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Elements of a Cultural Theory of Objectification[1]

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Summary: In this article, we present the general bases for a cultural theory of objectification. The theory in question deals with the teaching and learning of mathematics and takes its inspiration from some anthropological and historico-cultural schools of knowledge. This theory relies on a non-rationalist epistemology and ontology which give rise, on the one hand, to an anthropological conception of thinking, and on the other, to an essentially social conception of learning. According to the theory of objectification, thinking is not only characterized by its semiotically mediated nature but more importantly by way of its existence as a *reflexive praxis*. The learning of mathematics is thematized as the acquisition of forms of reflection on the world guided by historically formed cultural modes of knowing.

Key Words: Objectification, mathematical thinking, semiotics, sense, meaning, cultural signification, signs.

Résumé: Dans cet article, on présente les bases générales d'une théorie culturelle de l'objectivation. Il s'agit d'une théorie de l'enseignement et de l'apprentissage des mathématiques qui s'inspire de certaines écoles anthropologiques et historico-culturelles du savoir. Cette théorie s'appuie sur une épistémologie et une ontologie non rationalistes qui donnent lieu, d'une part, à une conception anthropologique de la pensée et, d'autre part, à une conception essentiellement sociale de l'apprentissage. Selon la théorie de l'objectivation, ce qui caractérise la pensée n'est pas seulement sa nature sémiotiquement médiatisée mais surtout son mode d'être en tant que *praxis réflexive*. L'apprentissage des mathématiques est thématisé comme étant l'acquisition communautaire d'une forme de reflexion du monde guidée par des modes épistémo-culturels historiquement formés.

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Mots clés : Objectivation, pensée mathématique, sémiotique, sens, signification, signification culturelle, signes.

Resumen: En este artículo se presentan los lineamientos generales de una teoría cultural de la objetivación –una teoría de la enseñanza y el aprendizaje de las matemáticas– que se inspira de escuelas antropológicas e histórico-culturales del conocimiento. Dicha teoría se apoya en una epistemología y una ontología no racionalistas que dan lugar, por un lado, a una concepción antropológica del pensamiento y, por el otro, a una concepción esencialmente social del aprendizaje. De acuerdo con la teoría, lo que caracteriza al pensamiento no es solamente su naturaleza semióticamente mediatizada sino sobre todo su modo de ser en tanto que *praxis reflexiva*. El aprendizaje de las matemáticas es tematizado como la adquisición comunitaria de una forma de reflexión del mundo guiada por modos epistémico-culturales históricamente formados.

Palabras claves: Objetivación, pensamiento matemático, semiótica, sentido, significado, significación cultural, signos.

Resumo: Este artigo apresenta as linhas gerais de uma teoria cultural da objetivação uma teoria da ensino e a aprendizagem das matemáticas que se inspira de escolas antropológicas e histórico-culturais do conhecimento. Tal teoria se apóia em uma epistemologia e uma ontologia não racionalistas que dão lugar, por um lado, a uma concepção antropológica do pensamento e, por outro, a uma concepção essencialmente social da aprendizagem. De acordo com a teoria, o que caracteriza o pensamento não é somente sua naturaza semióticamente mediatizada, mas sobre todo seu modo de ser como praxis reflexiva. A aprendizagem das matemáticas é tematizado como a aquisição comunitária de uma forma de reflexão do mundo guiada por modos epistémico-culturais historicamente formados.

Palavras chaves: Objetivação, pensamento matemático, semiótica, sentido, significado, significação cultural, signos.

Introduction

Even for empiricists, all learning presupposes some thinking activity. Thinking is considered as the substrate of learning: that through which a relationship is established between a being and the world. Strangely enough, despite its importance, and even when talking about mathematical or geometric thinking, etc., thinking—as a concept in itself—is not a part of current theories in education. Without a doubt, one of the reasons has to do with the general belief that thinking is unobservable. As the founder of Radical Constructivism affirms,

> Among the most intriguing human activities that can never be directly observed is thinking or reflecting. At times one can infer thoughts or reflections … but

> the actual process of thinking remains invisible and so do the concepts it uses and the raw material of which they are composed. (von Glasersfeld, 1995, p. 77)

The idea that thinking cannot be observed betrays the influence of rationalist philosophy and its concept of being. Thus,

> The Cartesian self inhabits a world in which material activity is impossible, for thought is construed as a relation between the self and mental entities, ideas, which are not possible objects of material activity. (Bakhurst, 1988, p. 35)

To the two elements that unite thinking—the subject and the object—educational theories add yet another element, the teacher, and this completes the famous didactical triangle. Gradually, nevertheless, the professor is relegated to a minor role: literally that of someone who facilitates learning. Inasmuch as didactical theories conceptualize the individual as a self-regulating and self-stabilizing subject, uprooted from his/her socio-cultural context, with the ability to reflect like a scientist exploring his/her surroundings in search of phenomena which confirm the viability of his/her knowledge, inasmuch as the individual is viewed—as Martin and his collaborators note—as someone who seems to somehow carry the conditions of his/her growth within his/her very interior, a being who only needs facilitating surroundings in order to reach, through personal experience, his/her full economic and intellectual[3] potential, the professor then appears to be, against the overwhelming evidence of daily experience, a mere catalyst of the encounter between student and object of knowledge.

