

## **SENSIBLE THINGS, ESSENCES, MATHEMATICAL OBJECTS, AND OTHER AMBIGUITIES<sup>1</sup>**

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The least that we can say is that mathematical objects are mysterious objects. Some argue that they are the product of the mind. Others contend that they are pre-existing transcendental entities. And many do not understand how one can get to know something about them if we suppose that they are beyond human experience. Historically speaking, conceptualizations of mathematical objects rest on two major long-standing and opposing categories of Western thought, namely the realm of sensual things, on the one hand, and the realm of essences, concepts and the like, on the other hand. In this paper, I discuss, against the background of Greek philosophy, the way in which Kant attempted to resolve the distinction between essences and the sensible. In the last part of the article, I sketch a view of mathematical objects within a cultural theory of signification where mathematical objects are seen as fixed patterns of activity embedded in the always changing realm of reflective, semiotic- and artefact-mediated social practice.

### **1. Introduction**

During my flight to Italy, sitting next to me, was a young lady. We talked about the colorful Canadian autumn. Then we talked about her job. It turned out that she was a lawyer. “And what about you?” she asked. “What do you do?” After I explained what a mathematics educator is, she said: “What about the mathematicians? I have always been curious about what mathematicians do. Probably you can tell me.” Her question reminded me of Enrico Giusti’s recent book about mathematical objects. Giusti (2000) says that: “one of the most embarrassing questions for a mathematician is to ask him what he deals with.”

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<sup>1</sup> This text comes from a lecture given at the University of Modena and at the University of Palermo in October 2002. In preparing the written text I have decided to keep the colloquial style of the lecture. I am grateful to Filippo Spagnolo and Nicolina Malara as well as to their colleagues and students for their insights and remarks.

The problem is not merely that mathematical objects are abstract, for justice and right are abstract concepts, too. Nevertheless, comparatively speaking, we can say that mathematics is the only domain in which objects can be clearly defined. For instance, we can define a circle in the usual way and no reasonable human being will disagree. I have not found a single person in disagreement with our definition of a circle, even among my more controversial students. In contrast, “the jurists”, as Kant noted in his *Critique of Pure Reason*, “are still without a definition of their concept of right.” (A 731/B 759, p. 588 n. a)<sup>2</sup>. The difference, as Kant saw it, is that to define the concept of right I would need to use other words, and then I would need more words to define the previous ones, and so on. Mathematics, Kant said, is the only science that has exact definitions (A 729/B 757, p. 587), and by this he meant that we can provide definitions that will not require an infinite regress. Mathematical definitions, he said, “present the complete, original concept of a thing within the limits of its concept.” (A 727/B 755, p. 586). In modern terms, we can say that, according to Kant, mathematical objects have the property of being semiotically seizable.

Of course, I did not mention anything of what Kant said about the jurists to the young lady sitting beside me. I was afraid that I would kill the conversation. I was not sure that she would be interested in hearing about Kant’s basic epistemological presuppositions, although perhaps I was completely wrong, given that, taken in its entirety, Kant’s work aims at providing a basis for the distinction between “being” and “ought” in order to resolve the central problem of Enlightenment, i.e. the problem of freedom, a problem that, since then, is at the core of jurists’ activities. Perhaps it was a bare and unfounded prejudice and I missed an excellent opportunity for a fruitful exchange concerning Kant’s view on conceptual objects. All in all, I don’t want to miss my chance a second time, so this afternoon I would like to elaborate on some conceptions that have been held concerning mathematical objects. I am sure that your input and feedback on my review of these conceptions will help me to improve my own view and will pave the way for a sustained exchange.

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<sup>2</sup> Following the tradition, references to Kant’s *Critique of Pure Reason* are given according to the first 1781 edition (noted “A” followed by the corresponding page number) and the 1787 edition (noted “B” followed by the corresponding page number). I have also added the page number of Kemp Smith’s 1929 English translation).

I take as my starting point the following remark. One can very well survive doing mathematics without adopting an explicit ontology, that is, a theory dealing with the nature of mathematical objects. This is why it is almost impossible to infer from a technical paper in mathematics its author's ontological stand. The situation has become very different when we talk about *mathematical knowledge*. Probably this has to do with the evolution of mathematics education as an academic discipline. It seems to me that the reason is related to the emergence of new paradigms and the refinement of previous ones in our field. Theoretical questions about the content of knowledge and the way such a content is transmitted, acquired or constructed, has led us to a point in which we can no longer avoid taking ontology seriously.

