Vita Mathematica. Historical Research and Integration with Teaching. Ronald Calinger, Editor. Washington DC: The Mathematical Association of America. xii + 359 pp. ISBN 0-88385-097-4

Essay Review by Luis Radford

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Over the last two decades, there has been an increasing interest to incorporate history into the teaching of mathematics. This interest has led to the creation of the International Group on the Relations between History and Pedagogy of Mathematics (HPM) and the Universités d'été sur l'histoire et l'épistémologie des mathématiques. The book under review is, in fact, mostly a selection of papers delivered in 1992 at the Quadrennial Meeting of the HPM held at the University of Toronto and at the Seventh International Congress on Mathematics Education (ICME) at the Université de Laval in Québec city. As its subtitle indicates, *Vita Mathematica* aims at integrating historical research with the teaching of mathematics.

The book is divided into three parts. Part one deals with questions about historiography and epistemology and contains 3 articles. Part Two is devoted to historical studies and contains 14 chronologically arranged articles. Part Three has 13 articles dealing with fundamentals and selected cases of integration of history with mathematics teaching.

Part One begins with an article on new trends in the history of mathematics written by David Rowe. In this article, Rowe discusses a shift in recent historiography. As it becomes evident through the article, this shift is seen by Rowe as promoted by new interests among scholars and underlain by changes in conceptions about the nature of mathematical knowledge itself. The changes, Rowe claims, go from a Eurocentric vision of a monolithic body of mathematical knowledge to a more rich perspective in which mathematics is considered an activity embedded in a large variety of cultures and periods (p. 4). As a result, changes in conceptions about mathematical knowledge entail changes in the accounts about its history and its conceptual development. Thus, contrary to the historical accounts underlain by Platonistic or unified structural views of mathematical knowledge, in the new historiographic trends that emphasize mathematics as an activity interwoven with the contexts from where mathematical ideas arose and were endowed with meanings, a history of mathematics is oriented to providing accounts of the particularities of these ideas and their meanings against the background of the contexts in which the mathematical activity was carried out.

Several articles in the book illustrate in different ways the historiographic shift in mathematics. For example, in her paper, Barbin, following epistemologist Gaston Bachelard, argues that the development of mathematics is related to the central role that problems play in mathematical activities — no problems, no questions; no questions, no knowledge. The approach that she offers is at odds with Platonistic conceptions of mathematics; for instance, in her perspective, mathematical objects are seen as non-existent prior to the mathematical activity; according to her theoretical view, the mathematical objects result, in fact, from the human actions and reflections required to solve problems.

In his contribution, D'Ambrosio clearly advocates for a "history of mathematics [that] needs to be examined in a broader context than is generally done today" (p. 245), and suggests to consider mathematics as a type of knowledge arising from the individuals' attempts to understand their physical and social environments in relation to their own economic and cultural contexts. In doing so, D'Ambrosio's theoretical position not only shatters the traditional distinction between science and technology, on the one hand, and mathematics, on the other, but also in claiming that the kind of theory of mathematical knowledge that he is proposing needs to take into account theoretical elements from domains such as cultural anthropology, he locates the history of mathematics in the intersection of research domains that go far beyond the realm of most of the contemporary historiograpic schools.

The aforementioned historiographic shift does not go without theoretical problems, as Rowe himself noticed. One of them is linked to the so-called problem of presentism, that is, the problem of providing biased accounts of past mathematical achievements in light of contemporary mathematics. To size the scope of this problem, we have to start by noticing that presentism cannot be encountered in historical accounts underpinned by Platonistic views of mathematics. In such accounts, it may be legitimate to conceive that Babylonian scribes in the 16th century B.C. were actually working on (or attempting at dealing with) the modern algebraic concept of system of equations. This problem cannot be found in André Weil's concept of history either —a concept that Rowe discusses to some length in his paper— since, in this case, the grasping of mathematical ideas of past cultures is not only legitimately accomplished through a reading of those ideas in terms of the modern ones but over all this reading is *the* method to ensure the historical understanding. Presentism, as a theoretical problem, is hence a very particular problem of the historiographic shift with which Rowe is dealing.