The theory of objectification which will be sketched here is based on completely different premises. In opposition to rationalist and idealist currents, we argue for a non-mentalist conception of thinking as well as an idea of learning thematized as the communal acquisition of forms of reflecting upon the world, guided by historically formed epistemic-cultural modes of knowing.

1. A non-mentalist conception of thinking

In a first grade class in elementary school, the pupils had to solve a problem about a numeric sequence. The teacher introduced the problem through a story in which a squirrel, at the end of the summertime, brings two nuts to his new nest every day in preparation for the coming winter. In one part of the problem, the pupils had to determine how many nuts the squirrel had collected in his nest by the end of the tenth day, given the fact that there were already 8 nuts in the nest when the squirrel found it and that the squirrel never ate nuts from his winter provision. Christina, one of the pupils, began counting two by two: ten, twelve, fourteen, sixteen. When she noticed that she was not keeping track of the number of days that had passed, she started the count again. However, doing things simultaneously ended up being

[3] Martin (2004), Martin, Sugarman and Thompson (2003).

quite a difficult task. Addressing herself to Michael, her group mate, Christina said: "let's do it together!" While the rest of the class continued working on the problem in small groups, Christina and Michael went to the blackboard and, using a large wooden ruler, Christina began counting two by two while Michael counted the days out loud.
In Figure 1, when Michael says "nine," Christina points with a wooden ruler to the number 26 on a number line placed above the blackboard, being the number of nuts the squirrel had collected by day 9. In Figure 2, Michael, who continued counting the days, says "ten," while Christina moves the ruler to the right and points to the number 28, which is the correct answer to the question.

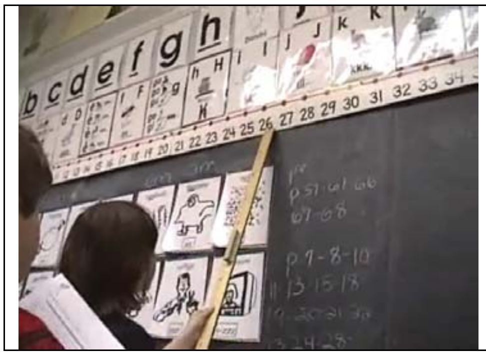


Figure 1 (left). Michael says 9, and Christina points to number 26. Figure 2 (right). Michael says 10, and Christina points to number 28.

Typically, thinking is understood as a kind of interior life, a series of mental processes on ideas carried out by the individual. According to this view, based on the data provided by the teacher, Christina and Michael would have been recuperating the necessary information from their memories in order to produce a mental representation of the problem. With the help of this representation, Christina and Michael's thinking would be moving through the stages of a space-problem, processing information that was perhaps codified in the form of propositional representations, through the rules of logical inference.
This conception of thinking, as "mental activity" (de Vega, 1986, p. 439), comes from Saint Augustine's interpretation of Greek philosophy at the end of the fourth century, an interpretation that brought about, in particular, a transformation in the original meaning of the Greek term *eidos*. While Homer, among others, used the term *eidos* in the sense of something external rather than mental—"that which one sees," for example, the figure, form or appearance[4]—for Saint Augustine, *eidos* refers to something situated *inside of the individual*.[5] Influenced by this transformation, seventeenth century rationalists such as Descartes and Leibniz believed that mathematics could be practiced even with one's eyes closed, given that the mind does not need the help of the senses or of experience to reach

[4] For example, in the English translation of Book VIII, lines 228-29, of the Iliad, Homer says: "Shame on you, Greeks! Contemptible creatures, admired only for your looks [eidos]." (translation by E.V. Rieu, revised and updated by P. Jones and D.C.H. Rieu, Penguin, 1950/2003). I am indebted to Eva Firla for her help with the etymology of the term eidos.
[5] A discussion about this transformation in Renaissance mathematics can be found in Radford (2004).

mathematical truths. As Leibniz put the matter, the principles that we need to understand objects or see their properties, the internal rules of reason, are "interior principles" that is, they are within our interior (Leibniz, 1966, pp. 34-37).

Anthropologists such as Geertz have demonstrated the limitations of the conceptualization of ideas as "things in the mind" or of thinking as an exclusively intracerebral process:

> The accepted view that mental functioning is essentially an intracerebral process, which can only be secondarily assisted or amplified by the various artificial devices which that process has enabled man to invent, appears to be quite wrong. On the contrary, a fully specified, adaptively sufficient definition of regnant neural processes in terms of intrinsic parameters being impossible, the human brain is thoroughly dependent upon cultural resources for its very operation; and those resources are, consequently, not adjuncts to, but constituents of, mental activity. (Geertz, 1973, p. 76).

Objectification theory takes off from a non-mentalist position on thinking and intellectual activity. Said theory suggests that thinking is a *praxis cogitans*, that is, a social practice (Wartofsky, 1979). To be more precise, thinking is considered to be *a mediated reflection on the world in accordance with the form or mode of the activity of individuals*.
In the rest of this section, I will discuss the different aspects of this definition.