However, my goal here is not to discuss the main ontologies that have been very influential in mathematics. Rather, I want to deal with one specific ontological problem, namely, the problem of the distinction between sensible things and essences. I consider this problem as one of the major ontological problems, not only because it shapes what we take mathematical objects to be but, more importantly for us, because our teaching attitudes depend upon it. Let me give you an example that happened to me a couple of weeks ago. In the course of teaching mathematics in primary school that I teach in our pre-service teachers program, we were watching a videotape in which young students were learning to subtract small numbers. One of my students was struck by how much the children were using their fingers. I took advantage to talk about body as a means of knowledge objectification (Radford 2002, 2003a). The student commented that when he was in school, his math teacher did not allow him to use his fingers: calculations must be carried out in the head. Teaching attitudes as this are shaped by stances that equate thinking with a process strictly intangible (supposedly dealing with pure essences). Body is equated here with the external counterpart of the "true" or genuine mental process.

To understand the ambiguities surrounding the distinction between sensual things, essences, and mathematical objects I shall go back to Plato and Aristotle. But I will not discuss them at length. I will spend most of my time discussing Kant. I want to elaborate the way in which Kant attempted to resolve the Greek distinction between essences and

the sensible, and then to see how this tension can find a solution in a cultural theory of signification embracing activity and knowing.

## 2. Platonism in Mathematics

As Brown notes in his recent book on the philosophy of mathematics, “Mathematics has always been Platonism’s strong suit” (Brown, 1999, p.24). It does not come as a surprise then that, in 1934, Bernays remarked that “it is not an exaggeration to say that platonism reigns today in mathematics”. And, if we believe Patras, who, not even a year ago, said that “There is almost no professional mathematician who does not recognize himself a Platonist”, we should conclude that things have not changed very much<sup>3</sup>.

Certainly, a great deal of the spectacular success of Platonic ontology is that it easily explains why, for instance, the proposition  $2+2 = 4$  is true in any place and in any time. The same holds for all the propositions contained in Euclid’s *Elements* and the whole bunch of Greek books. Thus, the Pythagorean Theorem was, is, and will always be true. For a Platonist it was true even before Pythagoras was born. Its trueness is beyond the vagaries of human experience and the caprices of cultures. For a Platonist, mathematical objects are unchanging objects, that is, *eidos* or *Forms*, with no variation at all, in any way, in any place and in any time.

The Platonic ontology of mathematicians has nevertheless, some intrinsic problems. As Resnik (1981, p. 529) noted:

since platonic mathematical objects do not exist in space or time the very possibility of our acquiring knowledge and beliefs about them comes into question ... Thus the Platonist seems to be in the paradoxical position of claiming that a given mathematical theory is about certain things and yet be unable to make any definitive statement of what these things are.

What I am calling here the Platonic ontology of mathematicians does not coincide exactly with Plato’s own ontology. Although in both ontologies, mathematical entities are considered as having no ties with the sensual realm, contemporary mathematicians find

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<sup>3</sup> See Bernays 1964, and Patras, 2001, p. 35.

difficult to endorse Plato's account of reminiscence –an account that asserts a theory of reincarnation. This is not the only difference. In section 6 I will deal with Plato's ontology. Now I turn to Aristotle's.

### 3. Aristotelianism

“By mathematical objects we mean”, Aristotle said, “things reached by abstraction” (*De Caelo*, quoted by Heath, 1998, p. 11).

Like Plato, he considered mathematical objects as ideal or non-sensible objects. In *De Anima*, Aristotle said that “it is not the stone which is present in the soul but its [immaterial] form.” (Aristotle, *De anima*, Book III, Part VIII). However, in contrast to Plato, he ascribed to mathematical objects an origin in the sensible world. In *Physics B2*, Aristotle said:

The mathematician is able to study surfaces, volumes, lengths, and points in isolation from their physical instantiations because [...] he is able to separate the two in thought [...] Having been separated by thought, mathematical objects are free from the changes which physical objects undergo (Aristotle, *Physics B2* 193b33-34).

