clearly goes beyond seeing, for as Kant (1781/1996) said in his endless struggle against Locke, Hume, and the empiricists, things do not present themselves directly to us: Our perception of things, Kant maintained, is woven into the knowledge with which we supplement our immediate vision. Furthermore, a generalization is not always a straightforward process but rather the actualization of one of the potential ways that particular cases may insinuate. Therefore, in the case of the generalization of a sequence of geometric–numeric objects, do we have to attend to the color of the objects or to their shape or to something else? Of course, as previous research has evidenced (e.g., see Arzarello, 1991; Arzarello, Bazzini, & Chiappini, 1994a, 1994b; MacGregor & Stacey, 1992, 1993, 1995; Rico, Castro, & Romero, 1996; Sasman, Olivier, & Linchevski, 1999), generalization tasks become even more complicated when students are asked to relate variables and to express their relation in algebraic language.

One of the questions that I previously attempted to investigate concerned the very semiotic nature of signs (like $x$ or $n$) that students use in their first algebraic expressions in generalizing activities (Radford, 1999a, 1999b, 2000c). Following Peirce's (1955) terminology, I suggested that the students' first algebraic signs were *indexical* in nature (as discussed in a later section).

In this article, I offer an exploratory investigation of presymbolic types of generalization in patterns and contrast them with the algebraic symbolic ones. It is my contention that a better comprehension of the specificities of the semiotic functioning and the cognitive requirements of presymbolic types of generalization may help us to understand the indexical meaning of the students' first algebraic symbols and the difficulties that students encounter when they engage in algebraic generalizing activities.

The differences in types of generalization are scrutinized in terms of the semiotic means of objectification that students use in mathematical generalizing processes.[1] The theoretical conceptualization of means of objectification that I am sconveying draws from Vygotsky's work and from phenomenology. It is intimately related to a semiotic cultural perspective developed elsewhere (Radford, 1998a, 1998b) and refined in the course of epistemological reflections about the nature of mathematical thinking and historical processes of symbolizing (Radford, 2000a, 2001a) and ethnographic classroom-based research (Radford, 1999a, 1999b, 2000c).

Inasmuch as the use of semiotics in mathematics education is still a recent event,[2] I begin by elaborating on some concepts that are central to this study. Next, I address some methodological features of the study and the rationale for the design of the classroom generalizing activities. This is followed by a discussion of three types of generalizations. Finally, in light of the presymbolic types of generalization, I reexamine the indexical meaning of students' first symbolic expressions to which I have referred in previous research.

## SEMIOTIC MEANS OF OBJECTIFICATION

### Attention, Awareness, and Objectification

Let us imagine that we are standing in front of a wall covered by shelves full of books without any particular intention to look at them. The books all look quite similar. Let us also suppose that, although we were looking at the books in this disinterested way, we suddenly remembered that we needed to check something in Aristotle's *Poetics.* The image of a small red book comes to mind. The nonreflective perception with which we began now gives rise to an intended perception. In scrutinizing the shelf, our attention will focus on some red books and, in practical terms, we will almost ignore the others. Suspecting that we will later need to find the same book, we decide to put a mark, or a sign of some sort, on the shelf so that the next time we enter the room the sign will mean something like, "Here's the book!" This mark, or sign, is achieving a particular task: In an elementary way, it is accomplishing an *objectification.*

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[1]There are other theoretical possibilities in which to investigate generalization. Nemirovsky (in press), for example, distinguished between moving away or toward the circumstances in which a generalization takes place and suggested that each one of these cases leads us to what he terms *formal generalizing* and *situated generalizing,* respectively.

[2]It would probably be fairer to say that the increasing interest in semiotics is, to a certain extent at least, the culmination of the studies dating back to the 1980s concerning the role of language in the learning and teaching of mathematics.

The term objectification has its ancestor in the word *object,* whose origin derives from the Latin verb *obiectare,* meaning “to throw something in the way, to throw before”(Charleton, 1996, p. 550). The suffix *–tification* comes from the verb *facere* meaning “to do” or “to make” (Charleton, 1996, p. 311), so that in its etymology, objectification becomes related to those actions aimed at bringing or throwing something in front of somebody or at making something visible to the view. It is in this sense that the sign, or mark, indicating the place of the red book on the shelf is objectifying it.[3] Of course, there are other means of objectification. We can put the book in a particular spot—for example, on the top shelf, or we can put it on our desk within arm’s reach. In the second case, the relation of our body to space plays the role of means of objectification. Unfortunately, this procedure will not work when it comes to dealing with mathematical objects. For one thing, they cannot be indicated. How then to proceed? How then to gain access to them?

It is at this point that the concept of representation has been useful. As Kant (1781/1996) noticed, the only way that an object can be given to us is by the mind being affected in a certain manner—in a sensible manner, by representations of the object.[4] Husserl (1931/1958), the founder of phenomenology, held a similar position. He suggested that conceptual things can be given to us in sensory “ways of appearance” (p. 160), for instance, through representations like formulas, drawings, and so forth. The concept of representation has been one of the most talked about concepts over the last 2 decades in mathematics education (e.g., see Janvier, 1987, or the special issue of *Mathematical Behavior* [Goldin & Janvier, 1998]). Recently, Duval (1999) introduced a distinction between semiotic and nonsemiotic representations. A semiotic representation is produced with signs and rules of use that bear an intentional character. Nonsemiotic representations do not have this intentional character. They may be produced by a physical or organic system, as in the case of a footprint in the sand (Duval, 1999, p. 42ff). According to Duval (1999), it is the category of semiotic representations that can play a fundamental role in objectification. He stated, “The representations produced semiotically can play the role of treatment (traitement) and objectification that are fundamental in any process of knowledge [production]” (p. 43).

I want to suggest, nonetheless, that as powerful as they are, semiotic representations are not sufficient to account for the complexity of processes of objectification in teaching and learning situations. In stating this, I do not want to minimize the pedagogical and epistemological role of representations. The colossal importance that we ascribe to writing in our culture (an importance that, not long ago, still sur-

[3]The phenomenological sense of objectification conveyed by its etymology is different from the sense of objectification that is understood as the actions undertaken to render something impersonal. We encounter this latter one in the analysis of the students’ dialogues. To avoid confusion between these two senses, we then talk about *desubjectification.*

[4]See Kant (1781/1996, pp. 71–72).

prised several tribes such as those visited by Lévi-Strauss, 1962, and Evans-Pritchard, 1937, and would have surprised the Pythagoreans of 6th century B.C., committed as they were to the oral practice of mathematics) makes futile any claim of that sort.[5] The point is that processes of knowledge production are embedded in systems of activity that include other physical and sensual means of objectification than writing (like tools and speech) and that give a corporeal and tangible form to knowledge as well.

Within this perspective and from a psychological viewpoint, the objectification of mathematical objects appears linked to the individuals' mediated and reflexive efforts aimed at the attainment of the goal of their activity. To arrive at it, usually the individuals have recourse to a broad set of means. They may manipulate objects (such as plastic blocks or chronometers), make drawings, employ gestures, write marks, use linguistic classificatory categories, or make use of analogies, metaphors, metonymies, and so on. In other words, to arrive at the goal the individuals rely on the use and the linking together of several tools, signs, and linguistic devices through which they organize their actions across space and time.

These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call *semiotic means of objectification*.

## Signs and Tools

A concrete example helps to illustrate these ideas. In a teaching episode about the concept of measurement reported by Seeger and Steinbring (1994), the teacher showed the students a transparency with a drawing of a house and a tree. In addition to this transparency, there was a short script involving two individuals, Irene and Karl. The first one claimed that the tree was 20 m high, whereas the second one disagreed with such a claim. There were no numerical clues accompanying the size of the house or the tree, so the students had to rely on indirect comparisons and information from real situations. The students then discussed Irene's statement and were encouraged to look for a method to discover who was right. In a passage of their analysis, Seeger and Steinbring stated the following:

> During this discussion, a student remembered that the door of the classroom was two meters high; he was convinced that the door in this picture was also two meters high. On the transparency, the teacher marked the height of the door by a red line indicating 2 meters. Then the ruler was used to estimate and to measure the height of the house. The following question arose: How often did the door fit into the house? First, the height of the door in the transparency is measured with the ruler and is determined to

[5] See Lévi-Strauss (1962) and Evans-Pritchard (1937).