1.1 Semiotic mediation

The mediating nature of thinking refers to the role, in the Vygotskian sense, played by artefacts (objects, instruments, sign systems, etc.) in carrying out social practice. Artefacts are not merely aids to thinking (as cognitive psychology would have it) nor simple amplifiers, but rather constitutive and consubstantial parts of thinking.[6] We think with and through cultural artefacts, so that there is an external region which, to paraphrase Voloshinov (1973), we will call the *zone of the artefact*. It is within this zone that cultural subjectivity and objectivity mutually overlap and where thinking finds its space to act and the mind extends itself beyond the skin (Wertsch, 1991).

According to the theory of objectification, Christina and Michael's thinking is therefore not merely something that passes through the students' cerebral plane. Thinking also occurs along the social plane, in the zone of the artefact. The wooden ruler, the number line, the mathematical signs on the piece of paper that Michael holds up as he reads behind Christina, all are artefacts which *mediate and materialize* thinking. These artefacts are an integral part of thinking.

1.2 The reflexive nature of thinking

[6] A critique of the conception of artefacts as amplifiers can be found in Cole (1980).

The reflexive nature of thinking means that the individual's thinking is neither the simple assimilation of an external reality (as the Empiricists and Behaviorists propose) nor an *ex nihilo* construction (as certain constructivist schools would have it). Thinking is a *re-flection*, that is, a dialectical movement between a historically and culturally constituted reality and an individual who refracts it (as well as modifies it) according to his/her own subjective interpretations and feelings.

In the previous example, the pupils' thinking processes were carried out during a complex coordination of perceptual activity and activities being semiotically mediated by the pupils' subjective interpretations and feelings (for example, reinterpreting the problem using a number line, counting two by two, etc.). At the same time, the problem upon which the children were reflecting is part of an historically constituted reality. Problems involving numeric sequences (arithmetic progressions) can be found in Babylonian mathematics and were later theorized by the Pythagoreans and members of other Greek numerological schools (Robbins, 1921). Not only does reality not impart itself in a direct or immediate way, as the Empiricists thought, but it also cannot be reconstituted through personal experience alone. So that,

> No one's personal experience, no matter how rich it might be, can result in thinking logically, abstractly and mathematically, and in individually establishing a system of ideas. To do this, one would need not just one lifetime, but thousands. (Leontiev, 1968, p.18)

One of the roles of culture (one on which we will spend a good deal of time in the next section) is to suggest to students ways of perceiving reality and its phenomena, literally, ways of setting one's sights (*viser*), as Merleau-Ponty (1945) would say, or ways of intuiting, as Husserl (1931) might have it.

To sum up in a more general fashion, the *re-flexivity* of thinking, from the phylogenetic point of view, consists in individuals giving rise to thinking and to the objects that thinking creates. However, at the same time, from the ontogenetic point of view, while thinking, any given individual is subsumed by his/her cultural reality and by the history of human thinking, both of which direct his/her own thinking. "The social being," says Eagleton, "originates thinking, but at the same time is dominated by it."[7]

1.3 The anthropological dimension of thinking

In the preceding section, it was said that thinking should be considered as a mediated *re-flection* of the world, in keeping with the form or mode of the activity of individuals. What is meant when we say that the *re-flection* that is thinking is achieved in accordance with the *form or mode of the activity of individuals*? This means that the way in which we come to think about and know objects of knowledge is framed by cultural meanings situated beyond the very content of the activities in whose interior the act of thinking itself occurs. These

[7] Eagleton (1997, p. 12).

cultural meanings act as mediating links between individual consciousness and objective cultural reality and they make themselves into prerequisites and conditions for individual mental activity (Ilyenkov, 1977, p. 95). Said cultural meanings *direct* activity and give it a certain *form*. It is for this reason that thinking is not something that we simply begin to do in a more or less unpredictable way and during which we suddenly come across a good idea. Even though it is true that practical sensual activity, mediated by artefacts, enters into the thinking process, in its very content, the way in which this occurs is subject to the cultural meanings in which the activity is being maintained.

Here is an example. The difference between the thinking of a Babylonian scribe and that of a Greek geometer cannot be reduced only to the kinds of problems with which they were respectively occupied, or to the artefacts they used to think mathematically, or the fact that the former was reflecting in a context tied to political and economic administration, whereas the latter was thinking within an aristocratic and philosophical context. The difference between the thinking of the Babylonian mathematician and that of the Greek one has to do with the fact that each one of these forms of thinking is underpinned by a particular *symbolic superstructure*—which, despite its great importance, has not been taken into account by contemporary theories dealing with the concept of activity[8]. This symbolic superstructure, which elsewhere we have called a *Semiotic System of Cultural Signification* (Radford 2003a), includes cultural conceptions surrounding mathematical objects (their nature, their way of existing, their relation to the concrete world, etc.) and social patterns of meaning production. The thinking of the Babylonian scribe is framed by a realist pragmatism where mathematical objects such as "rectangle," "square," and so forth –objects which the Greek geometer of Euclid's time conceptualized in terms of Platonic forms or Aristotelian abstractions (see Figure 3)– acquire their meaning.