Thus, one of the more important differences between Plato's and Aristotle's ontologies is that, in the former, the senses do not play an epistemological role. The senses are even discarded as a means to achieve true knowledge. In the latter, in contrast, they play an important role. In *De Anima*, Aristotle goes on to say that “no one can learn or understand anything in the absence of sense” (*De anima*, Book III, Part VIII). And this statement remains true even in the case of conceptual objects, for, to think, we need images, and images –Aristotle said– “are like sensuous contents except in that they contain no matter.” (Aristotle, *De anima*, Book III, Part VIII)

In a very important sense, since Plato and Aristotle, conceptions of mathematical objects amount to a tremendous confrontation between the tangible and the intangible, the material and the ideal. The dichotomy between these two poles has been the source of different attitudes in Philosophy. Sometimes efforts have been made to conciliate these poles, stressing key principles of one of them and borrowing elements of the other. Thus, in the 16<sup>th</sup> century, Josephus Blancanus and Pietro Catena elaborated two different

ontologies that combined in different ways Platonism and Aristotelianism. In the next section I will briefly discuss Blancanus' and Catena's ontological stances.

#### 4. 16<sup>th</sup> Century Italian discussion on mathematical objects

It was not unusual, among 16<sup>th</sup> century scholars, particularly among the teachers of Natural Philosophy to claim that mathematical objects do not exist. Since nature gives us only imperfect triangles and circles, these scholars vigorously maintained that mathematical objects can be found nowhere. In answering to his contemporary fellows, the Jesuit professor Josephus Blancanus –one of Clavius' disciples–, following Plato, argued that mathematical objects are *pure intelligible*. He answered that mathematical objects were not obtained from sensuous abstraction but appear rather as pre-existing universals. In other words, what he was saying is that it is vain to search mathematical lines, points and planes in the bush, in the street or in the market for they are not objects of the seen world, that is of the world of senses. Blancanus addressed the question of the existence of mathematical objects in his *De Mathematicarum Natura Dissertatio*. To defend the existence of mathematical objects against his rivals' calumny, Blancanus, in chapter 3 of his book, printed in Bologna in 1615, said that

even though these [perfect mathematical figures] do not exist in the nature of things, since in the mind of the Author of Nature, as well as in the human mind, their ideas do exist as the exact archetypes of all things, indeed, as exact mathematical entities, the mathematician investigates their ideas, which are primarily intended per se, and which are [the] true entities. (Blancanus, 1615; quoted by Mancosu, 1996, p. 180)

In the previous passage we see Blancanus adopting a 'mentalist' position in that he locates the mathematical objects *in* the mind. However, following one of the Renaissance interpretations of Aristotle, he took an abstractionist stance<sup>4</sup>. This led him right into the heart of the fabulous problem of the non-tangible (essences) and the

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<sup>4</sup> Hence, discussing the opposition between intelligible and sensible things, he said: "as a result of mathematical abstraction from sensible matter, this abstract matter acquires a certain perfection, which is called mathematical perfection." (ibid. p. 180).

sensible: “a triangle depicted in a chart is not a true triangle, but the true triangle is that which is in the divine mind.” (*op. cit.* p. 181)<sup>5</sup>.

Pietro Catena (1501-1576) –a lecturer of mathematics at the University of Padua from 1547 to 1576, who held the chair to be later occupied by Galileo– elaborated a conception of mathematical objects that was anti-abstractionist. Catena claimed that mathematical objects were ideal entities and they were innate<sup>6</sup>.

To contradict the Aristotelian thesis that knowledge rests on the apprehension of sensual data (a thesis that Aquinas summarized in the scholastic dictum “nihil in intellectu quod non prius in sensu”), Catena had recourse to the example of the soul. The mind, he argued, can conceive of the soul. Yet, we have never seen it with our eyes nor touched it with our hands. Men, he said, have a prior intellectual knowledge of conceptual things thanks to an autonomous activity of the mind, independently of the concrete world<sup>7</sup>. In this context, mathematical objects were seen as bearing an exclusive intellectual character and having no genetic ties with the sensible realm. Going against Renaissance theories of abstraction, he objected to the possibility to derive from inexact empirical lines, as drawn by the geometers, the existence of the concept of a perfect line with which deals the geometer. How, in fact, he asked, is it possible to discriminate between the multitude of accidents which are now of this kind, then of another kind, and to derive of these always changing accidents, the permanence of mathematical objects? How can we be sure that the accidents that we have subtracted and eliminated from material bodies are the same while the very nature of those accidents is to be always different? Catena’s argument lies in the contrast between the changing nature of the concrete world and the sober world of eternal and ideal mathematical objects.

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<sup>5</sup> Let me note in passing that in his short discussion about definitions Blancanus holds a position that predates Kant’s –the position to which I referred in the Introduction, namely, that mathematical definitions are “essential definitions”, that is to say, “definitions which explicate the whole nature [*quidditatem*] of the thing”. (*op. cit.* p. 181).

