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(This paper is a comment on the papers of a PME research forum on algebra)

SYNTAX AND MEANING

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A forum is certainly a multi-voiced dialogue, an example of what Bakhtin used to call heteroglossia, or the encounter of multiple perspectives in cultural interaction. With their own intonation and from their own perspective, the papers of the research forum engage in dialogue with each other about pedagogical, psychological and epistemological questions concerning two key concepts of school algebra, namely, equations and inequalities. They offer us valuable reflections on the search for new contexts to introduce students to inequalities (e.g. functional covariance) and a critical understanding of the limits and possibilities of these contexts. They also provide us with fine enquiries about urgent learning problems along the lines of key theoretical constructs that have played a central role since the 1980s in mathematics education (such as structure and the cognitive status of students' errors).

The papers tackle a general problématique against the background of the present context of discussions about cognition. In the past few years, there have indeed been important changes in conceptions of cognition in general, as witnessed by e.g., a recent interest in phenomenology, semiotics, and embodiment. We have become aware of the decisive role of artefacts in the genesis and development of mathematical thinking and we have become sensitive to theoretical claims from sociology and anthropology that emphasize the intrinsic social dimension of the mind. With their own intonation and from their own perspective, the papers of the research forum have engaged each other in a dialogue on the problem of algebraic thinking as set by the general stage of our current understanding of cognition. Since one of the key common themes of the papers is that of syntax and meaning, let me delve into it and comment on what the papers intimate in this respect.

1. Meaning

In the introduction to their paper, Boero and Bazzini find fault with the classical approach to inequalities and claim that the "purely algorithmic manner" that reduces the solving of inequalities to "routine procedures" limits students' understanding. This complaint is not new. In the seminal book edited by Wagner and Kieran (1989) the same reasons led Lesley Booth to object to the considerable attention paid to the syntactic aspects of algebra in the classroom. There is nevertheless a subtle but important difference in how solutions are envisaged one the one hand, by Booth and the structural perspective, and by Boero and Bazzini, on the other.

Booth claimed that difficulties in learning syntax were the result of a poor understanding of the mathematical structures underpinning algebraic representations: "our ability to manipulate algebraic symbols successfully requires that we first understand the structural properties of mathematical operations and relations", she argued, and added that "[t]hese structural properties constitute the *semantic* aspects of algebra." (Booth, 1989, pp. 57-58). I do not think that Boero and Bazzini disagree with the important role played by structural properties in the constitution of the semantics of algebra. Nevertheless, they seem to disagree with the idea that, ontogenetically speaking, the understanding of structural properties comes *first* as well as with the claim that these structural properties alone constitute the semantics of algebra. Indeed, in their approach (see also Boero, Bazzini, and Garutti, 2001), the study of the production of meaning is located in an activity that transcends mathematical structures. Their analysis traces elements of students' linguistic activity and body language in an attempt to detect metaphors, gestures and bodily actions that can prove crucial in students' understanding and use of algebraic symbolism. In their analysis of the way in which students make sense of a quadratic inequality, they emphasize the students' allusion to artefacts and to their understanding of symbols in terms of cultural linguistic embodied categories such as "going up" and "going down". As I see it, the covariational functional context that they propose is conceived of as a means for students to produce meaning and understand signs.

The idea that the production of meaning goes beyond mathematical structures and the claim that meaning is produced in the crossroad of diverse semiotic (mathematical and non-mathematical) systems is certainly one of the cornerstones of non-structural approaches to mathematical thinking. And yet, many difficult problems remain. Algebraic symbolism is undoubtedly a powerful tool. Even if some calculators and computer software are able to perform symbolic manipulations, algebraic symbolism is not likely to be abandoned in schools –at least not in the short term. Kieran's reflections on what happens to meaning when students translate a word-problem into symbolism, Sackur's interest in understanding the outcome of meaning in conversion between, and treatments within, registers and Dreyfus and Hoch's concerns about students recognizing the underpinning structures in equations thus appear to be more than justified. Certainly, one of the crucial problems in the development of algebraic thinking is to move from an understanding of signs having been endowed with a contextual and embodied meaning, to an understanding of signs that can be subjected to formal transformations. The meaning that results from noticing that a graph "goes up" or "goes down" supposes an *origo*, that is, an observer's viewpoint. This *origo* (Radford 2002a) is the reference point of students' spatial-temporal mathematical experience, the spatial-temporal point from where an embodied meaning is bestowed on signs. Algebraic transformations, such as those mentioned by Dreyfus and Hoch, require the *evanescence* of the *origo*. Does this amount to saying that symbolic manipulations of signs are performed in the absence of meaning? To comment on this question, let us now turn to the idea of syntax.

2. Syntax

One of the tenets of structuralism is the clear-cut distinction between syntax and semantics. From a structural perspective, the real nature of things is seen not in the world of appearances, but in their true meanings –something governed by the intangible but objective laws that Freud placed in the unconscious, and that structural anthropology, psychology and linguistics, after Saussure and Lévi-Strauss, thematized as "deep structures". Syntax was conceived of as lying on "surface structures", it was merely dead matter, the shadows of deep, structurally governed, mental activity. It is understandable that, in this context, in 1989 Kaput argued that instead of teaching syntax (which would produce "student alienation") we should be teaching semantics (Kaput, 1989, p. 168). Nevertheless, as I have already stated, we have become more sensitive to the claim that every experience, even the more abstract one found in mathematics, is always accompanied of some particular sensory experience, or –as Kant put it in the *Critique of Pure Reason*– that every cognition always involves a concept and a sensation.