In their interaction with activities (their objects, actions, division of labour, etc.) and with the technology of semiotic mediation (the zone of the artefact), the *Semiotic Systems of Cultural Signification* give rise, on the one hand, to forms or modes of activities, and, on the other hand, to specific modes of knowing or *epistemes* (Foucault, 1966). While the first interaction gives rise to the particular ways in which activities are carried out at a certain historical moment, the second interaction gives rise to specific modes of knowing which allow for the identification of "interesting" situations or problems and which demarcate the methods, reasoning, evidence, etc. that will be considered valid with regards to the reflection carried out on the problems and situations of a given culture[9]. The triangle in Figure 3 shows the complexity of activity as well as its diverse nature.

From our perspective, cultural diversity in the form of human activity explains the diversity of forms that mathematical activity takes on, something which is demonstrated to us by

[8] Leontiev did not theorize the symbolic superstructure that we are highlighting here even though it is fundamental for understanding thinking in its anthropological dimension. In the prolongation of Leontiev's Activity Theory, as carried out by Engeström (1987), said superstructure was not taken into account either.
[9] Henceforth, it is not only the action which constitutes the schema of the concept (Piaget)—or its seal or emblem (Kant)—but also the meaning of the action in a precise moment of the socio-cultural activity within which the action occurs (Radford, 2005).

history. Rather than seeing these historical forms as "primitive" or "imperfect" versions of a kind of thinking that is marching towards a perfected form as represented by current mathematical thought (ethnocentrism), the anthropological dimension of the theory of objectification considers these forms as belonging to human activity and thus resists privileging western rationalism as rationalism *par excellence*.

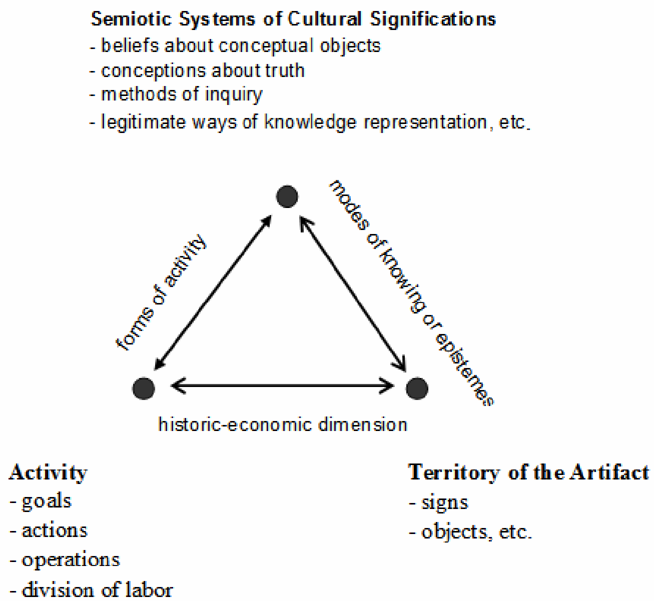

Figure 3. The arrows show the interaction between the Semiotic Systems of Cultural Signification with the activity and with the zone of the artefact. Said interaction generates the modes of both the activity and of the thought, modes which, in one dialectical movement, come, in their turn, to support the vertices of the triangle.

As Spengler (1948, p. 68 and p. 70) suggested many years ago, the mathematics of a culture are nothing but the style of the way in which man perceives his exterior world and, contrary to what is commonly held, the "essence" or mathematics is not culturally invariable. It is precisely this cultural diversity that explains the diversity of universes of numbers (ibid. p. 68).

The manner in which the Babylonian scribe, the Greek geometer and the Renaissance abacist end up thinking about and knowing objects of knowledge, the way in which they approach their problems and consider them to be solved, all are framed by the very mode of the activity and the corresponding cultural episteme (Radford, 1997, 2003a, 2003b).

2. The epistemological and ontological bases of the Theory of Objectification

Any didactic theory, at one moment or another (unless it voluntarily wants to confine itself to a kind of naïve position), must clarify its ontological and epistemological position. The *ontological* position consists in specifying the sense in which the theory approaches the question of the nature of conceptual objects (in our case, the nature of mathematical objects, their forms of existence, etc.). The *epistemological* position consists in specifying the way in which, according to the theory, these objects can (or cannot) end up being known.

Contemporary didactic theories that start from an application of mathematics, gradually adopt—even if it is not mentioned explicitly— a realist ontology and approach the epistemological problem in terms of abstractions. Naturally, the situation is not that simple, as Kant himself recognized.

As for Realism −which, in an important way, is the Platonist version of the instrumental rationalism (Weber, 1992) which emerged during the Renaissance− the existence of mathematical objects precedes and is independent from the activity of individuals. Like the Platonist, the Realist believes that mathematical objects exist independently of time and culture. The difference is that, while Platonic objects do not mix with the world of mortals, the objects of the Realist govern our world. According to realist ontology, this explains the miracle that is the applicability of mathematics to our phenomenal world (Colyvan, 2001). Naturally, in order to achieve this, Realism makes a leap of faith that consists in believing that the ascent from abstraction to objects is certainly possible. The faith which Plato placed in reasoned social discourse (logos) and which Descartes placed in cogitating with oneself are subjected to scientific experimentation by Realism.