<sup>6</sup> Catena’s neoplatonism cannot however be equated to the 18<sup>th</sup> century idealism which claimed that our minds or spirits are the only, or the fundamental, entities in the world, material things being unreal in some way.

<sup>7</sup> “human intellect understands the soul, even though the eye never saw it, or hand touched it. Therefore human intellect understands many things, of which it never had any perception by a sense”. Catena, *Universa Loca*, ivi, p. 16; from the Latin quotation by De Pace, 1993, p. 196, n. 26).

## 5. Plato's Platonism

Catena and Biancanus were opposed on several counts. They did converge on one crucial point though: both located mathematical objects *in* the mind. This is very different from Plato's ontological position. Although Plato (like Catena and Biancanus) considered that the objects with which the mathematician deals are not sensible objects (as he said in a much quoted passage of the *Republic* 510d), for him these non-sensual objects were *not* in the mind. To understand this point, we need to place Plato's conception of mathematical objects within his theory of ideas. I will now dwell on this point briefly.

The Greek term εἶδος (idea) meant initially "what one sees". In Greek early thought (e.g. in Homer's writings) the term εἶδος was related to "appearance" and to the "shape" or the "form" of the seen object. With Plato, the idea/form came to be understood as something suprasensible or, as Peters remarks, as the constitutive nature of things (Peters, 1967, p.47). Ontologically speaking, for Plato mathematical objects are "eternal and unchangeable", as Aristotle tells us at part A5, 987b of *Metaphysics*, where he comments upon his master's philosophy. They belong to the realm of essences or forms. This is why Plato said that the mathematical procedures move "solely through forms to forms, and finishes with forms."<sup>8</sup> That essences or forms cannot be *in us* is plainly asserted in one of Plato's dialogues, where Parmenides makes Socrates remark that "you, or any one who maintains the existence of absolute essences, will admit that they cannot exist in us", Socrates responds: "No [...] for then they would be no longer absolute."<sup>9</sup> Where are they, then? In the same Dialogue, Socrates says "the ideas are, as it were, patterns fixed in nature"<sup>10</sup>.

To better understand the mode of being of mathematical objects in Plato's ontology, we need to consider their anthropological dimension, that is to say, the way they relate to humans. There is an enlightening passage in the *Republic* 510e where Plato says that absolute objects "cannot be seen otherwise than by thought". Of course, statements as this have to be understood in the metaphorical sense of knowledge as vision

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<sup>8</sup> *Republic* 511c. I will not discuss here the problem of the place of mathematical objects within the hierarchy of forms (see Mueller, 1986)

<sup>9</sup> See Plato's dialogue *Parmenides* (tr. Jowett, B, eBook by netLibrary, p. 38).

<sup>10</sup> See Plato's dialogue *Parmenides* (tr. Jowett, B, eBook by netLibrary, p. 37).



that Plato inherited from the Eleatics. But what is important for our discussion is to notice that, in this “way of seeing”, mathematical objects stand in front of us. Gadamer’s comments on the original meaning of the word “*noein*” (thinking) and other derivative words that became central in Western thought (such as “*nous*” usually translated as “mind”) can help us understand the relationship between thinking and conceptual objects. Gadamer’s comments can also help us understand how far the Greeks were from conceiving thinking as something occurring in the mind. Gadamer says:

We usually render the word "*noein*" in translation as "thinking"; however we should not forget that the primary meaning of the word is not to become absorbed in oneself, not reflection, but, on the contrary, pure openness for everything. (Gadamer, 1998, p. 103)

In the early Greek thought thinking was related in a strong manner to seeing. Our senses tell us that there is something there, in our surrounding proximity. Thinking is a kind of communion between ourselves and our environment. This is why, thinking is pure openness. Gadamer continues:

In regard to *nous*, it is not, first of all, a question of one asking oneself what is seen to be there in each case but of observing that there *is* something *there*. The etymology of the word probably leads us back to the sensation of the animal, which notices the presence of something by its scent and without any more exact perception. This is how we must understand the relationship between "thinking" and being in Parmenides and also why [...] *noein* is mentioned with particular emphasis alongside the other features of being. It is as though the text wanted to say that it is the being of being itself that comes into presence in such a way that this being is as immediately there in its existence as the day is. (Gadamer, 1998, p. 103)

Plato hence transformed the original meaning of *eidos* as an external aspect (Bailly, 1950, p. 584) into the new meaning of *eidos* as essence. But the *eidos* (ideas) remained conceived as something external. How then, historically speaking, did we become conceiving of ideas as located *in* the mind? The answer to this question tells us another aspect of the problem between the essences/ideas and the sensual.