Many of the contributions of the book under review touch upon this delicate point in one way or another. Swetz, for example, emphasizes the difficulties that we may have in understanding Chinese mathematics –a difficulty that, as he noticed, led missionaries, teachers and translators in the 19th Century to see a lack of scientific accuracy in Chinese mathematics. In his contribution, Høyrup is very cautious when talking about Babylonian 'algebra'. Cooke, in his paper on Kovalevskaya states that "to reconstruct precisely the present state of any nineteenth-century mathematical topic is in a sense impossible. No mathematical problem is understood exactly as it was understood at the time of Kovalevskaya's death a century ago." (p. 177).

The shift from monolithic accounts of the history of mathematics to pluralistic accounts renders the transcendental teleology that underlies internal accounts inapplicable. But in doing so, the teleology of monolithic accounts that conceives of mathematics as unfolding linearly in time, needs to be replaced with something that will account for directions in mathematical research in a given period and culture. Although this problem is not tackled explicitly in any of the contributions of the book, Barbin's and D'Ambrosio's contributions nevertheless hint at a sketched solution: both of them consider mathematical activity as a key point to understand the growth of mathematics. By linking mathematical activity to the social settings, D'Ambrosio invites us to scrutinize mathematics and mathematical thinking in the web of social relations of the culture in which the activity is carried out.

Through specific case studies, some of the contributions of Part Two of the book can help us to see some links between mathematical thinking, activity and its sociocultural settings. Høyrup's paper is very informative in many respects. For instance, we can see a link between the mathematical conceptualizations underlying the Old Babylonian mathematics and the practical activities, such as surveying, with which the scribes were familiar. Cut-and-paste geometric procedures (that is, one of the main techniques underpinning Babylonian mathematics in Høyrup's account) can indeed be seen as the result of an intellectual and sophisticated reflection of practical techniques required in problems about fields and their areas. Knorr, in his investigation of the method of indivisibles on ancient geometry, suggests that Archimedes' technique for the determination of volumes and centers of gravity draws from heuristic and informal procedures characteristic of a mathematical tradition of practical geometry. In this tradition, heuristic, informal procedures may be convenient for those practical geometers involved in this mathematical practice. However, they do not fulfill the requirements of what is taken as valid in accordance with the Greek canons of formal proof. Even though consciously excluded from formal mathematical activity, the informal tradition nonetheless leaves its mark in the sophisticated Greek scientific mathematical activity. Katz's work also suggests that theoretical counting principles in combinatorics and induction in Medieval Hebrew and Islamic mathematics were related to situations of concrete practices (as the number of combinations that can be made of different tastes or other more intellectualized practices such as the words that can be made from a certain number of letters). Closer to our own epoch, Calinger's, Jahnke's and Aspray et al.'s contributions dealing with the Mathematics Seminar at the University of Berlin, the development of algebraic analysis in 19th century Germany and the rise of theoretical computer science and engineering, respectively, show how social institutions promote a certain kind of mathematical practice within which specific problems are tackled and methodologies developed.

Although in some of the papers in the first two parts of the book several (direct or indirect) ideas concerning the integration of historical research with teaching can be found (this is the case of e.g. Hitchcock's, Hughes', Lumpkin's, Fauvel's, Grabiner's, Hensel's and Kidwell's papers), it is in Part Three, as mentioned previously, that this point is explicitly addressed. However, contrary to what one may believe at first glance, such integration is far from trivial. One of the reasons is that history and pedagogy are two different research domains each having their own specific frameworks, methods of data collection and types of analyses. While historical research is based on e.g. the study of documents and past institutions, pedagogical research draws from theories of learning to envision types of classroom intervention in order to promote students' knowledge acquisition. Furthermore, their respective research questions are not necessarily the same.