The ontological and epistemological position of the theory of objectification moves away from Platonist and realist ontologies and from the Platonists' and Realists' conception of mathematical objects as eternal objects preceding the activity of individuals. By distancing itself from an idealist ontology, the theory also distances itself from the idea that objects are the product of a mind that works folded in onto itself or according to the laws of logic (the Rationalist Ontology). The theory of objectification suggests that mathematical objects are historically generated during the course of the mathematical activity of individuals. More precisely, mathematical objects *are fixed patterns of reflexive activity (in the explicit sense mentioned previously) incrusted in the ever-changing world of social practice mediated by artefacts*.

The circular object, for example, is a fixed pattern of activity whose origins cannot be found in the intellectual contemplation of the round objects which the first individuals would have encountered in their surroundings, but rather must be found in the sensual activity that led said individuals to take note of or notice the object:

> People could see the sun as round only because they rounded clay with their hands. With their hands they shaped stone, sharpened its borders, gave it facets. (Mikhailov, 1980, p. 199)

This sensual experience of labour has remained fixed in language which encapsulates original meanings, such that

> the meaning of the words “border”, “facet”, “line” does not come from abstracting the general external features of things in the process of contemplation. (Mikhailov, ibid.)

but rather comes from the activity of labour that has been taking place since the origins of humanity. Far from surrendering itself completely to our senses, our relationship with nature and the world is filtered through conceptual categories and cultural significations which make it so that

> man could contemplate nature only through the prism of all the social work-skills that had been accumulated by his predecessors. (Mikhailov, ibid.)

Let us end this section with a general observation about the evolution of mathematical objects which will later be necessary for our discussion about learning. Over time, the activity of labour gradually leaves its stamp on its conceptual products (Leontiev, 1993, p.100). Just like all mathematical objects, the concept of the circle, as a reflection of the world in the form of the activity of individuals, has been expressed in other ways throughout history. For example, it can be expressed through a word, a drawing, a formula or a mathematical table of numerical values. Each expression provides us with a different meaning tied to the previous ones and comes to constitute—as Husserl would say—the *noetic* stratum of the object. Just as it is the activity of individuals which comes to form the genetic root of the conceptual object, the object itself has a diverse expressive dimension which goes beyond simple conceptual “scientific” content. This expressive dimension also includes rational, aesthetic and functional aspects of its culture.

3. Learning as the cultural objectification of knowledge

3.1 Two sources for the elaboration of meaning

In the previous sections we have seen how human activity, from the phylogenetic point of view, can generate conceptual objects, which in term are transformed as a result of the activities themselves. From the ontogenetic point of view, the central problem is to explain how acquisition of the knowledge deposited in a culture can be achieved: this is a fundamental problem of mathematics education in particular and of learning in general.

Classical theories of mathematical education posit the problem in terms of a construction or re-construction of knowledge on the part of the student.[10] The idea of the *construction* of

[10] Naturally, there are various nuances, according to the conception that a given theory has of the individual who is learning (that is, the student). Starting from an extreme position, radical constructivism goes farther

knowledge originates with the epistemology elaborated by Kant in the eighteenth century. For Kant, the individual is not only an introspective thinker whose mental activity, if it is well carried out, will bring him mathematical truths as upheld by the rationalists (Descartes, Leibniz, etc.); nor is he only a passive individual who receives sensory information in order to formulate ideas, as proposed by the Empiricists (Hume, Locke, etc.). For Kant, the thinker is a being in action: the individual is craftsman of his/her own thinking (Radford, 2005). In reality, Kant expresses, in a coherent and explicit way, the epistemological change that had been gradually taking place since the appearance of manufacturing and the emergence of capitalism in the Renaissance and that Arendt (1958) summarizes in the following way: the modern era is marked by a displacement in the conception of the meaning of knowledge; the central problem of knowledge lies in a movement that goes from the *what* (the object of knowledge) to the *how* (the process), in such a way that, unlike medieval man, modern man can only understand that which he himself has made.

According to the theory of objectification, learning does not consist in constructing or reconstructing a piece of knowledge. *It is a matter of endowing the conceptual objects that the student finds in his/her culture with meaning*. The acquisition of knowledge is a process of active elaboration of meanings. It is what we will later call a process of *objectification*. For the moment, we need to discuss two important sources for the elaboration of meanings that underlie the acquisition of knowledge.

The knowledge deposited in artefacts

One of the sources of the acquisition of knowledge results from our contact with the material world, the world of cultural artefacts which surrounds us (objects, instruments, etc.) and in which is found the historically deposited knowledge from the cognitive activity of passed generations. Although it is true that some animals are able to use artefacts, nevertheless, for animals, artefacts do not end up acquiring a durable meaning. The wooden stick that a chimpanzee uses in order to reach a piece of fruit looses its meaning after the action has been executed (Köhler, 1951). It is for this reason that animals do not preserve artefacts. Furthermore—and this is a fundamental element of human cognition—unlike animals, the human being is profoundly *altered* by the artefact: by making contact with it the human being restructures his/her movements (Baudrillard, 1968) and new motor and intellectual skills are formed such as anticipation, memory and perception (Vygotsky and Luria, 1994).