## 6. On How Ideas ended up *in* the mind

To understand the post-Hellenic conception of thinking as something happening in our interior, we have to look back to an early Christian Church father and philosopher who lived in the second half of the 4<sup>th</sup> century and beginning of the 5<sup>th</sup> century AD – Augustine. It was Augustine, indeed, who, reinterpreting Plato’s philosophy, stated that truth comes from our interior, from the soul, and sketched what was to become the Western concept of the self. One of the famous phrases of Augustine conveying the radical change to come is the following: "Do not go outward; return within yourself. In the inward man dwells truth." (Augustine, *De vera Religione*, XXXIX, quoted in Taylor, 1989, p. 129)

Within the line of thought inaugurated by Augustine, Leibniz, for instance, will say in the 17<sup>th</sup> century that “our ideas, even those of sensible things, come from within our soul” (Leibniz, 1705/1949, p. 15). Taylor (1989, p.129) summarizes the distinction between Plato’s ontology and the ontology of idealism put forward by Augustine by alluding to the role of the vision in both. In Plato we cannot see the Ideas; we can only see them indirectly. Likewise, for Augustine, God cannot be seen and known directly. In both cases, the eye of the soul has to be turned in the right direction. But for Plato, the spiritual eye turns to that which goes beyond *doxa* (opinion) and appearance. For Augustine, in contrast, we have to follow the inner light and to do so we do not have to turn to the outer world: the route is *in* us. Thus, although both ontologies are based on the metaphor of vision, the direction of the gaze is radically different.

Augustine’s ontology conserved the Platonic distinction between reality and appearances. It maintained the Greek distinction between spirit and matter, between eternal and temporal and between the sensible and the intelligible. But this dichotomy was thematized as the opposition between the *inner* and the *outer*. Since then the philosophical traditions of Western thought that arose from Augustine’s ontology have returned time and again to the relationship between the ephemeral and the lasting, between what *is* (*esse* –to be) and what *is-not*. And, in an important sense, ontological traditions such as Blancanus’ and Catena’s are conceptual variations of the relationship

between *sensus* and *essentia*. Ontological traditions that followed in the footsteps of Augustine philosophy have made the first-person perspective fundamental to the search for truth, either as a sensual receptive process, like in Hume and the British empiricists, or as a process of abstract reasoning dismissing the world of senses and appearances as suggested by the Rationalist trend of Descartes, Leibniz and others.

The troublesome point with Platonism (in its original and the contemporary sense) is that, from the outset, subject and object (the knowing subject and the object of knowledge) belong to two opposite spheres. While the subject is supposed to live in a changing world of appearances, the ideas live in an unchanging world.

## 7. Kant: Between Senses and Reason

There is no system in the Western philosophical tradition in which the sensible and intelligible achieved such intense confrontation as in the system elaborated by Kant in the 18<sup>th</sup> century.

Following the empiricists, Kant asserted throughout the *Critique of Pure Reason* that things become objects of knowledge insofar as they affect our senses. The senses allow us to “intuit” the objects through what Kant called “intuitions” –a term that in Kantian vocabulary means an effected immediate relation of the objects on the subject (see A 19/B 33, p. 65). The Kantian concept of intuition corresponds to the original Greek concept of *noein* in that both convey the idea of immediacy. There is an important difference though: for Kant the immediate relation between our senses and the objects of experience do not belong to the realm of thought<sup>11</sup>. These intuitions are the brute material of knowledge. They still need to be regrouped by the mind thanks to the concepts of the pure intellect, which Kant called the a priori *forms* of knowledge. Without these forms, our perceptions and intuitions would remain dispersed.

These concepts of the pure intellect are not concepts of objects; they are logical skeletons without content; their function is to make possible a regrouping or *synthesis* of

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<sup>11</sup> Intuitions are not re-presentations but presentations of the objects (see von Glasersfeld, 1987). This is why Posy says that “[f]or Kant, intuitions are immediate presentations of their objects simply because they are nonconceptual.” (Posy, 2000, p. 171). They precede any mental representation.

intuitions. The synthesis is the responsibility of what Kant identified as the cognitive faculty of Understanding (see Figure 1).