> be 1 centimeter. The height of the house on the transparency was determined as 4½ centimeters, and consequently, 4½ times as high as the door. So, what is the height of the house, when the door is 2 meters high? (p. 155)

We do not need to go further in the episode to realize the variety of semiotic means of objectification that has been used so far to focus attention and to make intentions apparent. The red line is accomplishing two tasks, a phenomenological one and a representational one. Indeed, on the one hand, the red line is making something visible to the eye. On the other hand, the line is a sign: It stands for the height of the door. However, along with the red line, the activity also involved tools, like the transparency; the overhead project; the marker; and the ruler. These are not signs. Although tools and signs mediate activity, they cannot be equated. The marker and the ruler, for example, do not refer to anything else. They lack the referential character specific of signs. As Augustine (n.d.) said, "A sign is something which presents itself to the senses and something other than itself to the mind." It is the referential nature that distinguishes signs from merely tools and things. Tools command only some specific functions (Vološinov, 1973, p. 10). This does not mean, however, that tools do not possess meaning. In fact, by being inserted into the individuals' activities, tools acquire a functional meaning. It is in this sense that they are taken here as semiotic means of objectification. In this example, the teacher used the marker and the ruler in an intentional way to perform their *sui generis* meaningful function—that of writing and measuring, respectively.

The previous remarks suggest that signs and tools play different roles in objectification processes of knowledge and that a semiotic analysis may help us to disentangle the dynamics of the different means of objectification as used by teachers and students in the classroom. In particular, semiotic analyses of classroom settings can help us explore the linking of different sign systems. This is what we do later, when the focus is placed on the interaction of speech, gestures, and written (natural and symbolic) languages. Before we do so, we need to discuss some elements concerning the social nature of semiotic means of objectification.

## The Social Nature of Semiotic Means of Objectification

There is a sense in which semiotic means of objectification are obviously social. It applies, for instance, to language, which in itself constitutes a social practice. Nevertheless, there are other senses in which semiotic means of objectification are also social. One of them is related to their phylogenetic dimension, namely, that which pertains to their cultural and historical development. Certainly, the semiotic means of objectification that individuals find in their culture (language included) have been historically produced for some purpose. Borrowing a term from computer science, means of objectification appear like *macros,* that is, they keep us from undertaking the lengthy processes of having to reinvent things, for example, standard tools—such as a ruler for measuring. However, the more important point is not merely the economies of actions. It is precisely their historical dimension that makes semiotic means of objectification bearers of an "embodied intelligence," that is, they carry patterns of previous reasoning, as Pea (1993) cogently argued. The social nature of means of objectification is also apparent when we consider how their historically and culturally constituted use is progressively acquired by individuals.[6]

It is also important to note that the use of semiotic means of objectification appears rooted in systems of conventional signifying forms of the culture. The alluded cultural signifying forms account for the ways in which the individuals enter into contact with and use objects, tools, signs, and other means of objectification. They also account for the ensuing systemic character that human relationships can take.[7]

To sum up what has been said so far, let us come back to the aforementioned problem of the impossibility of any direct access to mathematical objects and the ensuing need for means to render them sensible. In the course of the discussion it was argued, following Husserl (1958), that their objectification is related to their "way of appearance." The latter, however, cannot be taken as something having the power to conjure mathematical objects and bring them to life. It is in this sense that the term, "ways of appearance," of the young Husserl (1958) needs to be supplemented with something that he himself came across when trying to answer the question of the constitution of intersubjectivity and the ideality of objects, namely speech and writing. As Husserl (1978) suggested in one of his posthumous works, in the form of a joint communal action, speech provides the means to release the objects of individual subjectivity. He ascribed to writing a capacity to transcend temporality.[8] Despite that, speech and writing are not disinterested human endeavours. They are driven by intentions. The example about locating Aristotle's *Poetics* in a bookshelf illustrates how the way we perceive and become aware of things is related to our intentions. This led us to envisage a broader context large enough to conceive of tools, things, gestures, speech, writing, signs, and so forth,

[6]Individuals placed in contexts where new signs and artefacts are to be used conceptualize them as things that signify something for others. "To see something for the first time, to realize something for the first time, already means to assume an attitude toward it: it exists neither within itself nor for itself, but for another (already two correlated consciousnesses)" (Bakhtin, 1986, p. 115). See also Sperber and Wilson (1986, p. 51).

[7]A clear example of a conventional signifying form is given by the attitude toward the manipulation of instruments of gentlemen scientists in 17th century England. As Shapin (1989) said, "Despite the clamor of seventeenth-century English scientific rhetoric commending a hands-on approach, natural philosophy was still overwhelmingly a gentlemanly activity, and the traditional contempt that genteel and polite society maintained for manual labor was pervasive and deeply rooted. English natural philosophers other than Boyle, like Thomas Hobbes, ridiculed practitioners who attempted to found science on the manipulation of instruments as opposed to the exercise of rational thought, comparing the experimentalists of the Royal Society to "quacks," "mechanics," and "workmen" (p. 561).

[8]See Derrida's Introduction in Husserl (1978).

in relation to the individuals' activities and their intentional goals.[9] In this broader context, we called semiotic means of objectification the whole arsenal of intentional resources that individuals mobilize in the pursuit of their activities and emphasized their social nature: The semiotic means of objectification appear embedded in socio–psycho–semiotic meaning-making processes framed by cultural modes of knowing that encourage and legitimize particular forms of sign and tool use whereas discarding others.

From an epistemological viewpoint, the semiotic means of objectification are already culturally endowed with specific ways of use. However, from an educational viewpoint, their use is not necessarily transparent for the students. The investigation of the way that individuals have recourse to and link them may shed some light on the problem of the social construction of knowledge.

Before analyzing students' means of semiotic objectification in presymbolic and in symbolic generalization of patterns, some elements concerning the classroom setting and the design of the instructional activity need attention.

## The Classroom Setting and the Generalizing Activity

Two schools in Ontario participated in a 3-year longitudinal study involving teachers and students from Grade 8 through to Grade 10. The students studied the Ontario Curriculum of Mathematics, which, like many programs around the world, gives considerable importance to the study of patterns as a way to introduce algebra in school. Although the curriculum stresses the role of communication and teachers are encouraged, more and more, to have recourse to teaching settings involving small-group activities, the question of how to implement the curricular guidelines is, of course, left to the education boards and the schools.

In the first year of the study, which I draw on here, three junior high teachers and I worked collaboratively to devise mathematical activities that would be meaningful for students, while at the same time, would allow us to investigate relevant developmental aspects of symbolic algebraic thinking. On average, I spent 1 week per month in the classroom of each school, videotaping the implementation of the activities. Prior to each new implementation phase, the teachers and I met for 1 to 2 days to discuss the videotapes and transcriptions of the previous month, to asses learning progress against the objectives given by the Ontario Curriculum of Mathematics, and to elaborate and plan the activities for the next month.

A review of the literature indicated that pedagogical approaches to generalization usually require students to perform a preliminary arithmetical investigation. One of the main underlying ideas is that the numerical structure of the arithmetical investigation will suggest the structure of the sought-after symbolic expression. Nonetheless, as our classroom observations revealed from the very beginning, the

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[9]*Activity,* here, is meant in Leont'ev's (1978) sense.

arithmetical experience is framed by the meaning of natural language. Although the role of speech in pattern activities has been encouraged in previous approaches to teaching algebra (e.g., the "seeing," "saying," "recording," and "testing" pattern approach of Mason, Graham, Pimm, & Gowar, 1985), students often do not spontaneously verbalize the pattern (Lee, 1996). Hence, in the case of our students, who were from a traditional classroom mode of instruction, we needed to develop some form of intervention that would foster this verbalization.

To illustrate our approach to encouraging this verbalization, the generalizing activity shown in Figure 2 is addressed. This activity, which is based on the classic toothpick pattern, was intended as an introduction to the algebraic generalization of patterns in a Grade 8 class. We decided to insert a question into our mathematical activities right after the arithmetic investigation of the pattern and just prior to the search of the symbolic algebraic expression. This activity thus comprised several tasks including, (a) finding the number of toothpicks required to make figure number 5 and figure number 25, (b) explaining how to find the number of toothpicks required to make any given figure, and (c) writing a mathematical formula to calculate the number of toothpicks required to make figure number *n.*

The mathematical activities were designed to be carried out by small groups of two to three students and were usually followed by general discussions conducted by the teacher. This allowed the students to share, analyze, and eventually revise their different solutions.