The world of artefacts appears, then, to be an important source for the process of learning, but it is not the only one. Objects cannot make clear the historical intelligence that is imbedded in them. This requires that they be used in activities as well as in *contact with other people* who know how to "read" this intelligence and help us to acquire it. Symbolic-algebraic language would otherwise be reduced to a group of hieroglyphics. The intelligence that said language carries would not be noticed without the social activity that

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than all other forms of constructivism. Brousseau (2004) summarizes the difficulties that this theory faces, stating, "En didactique, le constructivisme radical, est une absurdité," and adopts a more moderate Piagetian constructivism which, inevitably, leads the Theory of Situations to a series of paradoxes (e.g. in the concept of Didactic Contract).

takes place in the school. It is this social dimension which constitutes, for the theory of objectification, the second essential source for learning.[11]

Social Interaction

Even though the importance of the social dimension has been underlined by a great number of recent studies on classroom interaction, there are subtle differences with regards to its cognitive contribution (Cobb and Yackel, 1996; Sierpinska, 1996; Steinbring, Bartolini Bussi and Sierpinska, 1998). Often, interaction is considered as a negotiation of meanings or as a simple environment that offers the stimuli of adaptation that are required for students' cognitive development. The problem is that society and the classroom are not merely material spaces where students find an environment to adapt themselves; it is not only a matter of "external" conditions to which the subject must *accommodate* his/her activity. The crucial point is that activities and the material means that mediate them and their objects are saturated with scientific, aesthetic, ethical values, etc. that end up affecting the actions that individuals carry out and the reflections that these actions necessitate. Just as was mentioned in the first part of this article, the actions that individuals carry out are submerged in cultural modes of activity. It is for this reason that the classroom cannot be viewed as an enclosed space, folded over against itself, where knowledge rules are negotiated; in fact, these rules have a whole cultural history behind them and therefore pre-exist the interaction that takes place in the classroom. The classroom should also not be seen as a kind of biological environment where the individual operates according to his/her invariable mechanisms of general adaptation.

According to the perspective that we are suggesting, interaction plays a different role. Rather that performing a merely adaptive function—a catalyzing or facilitating one—according to the theoretical perspective that we are sketching, interaction is *consubstantial* to learning.

Therefore, we see that there are elements that play a basic role in the acquisition of knowledge and that these are the material world and the social dimension. The allocation of meaning that rests on these dimensions has a profound psychological importance inasmuch as it is both an awareness of cultural concepts as well as the process of development of the specific capacities of the individual. It is for this reason that, according to our perspective, learning is not merely appropriating something or assimilating something; rather, it is the very process by which our human capacities are formed.

3.2 Learning activity

A central element of the concept of activity is its *objective* (Leontiev, 1993). For example, the objective might be that students elaborate an algebraic formula in the context of an numeric generalization, that they learn an algebraic problem solving method, that they learn to demonstrate geometric propositions, etc. Even though the objective may be clear for the

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[11] The historical-cultural school of Vygotsky has expressed this point in a very forceful way. See, for example, Leontiev, 1993, pp. 58-59; 1968, pp. 27-29; Vygotsky, 1981b.

teacher, generally speaking, this is not necessarily the case for the students. If the objective were to be clear to them, then there would be nothing left for them to learn.
Within the didactic project in the class, the teacher proposes a series of mathematical problems to the students so that a given objective can be achieved. Resolving these problems becomes an *end* that directs the actions of the students. The given problems—loaded, from the beginning, with cultural and conceptual content—form possible trajectories for achieving the objective.

We have to underline the fact that, from the perspective of objectification theory, doing mathematics cannot be reduced to solving problems. Without devaluing the role of problems in knowledge formation (see, for example, Bachelard, 1986), for us, problem solving is not the end but rather *one* of the means for achieving the type of *praxis cogitans* or cultural reflection that we call mathematical thinking. So that, behind the objective of the lesson, there lies a greater and more important objective—the generally held objective for the teaching and learning of mathematics—namely, the elaboration on the part of the student of a reflection defined as a *common* and *active* relationship with his/her cultural-historical reality.
In other words, learning mathematics is not simply learning to *do* mathematics (problem solving), but rather it is learning to *be* in mathematics. The difference between *doing* and *being* is immense and, as we shall see later, it has important consequences not only for the designing of activities but also for the organization of the class itself and the roles that students and teachers play within it.

3.3 The objectification of knowledge

In a succinct way, the greatest objective of the teaching of mathematics is that the student learn to reflect according to certain historically constituted cultural forms of thinking that distinguish it from other types of reflection (for example, those of a literary or musical kind) inasmuch as in mathematical reflection, the individual's relationship with the world emphasizes ideas regarding form, number, measurement, time, space, etc. It is this emphasis which distinguishes mathematical thinking from other kinds of thinking.
In order to achieve this objective, we have to make recourse to practice for the simple reason that we do not have at our disposal a language which is able to enunciate and capture in its enunciation (in the classical sense of the term, that is, as an articulated conjunction of vocal sounds) mathematical thinking. In effect, there is no possible linguistic formulation for mathematical thinking that, if we were to do a reading of it—however careful that reading might be—would be able to render mathematical thinking comprehensible. Thinking, as we have said before (Radford, 2003b), is beyond discourse: it is a *praxis cogitans*, something that is learned by reflectively and critically *doing*.

The theory of objectification nevertheless does not see learning as a simple imitation or participation consistent with a pre-established practice, but rather sees it as the fusion between a subjectivity which seeks to perceive this linguistically impossible-to-articulate mode of reflecting which, in its turn, can but *reveal itself* through action.