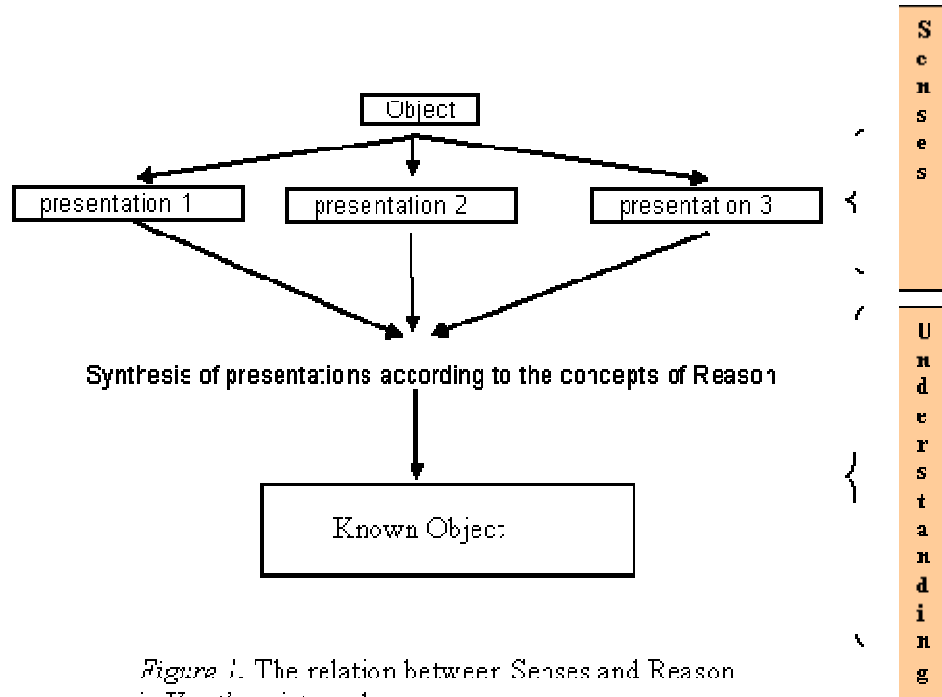


Figure 1. The relation between Senses and Reason in Kant's epistemology

We find hence in Kant a very interesting division of labor. On the one hand, we have the work of senses. On the other hand, we have the logical concepts whose work is to realize the synthesis of particulars. We should note that, for Kant, the work of senses is neither something merely cosmetic nor something that can be reduced to the pure passive receptivity of the empiricists. Even abstract objects have to be filtered through sensibility. In a letter to K. L. Reinhold, dated May 19, 1789, Kant says: “the mathematician cannot make the smallest assertion about any object whatsoever without exhibiting it ... in intuition”, that is without making it intuitable through a particular representation. (Zweig, 1967, p. 145). For Kant, our capacity of being affected by things acquires a constitutive role in cognition: it becomes an ingredient of knowledge. As Allison says, “Kant presents a conception of mind wherein the very content of knowledge is determined by sensible as well as intellectual conditions” (Allison, 1973, p. 76).

But the specific modes with which our senses provide us with the means to intuit things in the realm of phenomena, imposes also a limitation on what can be known. And this is an important difference between Kant and the Rationalists –a difference that scandalized many of Kant’s contemporaries when he said that we cannot get to know the things as they really are, that is, the things-in-themselves.

## **8. The Kantian unknowability of *things-in-themselves***

The rationalists contended that, in the best case, the phenomena that we observe may only suggest something true. And they did not consider it a problem. For them, phenomena were not the path to Truth anyway. The right path was Reason. And the way for Reason to reach truth was to proceed by inferences –inferences based on the pure concepts of thought. The Rationalists supposed that it is possible to have an inferential knowledge of *true things*, of things as they are in themselves. If we are tempted to find reasonable the rationalist position, it is, Kant would argue, because we are adopting a very deep epistemological principle. This principle, which turns out to be very problematic, is that there is *continuity* between the world of phenomena and the conceptual objects of the world of reason.

What Kant was doing was precisely challenging this belief. In other words, Kant questioned the belief that we can ascend from the sensible to the non-sensible by abstraction, and that the difference between the things-in-themselves and our intuitions or representations would just be a matter of degree. Kant insisted that the difference between *sensus* and *essentia* is not a difference of degree (as most theories of abstraction suppose). It is a transcendental difference<sup>12</sup>. Kant challenged the idea that the conceptual object is the extension of the concrete object, or that the conceptual object is an abstraction of the particular, that is, the particular version of the object but in a kind of idealized form, as it was conceived in the Aristotelian theory of abstraction. Between the thing-in-itself and the phenomenon, Kant argued, there is a gap. There is a gap that we cannot bridge. This is the problem of the unknowability of things-in-themselves. In other words, Kant claimed that what we come to know is not Reality itself: what we