The interpretative analysis that follows interweaves theoretical reflections with relevant passages from discussions had by one of the student groups (the students are identified as Josh, Anik, and Judith). Oblique reference is made to the work of other small groups. An interpretative, descriptive protocol analysis was used in exploring the data. Specifically, the Non-numerical Unstructured Data Indexing Searching and Theorizing program for qualitative research (Gahan & Hannibal, 1998) was applied. This lead to what has been termed a *situated discourse analysis,* which provides an organization of students' utterances in "salient segments," omitting (when necessary) students' repetitions. This analysis captures essential dialogic descriptions of the problem as discussed by the students and the teacher (details in Radford, 2000b).

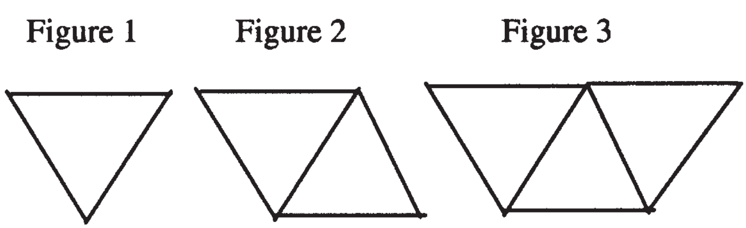


FIGURE 2 The toothpick pattern.

## FACTUAL GENERALIZATIONS

### Factual Generalization as a Scheme Abstracted From Actions

The topic of patterns was not new for the students. In previous years, they had some experience in continuing patterns and working on tasks like the one included in Item a, listed earlier (finding the number of toothpicks required for Figure 25). What was new for them in Grade 8 was the use of algebraic symbols (x, n, etc.) in generalizing the pattern.

To answer Item a, the students discussed the shape of the first figure of the sequence for awhile (see Figure 2), and then they made a drawing of the fourth and fifth figures, counting the number of toothpicks in doing so. Seeing was clearly good enough in this instance. However, when the students were asked to find the number of toothpicks in the 25th figure, they realized that drawing was not considered a practical option so they started looking in depth:

Episode 1 (Note that "[…]" indicates the omission of some words or lines, "…" indicates a pause of 3 sec or more, and "." or "," indicate a pause of less than 3 sec)

1. Judith: The next figure has two more than … look … […] [Figure] 6 is 13, 13 plus 2. You have to continue there. Wait a minute … [gets a calculator] OK. OK, it's plus … .
2. Anik: Well, you can't always go plus 2, plus 2, plus 2 … .
3. Judith: But of course. That's Figure 7, plus 2 equals Figure 8.
4. Josh: That will take too long!

Judith noticed the additive rule linking a term to the next, which in symbolic notations may be stated as, $u_{n+1} = u_n + 2$. Anik and Josh realized that this strategy requires going step by step and persuaded Judith that it would be better to find an easier way. The practical limitation borne by the additive rule (stressed by Anik and Josh in the aforementioned Lines 2 and 4, respectively) as a way to find the number of toothpicks in Figure 25 led the students to work on new possibilities. This change in intention led to a shift of attention. Some lines later Josh said:

Episode 2

1. Josh: It's always the next. Look! [and pointing to the figures with the pencil he says the following] 1 plus 2, 2 plus 3 […].
2. Anik: So, 25 plus 26 … .
3. Josh: Wait a minute. Yeah, 3 plus 4 is 7, 4 plus 5 … so it's 27 plus 26?

4. Anik: Well, because you always … like … look [and she stretches her arm to point to the figures—see the second picture in Figure 3), 3 plus […] it's 25 plus 26.

Line 1 shows the moment in which Josh realized that there is a pattern linking the number of toothpicks of a figure and the sum of the ranks of two consecutive figures. To do so, Josh used his pencil and accomplished a "crude pointing." Anik was then able to clearly see the pattern and provided the right answer. For Josh, however, it was still unclear which consecutive figures had to be taken into account. In Line 3 he was trying to find out the exact relation. In Line 4 Anik offered more information. After this, consensus was reached and the students wrote the answer as, 25 + 26 = 51.

As seen in this passage, the students did not have much trouble calculating the number of toothpicks in the 25th figure. What is more important is that they did so not by counting the number of toothpicks, figure after figure up to the 25th figure, but by a process of generalization, which I refer to as *factual generalization.* A factual generalization applies to objects at the same concrete level. In this case, the concrete level is the numerical one (1, 2, 3, … , 25, 26, etc.). That which is abstracted are the actions undertaken on objects at the concrete level; furthermore, these actions are abstracted in the form of a numerical scheme.

More specifically, a factual generalization is a generalization of actions in the form of an operational scheme (in a neo-Piagetian sense). This operational scheme remains bound to the concrete level (e.g., "1 plus 2, 2 plus 3" Episode 2, Line 1). In addition, this scheme enables the students to tackle virtually any particular case successfully.

FIGURE 3 Two examples of "crude" pointing. In the first picture, Josh is uttering Line 1 of Episode 2. He articulates the utterance, "It's always the next. Look!" with the drawn figures on the sheet and mathematical symbols; the pencil functions as the pointing object. In the second picture, Anik (the girl on the left) utters Line 4 of Episode 2. Judith (the girl on the right) follows the gesture–discursive–symbolic actions (Luis Radford appears in the background taking field notes). In both cases, the crude pointing accompanies the word *look* to clarify intentions and to achieve objectification.

At this point, we need to determine the semiotic means of objectification that the students used to accomplish the kind of generalization that we termed *factual.*

## Semiotic Means of Objectification in Factual Generalizations

*The term the next.* To investigate the semiotic means of objectification we need to pay attention to the varied linguistic devices and signs that students used to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions. Looking back at Episodes 1 and 2, we find a coordinated use of words, gestures, and drawings from which the factual generalization is achieved. Let us deal first with Episode 1. As seen earlier, in one of the students' first attempts, an abstracting scheme is made apparent by Judith's remark: "The next figure has two more than … look … […] [Figure] 6 is 13, 13 plus 2. You have to continue there" (Episode 1, Line 1). The scheme that abstracts the concrete actions now becomes visible to the other students.[10] To ensure that Judith's peers understood her, she illustrated the scheme with the help of concrete actions (i.e., when saying "6 is 13, 13 plus 2").

Notice that, to emphasize the actions, Judith did not calculate the total: She just said, "13 plus 2." Mentioning the total would not help her cause—it would shift attention away from the emergent scheme. Hence, instead of a total, Judith offered a colloquial expression of the generalizing scheme, and to secure the generalizing component she added, "You have to continue there."

The spatial positional term *the next* is a central semiotic means of objectification in Judith's factual generalization. This point is revisited in a later section, but it is worth noting that this term plays a central role here: Through it, a specific reading of the sequence of figures is made possible. It emphasizes the ordered position of objects in the space and shapes a perception relating the number of toothpicks of the next figure to the number of toothpicks in the previous figure.

*The adverb always.* I now consider Josh's factual generalization. In Episode 2, Josh stated, "It's always the next. Look! [and pointing to the figures with the pencil] 1 plus 2, 2 plus 3 […]." The abstracting scheme on which Josh's factual generalization is based is presented in a very similar form to the one that we just analyzed: The general verbal formulation is followed by illustrative concrete actions (in this case supplemented by pointing gestures). A closer look at Josh's utterance reveals that, in addition to the objectifying term, the next, Josh used the ad-

[10]This is so, despite the fact that the referential dimension of the students' words is still vague. For example, the students use the word *figure* not to refer to the figure itself but to the number of toothpicks in the figure. The role of vagueness has been recognized as an essential element in the constitution of students' mathematical discourse (see Rowland, 2000, particularly Chapter 3).

verb *always.* As reported elsewhere (Radford, 2000b), adverbs like "always" underpin the *generative functions of language,* that is, the functions that make it possible to describe procedures and actions that can potentially be carried out in a reiterative, imagined way. They are *ad hoc* linguistic expressions that convey the idea of the abstracting scheme underlying the generalization of actions.

*Rhythm and movement.* The semiotic means to objectify factual generalizations are varied. In another small group, one of the students summed up her group discussion by saying:

Episode 3

O.K. Anyways, Figure 1 is plus 2. Figure 2 is plus 3. Figure 3 is plus 4. Figure 4 is plus 5 [the student pointed to the figures on the paper as she utters the sentence].

In another small group the students stated the following:

Episode 4

Student 2: We'll go by 2 … 3, 5, 7, 9, 11. [pointed to the figures as she counted].
Student 3: [Echoing the counting] 1, 3, 5, 7.