My point in stressing the differences between historical research and mathematics education is not to mean that their integration is impossible. The point is that there are differences that need to be taken into account and to be dealt with if we want to maximize the opportunities for the integration of historical research with teaching. This can be clarified in referring to Heiede's paper.
In his contribution, Heiede offers an interesting view on history and teaching based on a diachronic conception of knowledge in which the issue is not merely realizing that knowledge has a past but that "the history of a subject becomes an inseparable part of the subject itself" (p. 231). Heiede gives clear examples that show how undue training in history can lead teachers to false interpretations of episodes in the history of mathematics. However, I want to argue that, to obtain a fruitful integration of historical research with teaching, we also need to consider, in a specific way, how to link results from contemporary theories of the learning of mathematics to available data concerning historical conceptual developments. Albeit we may have an extensive knowledge of, say, the rise of symbolic algebra in the 16^{th} century, if we do not know about the 14 year-old students' cognitive constraints and capabilities and the semiotic demands imposed by the use of specialized symbols during the students' construction of their first symbolic algebraic expressions, then the historical knowledge will not be exploited to its maximum pedagogical potential.

To make my point clearer, I want to distinguish here between two different kinds of pedagogical actions based on the history of mathematics. The first one seeks to improve students' *perception* of mathematics. The second one aims at enhancing students' *conceptual mathematical thinking*. Overlooking this distinction may lead one to believe that by inserting historical data into a teaching sequence the conceptual students' mathematical thinking will be automatically enhanced. Although accurate historical information will never harm any student, this information does not necessarily entail a gain in their conceptual understanding of mathematics. Thus, to continue with our example of symbolic algebra, making students aware of rhetoric algebra will not necessarily deepen the students' capabilities of handling and understanding polynomials.
Surely, through the insertion of historical anecdotes, biographical information and the history of some problems, many students can and have been made aware that mathematics, like literature and painting, did not arise from nothingness and, as other human intellectual enterprises, mathematics has a history, too. This pedagogical use of the history of mathematics (that for obvious reasons may be termed *humanistic*) is mainly related to improving students' perception of mathematics. Many of the articles in the third part of *Vita Mathematica* belong to the pedagogical use (see e.g. Rickey's and Dee Michalowicz's contributions). In contrast, the enhancement of students' conceptual mathematical thinking is related to a use of history that I want to call *epistemological*.

Although the humanistic and the epistemological use of history may be interrelated and can complement each other and that both seek to improve the students' learning of mathematics (see e.g. the contributions of Bero, Flashman, Jozeau & Grégoire, Kleiner, Man-Keung, Laubenbacher & Pengelley, Kronfellner, Tattersall and Dadić), a suitable integration of historical research and teaching for epistemological reasons requires a subtle theoretical coordination of historical and psychological issues. The difficulties of such coordination can be traced back to Cajori's *A history of elementary mathematics with hints on methods of teaching.* Despite the title of the book, the hints on teaching methods remained very tenuous. Probably one of the reasons for Cajori's limited success is the fact that the psychological component underlying the learning of mathematics remained undeveloped, if not untouched.

As a result of the previous remarks we see that to translate into effective practice Heiede's general statement, we need to further reflect on how past developments relate to contemporary students' mathematical thinking. I do not believe that there is a straightforward answer to this question. More often than not, this question has remained implicit, and when the question has been taken into account in the general literature, the link between the past mathematical developments and the contemporary students' mathematical thinking has often been conceived of in terms of recapitulationism.

As it is well known, recapitulationism, an idea introduced at the end of the 19th century, following Darwin's writings on the evolution of species, posits that the development of the individual (*ontogenesis*) *recapitulates* the development of mankind (*phylogenesis*). In this context, the students' mathematical thinking could be seen as a recapitulation of the conceptual development of mathematics. The ontology in which recapitulationism is couched requires neither the transcendental teleology of Platonism nor the philosophical idea of 'final cause' advocated by the early 19th century Naturphilosophen. Rather, its teleology is driven by an idea of efficient cause of the individual development seeing such a development in the context of the theory of evolution. Here the individuals and their ideas are seen as the result of an adaptation to their environment. And, in the interplay between the individuals and their environment, some of the biological and psychological functions and ideas may develop while others may be lost according to the natural selection. Yet, what could be a 'natural selection' in a cultural setting of a few centuries ago may not be natural in contemporary settings. As contexts change, new problems are posed and (as Barbin would say) the concepts required to solve them change too. Adaptations then become modified and 'natural selection' loses its recapitulative dimension. This is very clear in highly social organized settings like schools where calculators, computers and other modern tools are used as instruments to build knowledge. Our students are equipped with technological tools allowing them to come into contact with, say, decimal numbers in a completely different manner than the 16th century businessmen that Simon Stevin had in mind.