How, then, within this context, can we address e.g. Dreyfus and Hoch's legitimate concerns? Recognizing equivalent equations is one of the fundamental steps in learning algebra. The formal transformation of symbols in fact requires an awareness of a new mode of signification –a mode of signification that is proper to symbolic thinking (Radford, 2002b) and whose emergence only became possible in the Renaissance. As Bochner (1966) noted, despite the originality and reputation of Greek mathematics, symbolization did not advance beyond a first stage of iconic idealization where calculations on signs of signs were not accomplished. It is not surprising then that the problem of explaining the formal manipulation of symbols puzzled logicians and mathematicians such as Frege, Russell, and Husserl. While for Russell (1976, p. 218) formal manipulations of signs are empty descriptions of reality, for Frege and Husserl formal manipulations do not amount to manipulations devoid of meaning. In fact, for Frege, equivalent algebraic expressions correspond to a single mathematical object seen from different perspectives: they have the same referent but they have a different *Sinn* (meaning). Adopting an intentional, phenomenological stance, Husserl contended that manipulations of signs require a shift in attention: the focus should become the signs themselves, but not as signs *per se*. Husserl insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the "rules of a game" (Husserl 1961, p. 79).

These remarks do not solve the crucial problem raised by Dreyfus and Hoch, also present in the other papers of this forum. It would certainly be of little help to tell students that a seemingly rational equation is, after transformations, equivalent to a linear equation because they are both designations of the same mathematical object.

Perhaps Husserl's insight intimates that the change in the way we attend the object of attention (e.g. the modeled situation or the equation itself) leading to an *awareness* of the "rules of the game" rests on a process of *perceptual semioisis*, or a dialectical movement between perceived sign-forms, interpretation, and action. Hence, it may be worthwhile to consider the ontogenesis of new modes of signification required by algebraic symbolism as a back and forth movement between interpreting the symbolic expression in its diagrammatic form (Peirce) and the (mathematically structured) hypothetical generation of new diagram-equations.

It might be very well the case that the greatest difficulty in dealing with equations and inequalities resides in: (1) the understanding of the *apophantic* nature of equations and inequalities and (2) the *apodeictic* nature of their transformations.

Number (1) refers to the fact that, in contrast to a symbolic expression like x+1, an equation or an inequality makes an *apophansis* or *predicative judgment* (in Husserl's sense; Husserl, 1973): it *asserts* e.g. that P(x) = 0. Number (2) refers to the necessary truth-preserving transformations of equations and inequalities –if, for a certain x, it is true that P(x) = 0, then Q(x) = 0, etc., something that Vieta expressed by saying that algebra is an *analytic* art. What I want to suggest is that the predicative judgments $P(x)=0$ or $P(x)\leq 0$, etc. that rest at the core of solving an equation or an inequality should not be confined to the written register containing an alphanumeric string of signs. We need an ampler concept of predication (and of mathematical text) less committed to the written tradition in which Vieta was writing not many years after the invention of printing. We also need a better concept of predication capable of integrating into itself the plurality of semiotic systems that students and teachers use, such as speech, gestures, graphs, bodily action, etc., as shown clearly in the Grade 8 lesson mentioned by Kieran. Predicative judgments would be made up of a complex string of gestures, written signs, segments of speech and artefact-mediated body actions. Their transformations would not be confined to the realm of logic and formal symbol manipulation, for the passage from one step to the next in a semiotic process is not something predetermined in advance by the logic of deduction alone: what seems to be a formal manipulation is in fact continually open to interpretation. There is, in the end, no opposition between syntax and meaning. Every sign has a meaning. Otherwise, it cannot be a sign. Conversely, every meaning is an abstract entity –"a general" (Otte, 2003)– which finds instantiation in signs only.

References

Bochner, S. (1966). *The role of mathematics in the rise of science*. Princeton, NJ: Princeton University Press.

Boero, P. , Mazini, L. & Garutti, R. (2001). Revealing and promoting the students' potential. *Proceedings of the PME 25 Conference*. Utrecht.

Booth, L. (1989). A Question of Structure. In S. Wagner and C. Kieran (Eds.), *Research Issues in the Learning and Teaching of Algebra* (pp. 57-59). Reston: NCTM & Lawrence Erlbaum.

Husserl, E. (1961). *Recherches Logiques* (Recherches I et II). Paris: PUF.

Husserl, E. (1973). *Experience and Judgment*. Evanston: Northwestern University Press.

Kaput, J. (1987). Linking Representations in the Symbol System of Algebra. In Wagner and Kieran, (Eds.), *Research Issues in the Learning and Teaching of Algebra* (pp. 167-194). Reston: NCTM & Lawrence Erlbaum.

Otte, M. (2003). *Meaning and Mathematics*. Pre-print. Institut für Didaktik der Mathematik. Universität Bielefeld.

Radford, L. (2002a). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, **22**(2), 14-23.

Radford, L. (2002b). On heroes and the collapse of narratives. A contribution to the study of symbolic thinking. *Proceedings of the PME 26 Conference*, Vol. 4, pp. 81-88. University of East Anglia, UK.

Russell, B. (1976). *An Inquiry into Meaning and Truth*. London: G. Allen and Unwin.

Wagner and C. Kieran (eds.) (1989). *Research Issues in the Learning and Teaching of Algebra*. Reston: NCTM & Lawrence Erlbaum.

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