In Episodes 3 and 4, the objectifying process is different from the one used by Josh's group. In fact, in these two episodes we do not find adverbs such as always and spatial positional linguistic terms like the next. Actually, the abstracting scheme does not reach a verbal description. To obtain a generalizing effect, the students rely on the rhythm of the utterance, the movement during the course of the numerical actions, and the ostensive correspondence between pronounced words and written signs. Rhythm and movement here play the role of the adverb always.[11]

Although rhythm and movement are also present in Josh's utterance ("Look! 1 plus 2, 2 plus 3") we would say that, in the last two episodes, rhythm and movement create a cadence that, to some extent, frees the students from using other explicit semiotic linguistic means of objectification. Many studies on the relations between speech and gestures, conducted within an information theory paradigm, have suggested that gesturing while speaking serves to emphasize and comple-

[11]Sometimes, to emphasize the rhythmic cadence, even the students display an unusual number of terms. According to Mason (1990), "[t]he roots of sequence lie in our experience of rhythm, rhymes, seasons, moon phases, day and night and other repetitions of the same or similar events" (p. 25). Along the same vein, Gattegno (1983) attributed much of our algebraic experience to the awareness of such dynamics. Here we have the reciprocal case: To investigate the sequence, the students lend rhythm and movement to it.

ment the information that individuals convey in their interaction. Gesturing has also been seen as something unveiling cognitive representations (for an overview, see Kendon, 1981). In our case, in addition to facilitating goal attainment, the interaction and articulation of the semiotic means of objectification (such as gesturing, speech, sign drawings, sign digits, and rhythm) are seen as a means for constructing meaning (see Figure 3).

### Meaning in Factual Generalizations

So far, the students' construction of meaning has been grounded in a type of social understanding based on implicit agreements and mutual comprehension that would be impossible in a nonface-to-face interaction. Therefore, in the first two episodes, we saw how the students relied on the possibility of seeing. Judith and Josh both said "look" in making a case for their argument. In the second episode, the students understood Josh's brief utterance and agreed on the numeric actions to be performed. In the third episode, the students understood that "Figure 1 is plus 2" means "the total number of toothpicks in Figure 1 is equal to 1 plus 2," and so on. In the fourth episode, the students understood that "to go by two" means to add 2 to the previous number. Naturally, some means of objectification may be powerful enough to reveal the individuals' intentions and to carry them through the course of achieving a certain goal. However, these means may be inadequate in more complex situations where greater precision is required. This is what happened when the students turned to the next part of the classroom activity, as indicated next.

## CONTEXTUAL GENERALIZATIONS

### Abstracting Actions and Objects

The next task of the mathematical activity required the students to write an explanation of how to calculate the number of toothpicks for any given, although nonspecific, figure. In doing so, two new elements were introduced in the activity—a social-communicative one and a mathematical one.

*The social-communicative element.* To produce an explanation requires the students to take into account a new, generic individual. Indeed, the explanation is addressed to someone else (an addressee), someone who is not part of their group. Because the communicative contact with the generic addressee is done through a written explanation, the social understanding is shifted. Implicit and mutual agreements of face-to-face interaction (e.g., gestures, clue words) need to be replaced by objective elements of social understanding demanding a deeper degree of clarity.

*The mathematical element.* In addition to the social-communicative element, a new abstract object has been introduced into the discourse: The question, in fact, asks for any although nonspecific figure.

The two new aforementioned elements led the students to move into another stratum of discourse. The students tried the following strategy:

Episode 5

1. Anik: Yes. […] We can say, like, it's the number of the figure, right? Like, let's say it's 1 there. If … if … OK. You add … like, how do you say that? In order of … [Then, implicitly referring to Figure 2, she says the following] You add it by itself, like. You do 2 plus 2, then after this, plus 1, like. You always do this, right?
2. Judith: [Nods approvingly].
3. Anik: You would do 3 plus 3 plus 1, 4 plus 4 plus 1, 5 plus 5 plus 1. Do you know what I want to say?
4. Judith: Yes.
5. Anik: How do we say it then?

As we can see, in Line 1, Anik chose a different strategy from the one underpinning the factual generalization seen in Episode 2. Indeed, instead of taking the number and the next number (as Josh suggested when they were working on the 25th figure), she counted the same number twice and then added 1. A clear difference between these strategies is that Anik's included calculations with just one unspecified number. Josh's, in contrast, included calculations with two unspecified numbers. Let us call Josh's strategy $S(a,b)$ and Anik's strategy $S(a)$.

An examination of Episode 5 indicates that, in working the task, the students were still relying on the rhythm and movement means of objectification proper of factual generalizations. Not being able to find a satisfactory verbal description for Anik's $S(a)$ strategy, they finally gave up and came back to the $S(a,b)$ strategy:

Episode 6

1. Anik: Yes. Yes. OK. You add the figure plus the next figure … No. Plus the … […].
2. Anik: [She writes as she says the following] You add the first figure … .
3. Josh: [Iinterrupting and completing Anik's utterance says the following] … [to] the second figure.
4. Anik: You add … OK. You add the first [figure] … .
5. Josh: [To] the second figure.

6. Anik: So … [inaudible]. It's not the second figure. It's not the next figure?
7. Josh: Yes, the next one [figure].
8. Judith: Uh, yes, the next [figure] […].
9. Anik: [Summing up the discussion] You add the figure and the next figure.

Line 9 shows how the students formulated a new kind of generalization. As in the case of factual generalizations, the new generalization appears as the abstraction of concrete actions in the form of an operational scheme. Although, a difference is that the new scheme does not operate on the level of concrete numbers, as factual generalizations do. As the episode shows, specific figures (like the fifth, sixth, etc.) have been displaced and put in abeyance. Instead of alluding to specific figures, the students now talk about "the" figure and "the next" figure. Although in doing so, the students' generalization bears some confusion in that they did not distinguish between the number of the figure and the figure itself, the generic expressions "the figure" and "the next figure" allow them to attain a new level of generality. The new generalization encompasses an abstraction from actions and an abstraction from specific figures.

## The Semiotic Means of Objectification

Episode 6 shows that rhythm and ostensive gestures have been excluded. What, then, are the semiotic mechanisms of objectification that the students display in the new kind of generalization? To answer this question, we have to be aware of a transformation in the students' mathematical discourse. As we recall, the students were asked to explain how to calculate the number of toothpicks for any given although nonspecific figure. Because it is not possible to reason about any given although nonspecific figure without somehow pinpointing the object of our reasoning, the latter has to be specified without ever making it completely specific.[12] The students thus transformed the expression given in the task ("any given figure") into the expression "the figure," a linguistic generic term that conveys the sought generality.

In addition to this, the students introduced another linguistic term, namely, *the next figure,* which was previously used by Judith and Josh (Episodes 1 and 2). Here, however, it is used and exploited to a greater extent. In using the linguistic term "the next figure," the students focus their attention on one of the attributes of the sequence of figures. They ignore geometrical shape, color, and so forth, and—

[12]Thinking is always thinking about something. The problem, at this point of the students' activity, is that it would not be possible to talk about, for example, Figure 5 unless Figure 5 or any other particular figure is taken in a kind of metaphorical sense (see Radford, 2000b, pp. 247–248).

with words— point to an attribute of the sequence that appears relevant, that is, *proximity.*[13]

We are now at the heart of the students' objectifying process, which is carried out through these generic and locative terms. These terms allowed the students to refer to objects much in the same way as *deictic* or "demonstrative terms," like *this* and *that,* do in colloquial speech.[14] At the same time, there is a significant difference: deictic or demonstrative terms indicate something visually attainable.[15] In contrast, because the figure is not the 1st figure, the 5th figure, or any other specific figure, the generic and locative terms (the figure and the next figure) refer to something that cannot be materially seen. The interesting point is that, in using the generic and the locative terms, new objects have entered the discourse. The emergent objects have become detached from the students' potential sensory experience and have become emphasized in such a way that they are now put forward, as if they have become something visible. In doing so, the students have reached a new type of perceptual field.