In their different theoretical variants, the recapitulation of ideas has recently been subject to a deep revision—partly because of its limited epistemological scope, as evidenced by the epistemologist Jean Piaget (Piaget and Garcia 1989), but also because of the emergence of new conceptions about the role of culture in the way we come to know and think (a detailed discussion of this can be found in Furinghetti and Radford, in press). The internal/external distinction often made in accounts about the history of mathematics has somehow been a comfortable one in that it has demarcated borders where historiographic schools can function. Nevertheless, I, as many other mathematics educators, echo the claim of some historians who find such a distinction extremely artificial to account for the conceptual developments in mathematics.

The elaboration of, and the recourse to, rich theoretical frameworks capable of providing useful epistemological descriptions of the growth of mathematical knowledge is one of those actions required to bridge the gap between historical and pedagogical research. I fully agree with Rowe's claim that the investigation of the mathematical knowledge and its history needs a pluralistic viewpoint. But if such an investigation is to be carried out against the background of the study of the culture and the social and economic means that encompass the activities out of which the mathematical problems, concepts and methods arose, then Rowe's position does not appear ambitious enough. As I see it, this problem goes beyond the scope of the dialogue he wishes would occur between philosophers, mathematical historians, and historians working from the theoretical perspective of the history of science. Although this dialogue would, of course, be very useful, it would not be sufficient. Indeed, if knowledge is considered as an intellectual reflection of the external world in the forms of the individual's activity, as Ilyenkov suggested (Ilyenkov 1977, p. 252), then mathematical knowledge needs to be related to the forms of the individual's activities as they happen in the complexity of the human world. Within this context, the elaboration of the alluded theoretical frameworks, based on fresh conceptions of knowledge, seems to require conceptual categories and methods of analyses that go far beyond the realm of history, philosophy and mathematics alone. The issue is no longer to have philosophy as the provider of the mechanisms required for the normative evaluation of knowledge, as in traditional epistemology, but to see how mathematical concepts, methods of inquiry and their validation are interwoven with social practices. This view supersedes both the orthodox and non-orthodox distinctions of internal/external history, for the production of knowledge and its history are seen here as related to their diverse contexts and grounds. As a result, the internal and external distinction collapses and is replaced by a new view, where without reducing the study of mathematical knowledge to its sociology, knowledge is conceived of as intrinsically social and cultural.

It is at this point that D'Ambrosio's suggestion that anthropology has a role to play in our conceptual and epistemological analyses concerning the growth of mathematical knowledge seems very appealing to me. Certainly, to a great extent, anthropological research has, for the past 30 years or so, offered comparative studies of cultural processes that insist on the relation of the production of knowledge to its cultural settings (an example that I want to mention here is Lizcano 1993). Although I do not believe that we can transpose the anthropological reflections to the history of mathematics *as is*, anthropologists have a wide experience and have developed a range of concepts to deal with problems that are becoming urgent to address; such as problems of interpretation and understanding (see Unguru 1991). It is in this context that I believe the anthropological reflexion can be useful for a non-monolithic and non-teleologically transcendental history of mathematics. However, this task is not easy, as the different contributions of *Vita Mathematica* have hinted.

As it became transparent over the course of this review, I find a reflective attempt to overcome traditional historiographies of mathematics in many of the contributions of the book. The fine-grained historical analyses and the rich range of pedagogical application of history to teaching makes this book a valuable one for a large audience. Despite some typographic problems due to probably the lack of a pre-print proofreading process, the book is highly recommended to historians of mathematics and sciences, mathematics teachers, mathematics educators and research mathematicians.

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