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<sup>12</sup> For details, see Allison (1973, p. 75).

come to know is but only what is given to us in sensations. This is why he criticized Leibniz and Lock. He considered that Leibniz, in believing that he could obtain knowledge of the inner nature of things through a formal logical calculus only, without paying attention to the content of the propositions, had “erected an *intellectual system of the world*” and that he was misled in confusing appearances with the thing in itself. Lock, he said, in limiting himself to the realm of senses, ended up “sensualizing” all theoretical concepts.<sup>13</sup>

The architecture of mind that Kind offered was an attempt to *salvage* (to borrow Adorno’s term) the very ontological status of conceptual objects from the bad position in which the empiricist had put it. It is an effort to salvage the conceptual objects from the vicissitudes and vagaries of human experience and the ephemeral information that we collect from our senses. For Kant, conceptual objects are a priori objects. And this means independent of *all* experience. As such, they are beyond the realm of senses. This is the influence of the rationalist heritage in Kant’s work. But they exist only in their relation to experience –and this is the influence of the empiricist heritage in Kant’s system.

Of course, Kant’s system attains here a tremendous tension. The old Greek distinction between the intelligible and the sensible reaches at this point its most profound contrast.

## 9. Subject and Object

Kant conceived of an object of knowledge as a *relation* between (re)presentations or intuitions as given to us by our senses and synthesized by the individual through the a priori logical concepts of the faculty of understanding.

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<sup>13</sup> See A 270/B 326 – A 271/B327, p. 282-83. There is a letter written on March 22, 1801 by Heinrich von Kleist, one of Kant’s contemporaries, that gives us a good idea of the impression that Kant’s ontological position made in his time. Commenting on Kantian philosophy, von Kleist says: “If men had green glasses instead of eyes, they would have to believe that the objects they saw were green – and they would never be able to decide whether their eyes showed them things as they were. Or whether they did not add something to them that belonged not to them, but to the eye. The same is true of the mind. We cannot decide whether what we call truth is truly true, or whether it only appears so to us.” (Quoted in Adorno, 2001, p. 252).

From an epistemological viewpoint, by introducing the idea of the synthesis of the understanding Kant's system accomplished something new in comparison to the previous systems. Indeed, the individuals neither play a receptive role (as in Empiricism), nor do they merely instantiate the logical rules of an innate reason that, in a generous act, God supposedly granted us (as in Rationalism). In Kant's constructivism the individuals are conceived as actively synthesizing the intuitions and impressions. They now play a dynamic role: They have become the producers of their own knowledge. But the way in which Kant elaborated this point led us to a conception of the world in which it appears as a *subjectified* world, that is, as a world constructed from the first-person perspective coming out from the filter of the synthesizing activity of the individual.

At the opposite side, in contrast to the subjectivization of the world, there is a process of *reification*. It results from the fact that subjectivization leads to ascribe to the world a unified reference point, that of the synthesizing consciousness. Reification results from the "not-I" (that which is not me) conceived as a pale "Other" discarded by the all embracing first-person perspective and leading to a view in which the world appears as subtracted from its cultural context –a world that ends up as a world-for-myself. Reification goes so far that even complete knowledge of the subject is impossible: the unstoppable synthesizing machinery can only synthesize what is intuitable in the realm of phenomena and the subject becomes as unknowable as the things-in-themselves.

Adorno links the emergence of Kant's philosophy and the phenomena of subjectivization and reification to the bourgeois element that asserts that the world of experience is the product of the individual's labor. Taking the word "bourgeois" as a sociological category whose etymology means "those who live in the burgs" as opposed to "those who live working in the fields", Adorno says that the

growth of subjectivism and reification expresses [...] the essential antinomy of bourgeois society in general [...] human beings have increasingly made the world in their own image, and the world has become progressively theirs. At the same time, however, the world has increasingly become a world that dominates them. (Adorno 2001, p. 115)