To better understand the role of the semiotic means of objectification in this phase of the students' mathematical activity, we have to discuss the kind of visibility that emergent objects have attained. The way in which I consider linguistic terms capable of making something visible is derived from the phenomenological nature of language. This is related to Husserl's (1958) idea of the "way of appearance" of objects that were discussed earlier. In one sense, linguistic terms call our attention to certain objects of our environment.[16] In another sense, linguistic terms and the various signs used in social intercourse allow the individuals to go beyond what is offered visually and to create conceptual worlds. Of course, the appearance of the objects in these conceptual worlds are different from those of concrete ob-

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[13]The linguistic term *the next* belongs to what Miller and Johnson-Laird (1976, p. 380) called *spatial locatives,* that is, a system of categories conceptually related to the experience of space through which individuals mark their reality and the ensuing perception of it. The ordinal terms *first* and *second,* used in Lines 2 through 5 in Episode 6, have the advantage of somehow specifying the position of terms (the first figure is the one that is being considered in the action). However, the first figure may also be understood as the first figure of the pattern and consequently may be open to misunderstandings. The students did not feel comfortable with the choice of ordinal terms and decided to abandon it. They chose the more appropriate terms: the figure and the next.

[14]Deictic terms (e.g., this, that, you, I) are linguistic expressions that refer to objects in the universe of discourse by virtue of the situation where the dialogue is carried out. Their referents are determined according to the contextual circumstances. For example, *this* may refer to something in one context and to something completely different in another context (see Nyckees, 1998, p. 242ff.).

[15]For example, when someone says to me "this chair," the chair is supposed to be in sight; otherwise, instead of relying on the contextual power of the deictic term (this), my interlocutor will need to describe the chair to which he or she is referring.

[16]Evidently, linguistic terms are not the only way to accomplish this. The example about the sign indicating the place of Aristotle's *Poetics* on the bookshelf applies here. As to mathematics, Dörfler (1986, p. 151) stressed the role that naming has in the subjective and cognitive creation of mathematical objects.

jects; these objects can be perceived through signs and artefacts only. Sfard (2000) contended that symbols, "have the power to turn the extended-in-time, transitory, and invisible into contained-in-space, permanent, and perceptually accessible" (p. 321). Notwithstanding the semiotic nature of their appearance, these "invisible" objects constitute a world that, in a certain way, is as real as the physical world. Many years ago, Leslie White (1942) commented that, "With words man creates a new world, a world of ideas and philosophies. In this world man lives just as truly as in the physical world of his senses" (p. 372).

These remarks now allow us to clearly state the question of the students' semiotic means of objectification as follows. We see that, instead of a coordination of utterances and rhythmic pointing, the abstract objects appear in this part of the students' mathematical activity as being objectified through refined linguistic, generic, and locative terms referring to nonmaterially present objects. As a consequence of this linguistic objectifying process based on a refined, although still ostensive, way of functioning, the abstract objects are contextually conceptualized in reference to the features of the given concrete objects (i.e., Figure 1, Figure 2, and Figure 3). The abstract objects are hence abstract while bearing contextual and situated features that reveal their very genetic origin.

## Contextual Generalizations

All in all, without using letters and capitalizing on factual generalizations, the students succeeded in objectifying an operational scheme that acts on abstract—although contextually situated—objects and specifies temporally situated mathematical operations on them, thus ensuring the attainment of a new level of generality.[17] These contextually situated objects abound in classroom discourses, where they become part of the process of construction of nonsituated, mathematical objects. This, I think, is a notable reason for paying careful attention to their genesis and their functioning.[18]

Let us call these nonsymbolically based types of generalizations, performed on conceptual, spatial temporal situated objects, *contextual generalizations.* Contex-

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[17]The temporally situated mathematical operation is clear in the sequenced actions indicated in the utterance, "You add the figure and the next figure."

[18]To give a different example of contextually situated objects, let me refer to Carraher, Schliemann, and Brizuela's (2001) work. In the reported lesson with 8- and 9-year-olds, the teacher introduced a story problem involving an unknown amount of money. Throughout the story, specific amounts of money were added or subtracted (e.g., \$5 or \$3). The children kept track of this amount of money by placing marks on a number line (that, along with gestures, body movement, words, and signs, functioned—in terms of my framework—as semiotic means of objectification). In doing this, the students encountered the "cancellation of terms" ($N + 3 – 3$ was identified by one of the children with N). In my interpretation of the classroom episode (see Radford, 2001b), amounts of money to be subtracted appear as contextually situated conceptual objects that may prove useful (although I am not saying sufficient) for the construction of the abstract concept of negative numbers.

tual generalizations differ from algebraic generalizations on two important, related counts. First, algebraic generalizations involve objects that do not have spatial temporal characteristics. In fact, algebraic objects are nonsituated and nontemporal. Second, in algebraic generalizations the individual does not have access to a (figurative) point of reference to "see" the objects. Certainly, the crucial term, the next figure, in the aforementioned contextual generalization supposes that the individual has a privileged view of the sequence, a point of reference: She or he sees the figure (in a figurative way), and this allows him or her to talk about the next figure. In contrast, when dealing with sequences through algebraic symbols, the individual has to dissociate himself or herself from the terms of the sequence temporally and spatially (see Traugott, 1978, pp. 380–381). This point was made by Bertrand Russell (1976). He observed that in the world of mathematics (and of pure physics), space and time are seen impartially "as God might be supposed to view it" (p. 108). Also, to emphasize the nonsubjective character of space and time in descriptions of mathematical objects, he added that in such descriptions, "there is not, as in perception, a region which is specially warm and intimate and bright, surrounded in all directions by gradually growing darkness" (p. 108).

How then will the students proceed to produce meanings that will not rely on time and space—meanings that, so far, have resulted from their spatial temporal embodied situated experience? What are the semiotic means of objectification that will sustain the students' production of voiceless, symbolic algebraic expressions? Although I do not have a clear answer for these difficult questions, I now scrutinize the way students attempt to reach the objective dimension required by symbolic algebra. To do so, I target the respondent's desubjectification process—a process that emphasizes changes in the relation between the object of knowledge and the knowing participant. The detection of students' semiotic means of objectification underlying the production of symbolic algebraic expressions then allows us to contrast presymbolic and symbolic types of generalization.

## SYMBOLIC GENERALIZATIONS

### Bypassing the "Positioning Problem"

In the next passage, the students did not symbolize the contextual generalization based on the strategy $S(a,b)$. They worked out an algebraic symbolization of the $S(a)$ strategy:

Episode 7

1. Josh: It would be $n$ plus $n$ … .
2. Annie: $n$ plus … OK. Wait a minute! … $n$ … .
3. Judith: Yes. $n$ plus … yeah it's $n$ … […] plus $n$ plus 1.

4. Annie: Yes! *n* plus *n* plus 1! [i.e., $(n + n) + 1$, as it will become clearer in Line 9] […].
5. Judith: Yes. Because, look! Look! … .
6. Annie: Your first figure is "*n*" right? Plus you have *n* because it's the same number … .
7. Judith: Because, look! Look! 4 plus 4 equals 8 plus 1.
8. Josh: *n* plus *n* plus 1.
9. Annie: Bracket plus 1 [they write, (n + n) + 1].
10. Judith: OK. Let's try it. Example … [Josh says, "4 plus 4 equals 8," and Judith adds, "4 plus 4 equals 8 plus 1 equals 9." At this point the students are satisfied with their result and start working on the next question].

There is an aspect of the desubjectification process in which the students have succeeded so far, namely, the insertion of a speech genre based on the impersonal voice. This is evidenced by the students' utterances produced in Lines 8 and 9. Therefore, "your first figure" in Line 6 becomes *n* in Lines 8 and 9. Furthermore, in contrast to the subjective utterance in Line 6, Lines 8 and 9 no longer make any allusion to an individual owning or acting on the figures. With this, the traces of subjectivity start fading in a process where personal voices (e.g., "I add," and "you put") and the general deictic objects (e.g., "this figure") underpinning the previous mathematical experience shift to the background thereby providing room for the emergence of objective scientific and mathematical discourse.

Still there were other aspects of the desubjectification process that proved to be more difficult. To understand this, we have to raise the following question:

> Why did the students not symbolize the $S(a,b)$ generalizing strategy based on "the figure plus the next figure" that they objectified before?

Actually, before the students engaged in the $S(a)$ strategy in Episode 7, they began to work on a symbolization based on the $S(a,b)$ strategy. The latter strategy, however, requires finding a way to forge a symbolic link between the figure and the next figure and their corresponding ranks. The difficulty to accomplish this was clearly expressed by Anik when she stated the following:

> Episode 8
>
> Ok. You can say … you make … OK you add the figure … oh my God, how do you say it [in algebraic symbols] … the figure plus the next figure?