Without a doubt, there is a close relationship between mathematical thinking and its objects, in the sense that these objects cannot be perceived except through some kind of thinking which, in its turn, at the moment of its ontogenetic formation, has to point toward one or more of these objects. But, how is this possible? In order to form itself, thinking seems to presuppose the existence of the object. On the other hand, the object cannot come into being without the thinking (understood as a *praxis cogitans*) that gives rise to it.
The mystery surrounding this relationship dissolves if we go back to that which we said in the first part of this article. The mathematical object—conceived of as a *fixed pattern or patterns of reflexive activity incrusted in the ever-changing world of social practice*—cannot be perceived, except through reflexive activity itself.
Hence, in order to get to know objects and products of cultural development, it is "necessary to carry out a determined activity around them, that is to say, a kind of activity that produces its essential characteristics, embodied, 'accumulated' in said objects." (Leontiev, 1968, p. 21).

Teaching consists of generating and keeping in movement contextual activities which are situated in space and time and which are heading toward a fixed pattern of reflexive activity incrusted in the culture.
This movement, which could be expressed as the movement from process to object (Sfard, 1991; Gray and Tall, 1994) has three essential characteristics. First, the object is not a monolithic or homogenous object. It is an object made up of *layers of generality*. Second, from the epistemological point of view, said layers will be more or less general depending on the characteristics of the cultural meanings of the fixed pattern of activity in question (for example, the kinaesthetic movement that forms a circle; the symbolic formula that expresses it as a group of points at an equal distance from its centre, etc.). Third, from the cognitive point of view, the layers of generality are noticed in a *progressive* way by the student. The "Aha!" that became so popular, in part thanks to Gestalt theory, is the final point in a long process of growing awareness.
The learning process consists in finding out how to take note of or how to perceive these layers of generality. Just as learning is a *re-flection*, to learn presupposes a dialectical process between subject and object mediated by culture; a process during which, through his/her actions (sensory or intellectual) the subject takes note of or becomes aware of the object.

Objectification is precisely this social process of progressively becoming aware of the Homeric *eidos*, that is, of something in front of us—a figure, a form—something whose generality we gradually take note of and at the same time endow with meaning. It is this act of noticing that unveils itself through counting and signalling gestures. It is the noticing of something that reveals itself in the emerging intention projected onto the sign or in the kinaesthetic movement which mediates the artefact in the course of practical sensory activity, something liable to become a reproducible action whose meaning points toward this fixed eidetic pattern of actions incrusted in the culture which is the object itself.[12]

[12] See Radford, 2002, 2003c, 2004.

4. The classroom as a learning community
4.1 Being-with-others

The classroom is the social space in which the student *elaborates* this reflection, defined as a *common* and *active* relation with his/her historical-cultural reality.[13] It is here that the encounter between the subject and the object of knowledge occurs. The objectification that allows for this encounter is not an individual process but a social one. The sociability of the process, nevertheless, cannot be understood as a simple business interaction, a kind of game between adversaries during which each player invests some capital in the hopes of ending up with more of it. Here, sociability means the process of the formation of consciousness which Leontiev characterized as *co-sapientia*, that is to say, as knowing in common or knowing-with others.

Naturally, these ideas imply a re-conceptualization of the student and his/her role in the act of learning. Insofar as current theories in mathematics education draw on the concept of the individual as formulated by Kant and other Enlightenment philosophers, education justifies itself by guaranteeing the formation of an autonomous subject (understood in the sense of being able to do something for oneself without the help of others). Autonomy is, in effect, a central theme of modern education that has served as a basis for the theorizing of socio-constructivism (see, for example, Yackel and Cobb, 1996) and the theory of situations (Brousseau, 1986; Brousseau and Gibel, 2005, p.22). The rationalism that weighs on this concept of autonomy comes from its alliance with another key Kantian concept: that of liberty. There can be no autonomy without liberty and, for Kant, liberty means the convenient use of Reason according to its own principles so that “it is through reason that we get an insight into principles” (Kant, 1900, p. 34).

Since the Enlightenment did not put forward the possibility of there being a multiplicity of reasons, but rather postulated that western reason was *the* reason, community coexistence implies respect for a duty which, in the end, is nothing but a manifestation of that universal reason, whose epitome is mathematics. It was this supposed universality of reason that led Kant to fuse together the ethical, political and epistemological dimension and to affirm that “to do something for the sake of duty means obeying reason.” (Kant, 1900, p. 37).

For the theory of objectification, classroom functioning and the role of the teacher are not limited to trying to achieve autonomy. It is more important to learn how to live in the community that is a classroom (in its fullest sense), to learn to interact with others, to open oneself up to understanding other voices and other consciousnesses, in brief, to *be-with-others* (Radford, in press).
Just as "the social is irreducible to individuals, however numerous they might be" (Todorov, in Bakhtine, 1984, p. 19), sociability in the classroom means a coming together through links and relations that are prerequisites for that kind of reflection that we mentioned earlier, defined as *common* and *active* and which is elaborated by the student along with his/her

[13] The term *elaborate* has to be understood in its medieval etymological sense, as *ēlabōrātus* (from ex-labōrāre), that is to say, *joint labour or sensual work*.

historical-cultural reality. This sociability not only leaves its mark on the conceptual content being pursued but is furthermore an integral part of it.