Of course, Kant's concept of object also has implications in the way objectivity is defined. Since knowledge now appears as defined in terms of the results of a synthesizing consciousness, it results impossible to reach objectivity, as understood by the rational program. Truth becomes synonymous of coherence. Yet, Kant's work has to be placed within the grandiose Platonic paradigm that opposed the world of truth, of the unchanging things, the paradigm that opposed the world of that which has always been, to the world of senses and phenomena, that is, the world of delusion and deception. Following Descartes, Leibniz and other rationalists, Kant assumed the theory of truth according to which "truth is whatever remains once everything sensory, everything ephemeral and hence deceptive has been subtracted." (Adorno, 2001, p. 25). Kant adopted this conception of eternal truth and this is why the concept of time will remain central in his philosophy. We reach here a second tension in Kant's work. The first one concerned the tension between the sensual and the intelligible or between the senses and Reason. The second one concerns truth. It results from a Reason that seeks to reach a truth that escapes its own possibilities. I often think that if by an ingenious procedure it was possible to perfectly translate the *Critique of Pure Reason* into music, Kant's *Critique* would sound like the tormented second movement of Gustav Malher's 5<sup>th</sup> symphony.

## **10. Concluding Remarks: The demystification of mathematical objects**

I want to finish this talk by providing a rough sketch of what I think needs to be taken into account if mathematical objects would be considered from an anthropological epistemology. Since this talk has mostly been about tensions and scandals, I dare to think that I will not shock my audience in saying that, to theorize about mathematical objects from an anthropological epistemology, one has to proceed, first of all, to a demystification of mathematical objects. This is why such an epistemology should start by acknowledging that –notwithstanding Augustine– Plato was right. Plato was absolutely right in affirming, in the *Parmenides*, that ideas are not in the mind. And that



he was equally right in conceiving of ideas as “fixed patterns”. But instead of seeing these patterns “as fixed in nature” we should say “as fixed in *social practice*”.

In doing so, the wall that divided the seen and the unseen worlds that Plato mentioned in the *Phaedo* falls into pieces and mathematical objects lose their eternal aura and their atemporality. They become part of the always changing world of the individuals. This does not mean that we will confound the bronze triangle and the mathematical triangle –to take one of Aristotle favorite examples. In the anthropological epistemology that I am considering, mathematical objects retain an aspect of their platonic ideality. But what this ideality is about is an ideality resulting from a reflection that the individuals carry out of their world in the forms of their actions and activities. Kant was hence correct in stressing the synthetic, constructive nature of knowledge. And Piaget was even more correct in saying that the constructive activity is not limited to a synthesis of presentations but also includes the concepts of reason that Kant unduly took as given a priori and that Kant consequently conceived as already given. Nevertheless, both Kant and Piaget were wrong in seeing knowledge as a process that ascends from the concrete to the abstract, from the tangible world to the world of the intangible, leading, in the case of Kant, to an embedding theory in which the sensual object is subsumed into the concepts of reason (see Figure 1 in Section 6 above) and, in the case of Piaget, to an emphasis in activity with concrete objects in the sensori-motor stage which vanishes into thin air in later ‘development stages’. The case is that, in our pursuit of knowledge, we have recourse to several semiotic systems. For instance, in knowing, we see, we interact with people, we talk, we gesture, we write, we grip objects and use artefacts, too. Perhaps it would be fairer to see knowledge and its objects as a continuous dialectical process between the tangible and the ideal, a process intrinsically related to artefacts and signs.

In this line of thought, an anthropological approach cannot avoid taking into account, I think, the fact that the manners in which we use the diverse kinds of signs and artefacts during our acts of knowing are subsumed in cultural prototypes of sign and artefact usage<sup>14</sup>. What is relevant is that the use of signs and artefacts alter our modes of reception of the objects of the world, that is to say, signs and artefacts alter the way in

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<sup>14</sup> In Radford (2003b), I conceptualize this point through the construct of *Cultural Semiotic Systems of Signification*.

which the objects are given to us through our senses<sup>15</sup>. The inescapable result is this: mediation alters our cognitive capacity of being affected by things and makes this cognitive capacity (and hence knowledge) culturally dependent<sup>16</sup>.

To summarize: From the viewpoint of an anthropological epistemology, the way in which I see that the riddle of mathematical objects can be solved is to consider mathematical objects as fixed patterns of activity embedded in the always changing realm of reflective and mediated social practice.

### Acknowledgments

This article is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH). Previous versions of sections 8 and 9 appeared in my paper: “Rescuing Perception: Diagrams in Peirce’s theory of cognitive activity”.

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<sup>15</sup> See for instance Luria and Vygotsky (1998), Vygotsky and Luria (1994).

<sup>16</sup> It would be a mistake, nevertheless, to say that the anthropological view on mathematical objects that I am sketching here is in line with relativism. Indeed, among the long list of theories that have attempted to solve the problem of knowledge, there is no other theory that is more acultural than relativism.

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