The fact that the symbolization of the terms of a pattern requires symbolizing them according to their position constitutes a key conceptual difference between symbolizing an unknown in an elementary equation (as in word problems involving a single unknown) and symbolizing the general term of an elementary pattern. Also, as previous research suggests, the positioning of the terms poses a semiotic problem for the students in other generalizing activities such as elementary number theory. For instance, Lee (1996) found that more than 50% of a group of 113 high school students could not provide suitable symbolic expressions (like $x$ and $x + 1$) for two consecutive numbers, with many of them opting for nonpositional symbolizations, like $x$ and $y$ that remained unconnected.

The analysis of the students' dialogues carried out in my previous work gave evidence of a series of difficulties consisting of designating, through algebraic symbolism, the elements of a sequence according to their rank (Radford, 1999b, 2000b, p. 250). I termed this problem the *positioning problem.* Let me add here that the semiotic difficulties related to the positioning problem result from the dramatic changes in the mode of designation that the disembodied algebraic language brings with it. The changes in the mode of designation include, (a) the exclusion of linguistic terms conveying spatial characteristics (e.g., the next); and (b) the suppression, in the symbolic language, of the acting individual (algebraic symbolism uses signs as ×, +, etc., but it does not allow us to insert within the symbolic expressions personal pronouns like I or you). As such, the positioning problem is part of the desubjectification process that the mastering of the algebraic language requires, and its presence here is illustrative of the difficulties that the students encountered while engaging in this process.

## The Teacher's Intervention

When the teacher came to see the students' work, she asked the students to explain the formula to her. Referring to the symbolic expression of the $S(a)$ strategy, the students responded:

Episode 9

1. Anik: … You add the figure plus the number of the figure … .
2. Judith: … [Completing Anik's sentence] again, after that, plus one more.
3. Josh: Then we can say how many toothpicks there are.

These responses show, in an amazing way, how Anik, Judith, and Josh complemented each other's utterances. The teacher, however, noted the discrepancy between the students' explanation written in natural language in the previous task and based on the $S(a,b)$ strategy (see Episode 6), and their current algebraic expression based on the $S(a)$ strategy. She decided to further immerse the students in the objectifying process by commenting that the symbolic expression did not say the same thing as their explanation in natural language, and asked if they could provide a formula that would say the same thing. They said:

Episode 10

1. Anik: How do you say it's the next one?
2. Josh: You do 1 plus 2 equals 3, 3 plus 2 equals 5, 3 plus 4 equals 7.
3. Anik: So we'd do *n* plus the next one, the next figure … .
4. Josh: That would be like $n + a$ or something else, $n + n$ or something else.
5. Anik: Well [no] because "*a*" could be any figure […] . You can't add your 9 plus your … like … […] . You know, whatever you want it has to be your next [figure].

When the students reached an impasse, the teacher intervened again: "If the figure I have here is *n*, which one comes next?" Thinking of the letter in the alphabet that comes after *n,* Josh replied, "*o*."

The teacher's utterance shows how her attempt to help the students overcome the positioning problem is underpinned by the spatial temporal dimension of the general objects alluded to earlier (e.g., the figures are dynamically conceived of as coming one after the other). Finally, after reworking the case for the 5th figure, the students noticed that 6, that is, the number of the figure that comes next, can be written as $5 + 1$, which was then reinterpreted as $n + 1$. The reinterpretation of $5 + 1$ as $n + 1$ was done through a format of questions and answers similar to those documented in interactionist studies (e.g., Voigt, 1985, 1989):

Episode 11

1. Teacher: Do you all agree that it will be *n* plus something?
2. Annie: Yes.
3. Teacher: When we were on Figure 5, right? What was the "something" that we added?
4. Annie: It was 6.
5. Teacher: 6, huh? If it was 10?
6. Annie: It was 11.
7. Teacher: What was the pattern, then?
8. Julie: Well, one more … [inaudible] […] . It is one more than … .
9. Teacher: [Taking over Julie's statement and trying to complete it] One more.

In an attempt to recapitulate the discussion, the teacher asked the following:

Episode 12

1. Teacher: This would be … ? [referring to the expression, (n + 1), that the students had previously written on their page].

2. Anik: It's the next [figure]!
3. Teacher: [Approvingly] Ah!
4. Anik: OK! There, now. I understand what it is I'm doing.
5. Judith: OK.
6. Anik: You put your *n, n* is your figure, right?
7. Judith: Yes.
8. Anik: OK. So, what we can do is *n* equals the figure … […] $n + 1$ equals the next figure, right?
9. Judith: Right. [Anik writes the answer $(n + 1) + n$].

Because of the teacher's intervention, the students were able to overcome the positioning problem. In doing so, they started tackling a delicate aspect of their process of desubjectification. However, this does not mean that the students had acquired new modes of denotation. For instance, the students were not able to identify any commonality between the expression reached here, that is, $(n + 1) + n$ and the expression $(n + n) + 1$. The expressions $(n + 1) + n$ and $(n + n) + 1$ remained essentially different for these students. I address this point in the next section.

## The Semiotic Means of Objectification in Symbolic Generalizations

To understand why the expressions $(n + 1) + n$ and $(n + n) + 1$ were perceived as different, we need to discuss the students' semiotic means of objectification. I claim that the students considered these expressions to be different because these expressions objectify two different sets of actions. In fact, over the course of the mathematical activity, we found that the algebraic expression $(n + n) + 1$ was the final point of a process of symbolization extended across three discursive strata (see Figure 4).

A similar inspection of Episodes 2, 11, and 12 shows that the same symbolizing process led the students to the algebraic expression $(n + 1) + n$.

Figure 4 illustrates the connection between (additive) numerical actions and the structure of the novice students' algebraic expression. It neatly illustrates the fact that the various signs (n, 1, +, and brackets) in the symbolic expression are concatenated in a nonarbitrary way. The signs, indeed, conform to the flow of the numerical actions undertaken in the first stratum and objectified further in the second stratum of discourse. The first action or event is symbolized first, then the second, and so on. Because the symbolic expressions correspond to different sets of actions, they are perceived as different. This is what leads the students to insist, so tenaciously, that brackets have to be written, as in Line 9, Episode 12 and in Line 9, Episode 7.

These remarks suggest that the semiotic means of objectification sustaining the algebraic expressions remain related to the sequenced order of numerical actions. To go further and to better understand this phenomenon, consideration is now

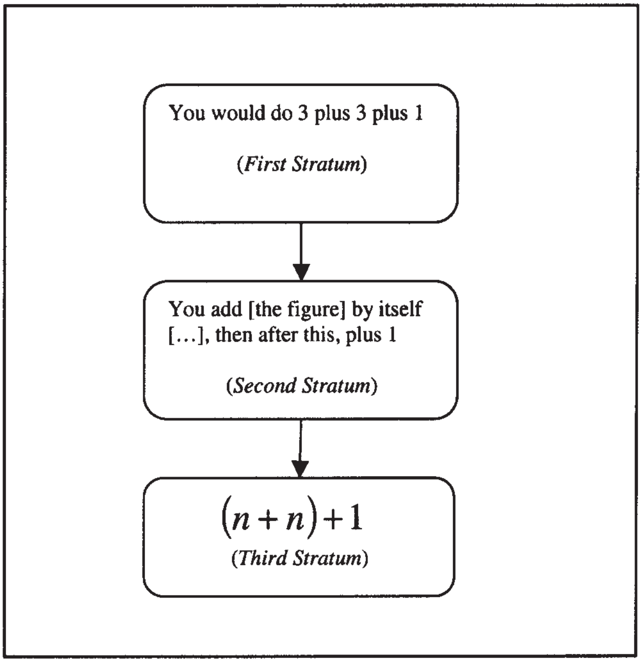


FIGURE 4 The additive numerical actions carried out in a first stratum of speech and objectified further in a second stratum induce the structure of the symbolic expression (third stratum).

given to the question of the semiotic nature of sign letters in the students' mathematical activity. In doing so, I draw from a distinction made by Duval (1999, in press) between sign and function; that is, I distinguish between the type of sign that is used in a certain semiotic activity and the functions that a type of sign allows one to fulfill.

As to the type of sign that the students used to build their first symbolic expressions, Figure 4 plainly shows that the sign *n* appears as an abbreviation of the generic linguistic term *the figure* embedded in a previous layer of discourse.[19] Due to the kind of existential connection between an abbreviated word and the sign abbreviating it, the latter can be seen as if it is pointing to the former (or, better, as if the abbreviating sign is showing or indicating the emplacement of the abbreviated

[19]In other groups, the students further stressed the intention of abbreviation with which they endowed their first algebraic symbols by using the letter *f*, that is, the first letter of the word figure. Diophantus also used letters to abbreviate words in his *Arithmetica* (see Heath, 1910). In their algebra treatises, many Italian mathematicians of the 14th and 15th centuries did the same as well (e.g., see the geometrically oriented abbreviation style of Piero della Francesca in Radford, 1995, p. 35).

word by exposing a part of it).[20] This is why I suggested elsewhere that the sign *n* can be seen as an index in Peirce's (1955) sense (Radford 2000b, 2000c).[21]

Regarding the functions that a sign can serve, it is worth considering the tasks that can be carried out with these kinds of degenerative indexes. As abbreviations, the mode of reference of indexes is very limited. In an almost physical connection, an abbreviation refers to a word, which in turn, refers to an object. As a result, in the students' symbolic expressions the abbreviated term *the figure* has not evaporated. It still exists there, underneath the sign *n* that is abbreviating it. The symbolic expressions, built on the basis of indexes, thereby inherit the spatial temporal dimension of contextual discourse that precludes the attainment of formal calculations. A formal calculation allows one to use the associative law of addition and eventually to get rid of the brackets. This, however, cannot work here simply because the sign *n* is pointing to a specific term of the student's discursive activity—the figure. What could $2n + 1$ be pointing to?

As Duval (in press) recently stressed, one of the crucial points in the learning of algebra is that, although natural language offers rich expressions to designate objects (like Figure 25, the previous figure, the next figure, the first figure, etc.) and to talk about them (e.g., "Figure 25 is composed of a string of adjacent equilateral triangles where … "), algebraic symbolism proceeds to a remarkable reduction of vocabulary. As a result, the mode of denotation of algebraic symbolism (at least from the 16th century onward) demands that a small number of signs be combined to offer denotations and descriptions of mathematical objects. It is in this sense that signs in algebra rely on a functional designation. Duval (in press) observed, "The functional designation appears precisely when the vocabulary becomes insufficient vis-à-vis the number of objects to be designated." That is, when we face *penuria nominum*—a lack of names to designate the objects of discourse.

Algebraic language presents one of the more striking cases of penuria nominum. This, of course, is not only specific to algebra (e.g., we do not have names for each irrational number either). Nevertheless, this problem acquires a tremendous significance in the classroom genesis of the symbolic expressions because the stu-

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[20]The abbreviation is playing a similar role as the mark placed on the bookshelf indicating Aristotle's *Poetics* in the introductory example.

[21]Peirce (1955) distinguished between two different kinds of indexes: *genuine* and *degenerate.* A genuine index relies on an existential relation with its object. A degenerate index involves a referential relation (see Peirce, 1955, p. 108). A paradigmatic example of genuine indexes is the smoke that one sees in a field; the smoke is an index of fire. There is a causal relation between them. In degenerative indexes, the causal relation is no longer required. Clearly, the students' use of signs in their emergent understanding of algebraic activity corresponds to the second kind. Many years ago, Burks (1949) attempted to unify Peirce's concept of index. He claimed that Peirce's "definition of an index would have to be revised to read: an index is a sign which signifies its object through an existential connection to this object or to a sign of this object" (Burks, 1949, p. 678; see also, Goudge, 1965).

dents tend to use indexes and indexes lack the capacity of being combined with other signs, as required by the functional designation.

In the next section, I push a bit further the reflection about the indexical nature of students' signs. Briefly, I submit that indexical (as well as iconic) behavior is not an unusual conduct in the learning of a new sign system.

## THE INDEXICAL NATURE OF STUDENTS' SIGNS

In stepping into the realm of algebra, the students—we have seen—tend to produce signs in a way that reflects the flow or movement of previous numerical actions. Brackets, for instance, become essential because they help the students mark the rhythm and motion of the actions. Given the strong connection between action and symbol (a connection ensured by a kind of indexical semiotic behavior), the acting participant produces symbolic expressions that are still contextual in nature. Symbolic expressions have not yet reached God's noncontextual view—to borrow Russell's (1976) metaphor.

To try to understand the roots of this process of production of signs and meanings, we need to recognize that indexes are one of the more elementary kinds of signs. From an ontogenetic viewpoint, the child's prelinguistic activity is largely composed of articulated sounds that aim to signal objects of his or her surroundings. The child's prelinguistic activity is also comprised of pointing gestures that serve to drive the focus of attention. Idiosyncratic as they are, these diverse signs of a communicative nature constitute the antecedents of language, that is to say, a "protolanguage." Therefore, this protolanguage appears, to a great extent, as essentially indexical.[22] The indicative function of language in small children was pointed out by Vygotsky and Luria (1994):

> A series of observations relating to very small children showed us that the primary function of speech as used by the infant is, in fact, limited to indication, to the singling out of a given object from the entire situation perceived by the child. The fact

[22]It can be noted that language enables a dialectical movement between the general and the particular. It allows us to generalize and to particularize. Nevertheless, the particular, the general, and the relation between them, exceed the realm of language. Although "what is thinkable will of course be constrained by language itself" (Eagleton, 1983, p. 175), the particular, the general, and their relation are conceptually shaped by the material, spiritual, economic and other aspects of the individuals' social life. For example, a particular is not merely a thing. It is a particular inasmuch as it is an object of attention and reflection within the frame of the practical activity of human beings. This is why, from an ontogenetic viewpoint, indexes have to be understood not only as isolating particulars or as actualizing the general but also as mediators. They are genetic mediators between the cultural–conceptual categories of the general and the particular.

> that the child's first words are accompanied by very expressive gestures, as well as a number of control observations, convinces us of this. (p. 125)

With the progressive mastery of speech, words become signs capable of being used with a certain autonomy regarding the objects they denote. Clark (1978) discussed cases where a mother and child look at a picture book together and interact in such a way that the mother points to or touches some part of the picture; the child follows the same gesture and touches the same part of the book. Meanwhile, in addition to the gesture-based actions, speech accompanies the interaction. The mother pronounces words referring to the picture in the book and the child imitates the mother's sound word for the named object. Words start replacing gestures. Therefore,

> When the naming relation is firmly established for any word–object conglomerate, then the pointing becomes redundant in some contexts for the word itself implies the activity of pointing. Therefore, there opens up the possibility of talking about (i.e., verbally pointing to) objects and relations that are not immediately present to the senses. (p. 257)

The child grasps the words and their meaning in rich interactive contexts in which indexical gestures become interwoven with other semiotic elements. Along with the indexical actions, the child's vocal gestures of his or her protolanguage start being replicas of the mother's or other adults' sounds. The child's sound–word replicas are *iconic* verbal signs of the adult's speech. The child's vocal gestures, quite early in fact, tend to reproduce the adult's words as much as possible. Also, as Clark (1978) suggested in the previous quotation, with the progressive mastering of oral language, words do not lose their first indexical nature. They acquire a second nature that dispenses them of their existential denoting connection (for a phylogenetic analysis of this problem see Leroi-Gourhan, 1964).

A similar phenomenon occurs in other semiotic areas like in the development of pictorial repertoires (see Kindler & Darras, 1998) and in writing (see Dagognet, 1973). In the latter case, children's learning of writing is heavily mediated by oral speech. There is a well-documented phase in their learning process where they write according to the way they speak. The understanding of written language is carried out by an indexical one-to-one translation into speech. Also, the whole structure of the written language follows in an iconic way the structure of speech. During a critical part in the process of learning to write, the written words will indexically and iconically refer to words of the spoken language and only later will written words reach a certain autonomy. Vygotsky (1997) wrote:

Understanding written language is done through oral speech, but gradually this path is shortened, the intermediate link in the form of oral speech drops away, and written language becomes a direct symbol just as understandable as oral speech. (p. 142)

Let us summarize these remarks in more general terms and correlate them with our classroom problem under discussion. For as long as a sign system, $S_1$, is still heavily dependent on other sign systems, $S_{01}$,$S_{02}$, … , from which $S_1$ arises, iconizing, pointing or other indexical devices play a fundamental role in ensuring the connection between the emergent system $S_1$ and the source systems $S_{01}$, $S_{02}$, and so on.

The aforementioned semiotic connection between the emergent and the source sign systems, I suggest, is what happened during the sprouting of algebraic language in the classroom. The link relating the algebraic letter symbols to the students' actions serves as the semiotic means of objectification underpinning the students' production of signs. This link makes indexes meaningful. Delete the action and the sign will lose its semiotic power and become an unrecognizable hieroglyphic-like mark.

Obviously, the heavy dependence of indexical signs on the actions they symbolize imposes severe limits on their use. For one thing, the signs cannot be separated from their whole symbolic expression, which makes the emergent algebraic expression impossible to break down into elementary parts. As mentioned previously, the students see their symbolic expressions as a whole, with the consequence that similar terms are seen as impossible to regroup. The impossibility to collect similar terms appeared clearer in another one of our students' small groups. The teacher went to see their work and tried to help them to simplify their symbolic expression by saying, "Then, $n + n$ is equal to … ?" The students promptly answered "*n*." The difficulty resides in that indexical signs cannot be added. As long as they are still pointing to their objects, one cannot collect them and merge them into a single new symbolic expression.

The indexical nature of students' signs may also help us to understand the vagueness that the students displayed in the semiotic act of reference. It became apparent that often there was a kind of disregard for precision in referring to some objects (see Footnote 10). Again, this phenomenon is frequent in early language acquisition. As Bruner (1975) said, "The objective of early reference … is to indicate to another by some reliable means, which among an alternative set of things or states or actions is relevant to the child's line of endeavour" (p. 268). Also—as in the case of our students, who took $n$ to stand for the figure instead of the number of toothpicks in the figure—, Bruner, still addressing the question of early reference, added, "Exactitude is initially a minor issue" (p. 268). Naturally, from the viewpoint of learning algebra, exactitude in referencing becomes a major issue.

## SYNTHESIS AND CONCLUDING REMARKS

This article has focused on the investigation of types of generalization of geometric–numeric patterns in novice algebra students. The investigation was motivated by the idea that the comparison between presymbolic and algebraic types of generalizations may shed some light on the difficulties that the students encounter when they are required to express generalizations using algebraic symbolism.

Because, from a psychological viewpoint, a generalization implies that something new has been made apparent (e.g., that a relation between certain concrete objects applies to other concrete objects or even to new objects), I undertook the task of looking into the students' processes of objectification, that is, the way in which the students made apparent the new relations and objects that needed to be put forward in the generalizing activity. Drawing from Husserl's (1958, 1978) phenomenology and adopting a semiotic cultural perspective, I introduced the concept of semiotic means of objectification and identified two types of presymbolic generalization in students. The first one was called factual generalization and consists of a generalization of numerical actions in the form of a numerical scheme. The semiotic analysis of the students' interaction suggested that central to the objectification of the numerical scheme is a bodily gesture (a type of crude pointing) that, despite its sensuous nature, allowed the students to go beyond the three figures that they had in front of them and to make apparent a pattern enabling them to determine the number of toothpicks in any specific figure (as Figure 25).

I called the second type of generalization contextual. In contrast to factual generalizations, contextual generalizations generalize not only the numerical actions but also the objects of the actions. They go beyond the realm of specific figures and deal with generic objects (like the figure) that cannot be perceived by our senses. They have to be objectified and produced within the realm of reasoned discourse, that which the Greeks called *logos.*

The classroom mathematical activity as well as language allowed the students to carve and give shape to an experience out of which new general objects emerged. However, the linguistic terms used by the students were such that the new general objects were still seen contextually. The means of objectification moved from a crude pointing to refined linguistic sentences revealing the flow of arithmetic actions in time and space. In this sense, the general mathematical objects emerged during a culturally embodied mathematical experience.[23]

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[23]Many discussions about embodiment in mathematics education come from Lakoff and Johnson's (1980) work on metaphors. Although I consider the path followed by Núñez (2000) and Lakoff and Núñez (2000) rather enlightening, I want to emphasize that the students' embodiment of experience, as I take it here, is not seen as resulting from a biological origin. Without dismissing the importance of our biological constraints and possibilities in cognition, in the semiotic cultural approach that I have been advocating, the embodiment of experience results from socially constituted practices semiotically mediated by language and other cultural and historical products. Instead of being the origin, the body — as Foucault (2001, p. 1011) contended — is a surface of inscription of historical events marked by language. From this perspective, instead of embodied experience, I would probably do better talking about empracticed experience; but, at this point, I do not want to further suffocate the discussion, and I limit myself to referring the reader to the work of Lizcano (1999) for an interesting critique of Lakoff's work and the discussion of the body as a cultural metaphor itself.

Regarding symbolic generalizations, I paid attention to the way that students tackled the desubjectification process that imposes the functioning of the algebraic language. When investigating the types of presymbolic generalizations, I stressed the fact that natural language accounted for close dialectical forms of relation between participant and object. In algebraic language, nevertheless, the relation between participant and object cannot be maintained. The dual reference participant–object becomes lost and it is no longer possible to talk about, for example, "your first figure." In entering symbolic algebra, the students are deprived of indexical and deictic spatial temporal terms and have to refer to the objects in a different way. Attention was then focused on the tensions caused by the shifting in the relation between the knowing participant and the object of knowledge.

The results suggest that students succeeded, to some extent, in devoicing subjectivity. Their symbolic expressions achieved the effacement or erasing of I, you, and so forth.[24] Also, in bypassing the positioning problem with the teacher's help, they attempted to produce nonspatially based symbolizations.

The suspension of subjectivity (related to objectivity) was recognized by Kant (1781/1996) as one of the two conditions for knowledge. The second condition to attain knowledge, he suggested, was the exclusion of time (that he related to logical necessity). It is in regard to Kant's second condition that the major cognitive and epistemological problems appear. Indeed, as the analysis of the students' mathematical activity clearly shows, the semiotic means of objectification to which the students have recourse to produce their very first algebraic expressions remain related to actions and movement, and hence to time. The comparison made here, between presymbolic and symbolic generalizations, intimated that the students' indexical signs are implicated in the numerical actions that they symbolize; this is why the whole structure of the symbolic expression captures and reflects the flow or movement of the numerical actions. We can say that the phantom of the students' actions still haunts the algebraic symbols.

There is a substantial consequence of this in terms of the limitations that indexical signs impose on a formal calculation of expressions. During the analysis, I stressed the fact that the relation between the meaning of signs and the actions that the signs are symbolizing impeded the students from seeing the algebraic expressions $(n + 1) + n$ and $(n + n) + 1$ as sharing a relevant commonality. This, I suggest, is a key point in the understanding of algebraic language.

For Frege (1971), expressions such as $(n + 1) + n$ and $(n + n) + 1$ refer to the same object. However, they do not refer to this common object in the same manner. These expressions name or describe the same object in a different way. In other words, they have a different sense (or, in our terminology, meaning). What the ontogenesis of the algebraic language suggests is that, in the meaning-making pro-

[24]The vital role played by deictic terms like *I* and *you* in school mathematical discourse has been stressed by Rowland (2000).

cess accompanying the production of signs, the primacy is action—action objectified through different semiotic means, starting from the form of a crude pointing. Vygotsky (1997) seemed to be right in saying that

> A gesture is specifically the initial visual sign in which the future writing of the child is contained as the future oak is contained in the seed. The gesture is a writing in the air and the written sign is very frequently simply a fixed gesture. (p. 133)

We saw here how rhythm and movement were the seeds that grew up into symbols when students acted and interacted in socially established mathematical activities.

The question of the individuals' actions and their semiotic objectification—discussed from other theoretical perspectives and in a different context by Arzarello (2000) and Dörfler (2000)—appears as a key element in contemporary understandings of the ontogenesis of algebraic language.[25] The requirements of the algebraic language are such that the algebraic objects have to be denoted in a layer of discourse where they bear a different kind of existence and where the participant denoting them has to become (to use a term from Lacanian theory of discourse) *decentred* (e.g., see Bracher; Massardier-Kenney, Corthell, & Alcorn, 1994). The epistemological and didactic understanding of the decentration of the participant urges us to reflect on and envision new dialogical and semiotic forms of action in the activities that we propose to students during their immersion in the phylogenetically constituted practice of algebra.

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[25]Arzarello (2000) paid attention to the way in which language, actions, and time become essential ingredients in the genesis of theoretical knowledge in technologically mediated environments related to processes of proof. Dörfler (2000) stressed the genetic role of actions and symbolization in the rise of mathematical concepts, something that is studied through the notion of *protocol* (p. 111ff).

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