The intrinsic social nature of knowledge and mathematical thinking has brought us then to conceiving of the classroom as a learning community whose functioning is oriented toward the objectification of knowledge. Its members work in such a way that:

- the community allows for the personal achievement of each individual;
- each member of the community has his/her place;
- each member is respected
- each member respects others and the values of the community;
- the community is flexible in its ideas and its forms of expression;
- the community opens up space for subversion in order to insure:
    - modification
    - change
    - and its transformation

Being a member of the community is not something that comes as a matter of course. In order to be a community member, students are encouraged to:

- share in the objectives of the community;
- involve themselves in the classroom activities;
- communicate with others.

We would like to insist upon the fact that the abovementioned guidelines are not simply codes of conduct. On the contrary, they are indexes of forms of *being* in mathematics (and, as a consequence, of *knowing* mathematics) in the strictest sense of the term.

In order to summarize the previous ideas, let us underline the fact that, for the theory of objectification, autonomy is not sufficient to account for the way of being in mathematics. The student who successfully solves problems but who is nevertheless incapable of explaining himself/herself, or of understanding or finding out about the solutions of others or of helping others to understand his/her solution is barely half way along the road to that which we consider to be success at mathematics. It is for this reason that the teacher must have at the ready a series of *actions of inclusion*. These actions are conceived of in such a way that the student who correctly solves mathematical problems without being able to attend to the interpersonal dimension of the community will little by little make space for himself in that same community.[14] The idea of autonomy as being able to achieve auto-sufficiency is replaced by the idea of *being-with-others*. Instead of conceiving of the class as a space for the personal negotiation of meanings or as a means for confronting the student, the class collaborates and cooperates with the student so that he/she can become part of the community.

[14] See our book, *Communication et apprentissage* (Radford and Demers, 2004).

4.2 Three phases of classroom activity

Work in small groups

In order to implement the learning community, the teacher favours work in small groups which later can, over the course of the mathematics lesson, exchange ideas with other groups. In this way, didactical engineering (Artigue, 1988) is not limited to designing mathematical problems but also includes a classroom management operational with the communitarian principles mentioned before. In each small group, the students support each other in order to arrive at the solutions to the problems they have been given. The students and the teacher are conscious of the fact that there are individual differences that lead to different forms of participation. Even types of participation that seem to be "less profound" (such as peripheral participation, in the sense used by Lave and Wagner, 1991) are welcomed, on the condition that the student in question *is-with-the-group*, that is to say that the student, for example, is paying attention to the group discussion, soliciting explanations that help to keep the discussion and actions going, is collaborating with his/her group, etc.

The teacher has to suggest tasks and problems that involve the objectification of knowledge. Some conditions have to be fulfilled. For example, in order to maintain a sustained reflection between the members of the group, with the teacher and later with other groups, the problems have to be complex enough to favour the appearance of diverse forms of tackling the problem and also engendering the discussion.
In our model, the teacher circulates between groups and discusses with the students. Although, in general, the teacher lets the students discuss amongst themselves without intervening unnecessarily, the teacher should nevertheless intervene at times when, for example, he/she believes that the discussion has come to a standstill or the students have not gone as far as was hoped.

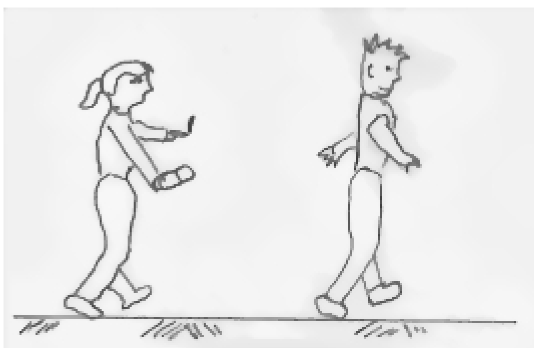
In order to illustrate these principles, let us take a look at an extract from a lesson on the interpretation of movement in a tenth grade class (15-16 year olds). The lesson included an artefact which measures the distance to an object through the emission-reception of waves (Based Ranger Calculator or CBR; see figure 4).

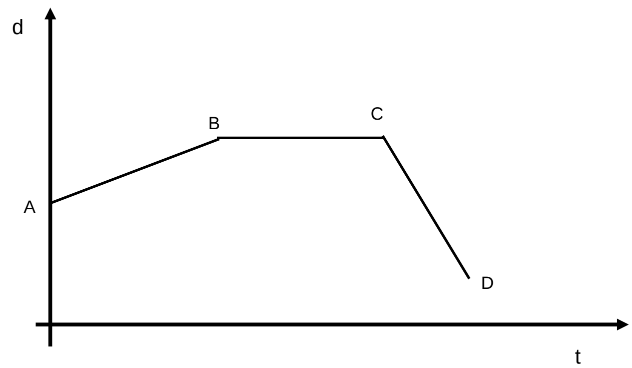
Figure 4. The Based Ranger Calculator® (to the left) is an artefact conceived of in order to study objects in movement: through the emission of waves, the CBR collects data about the distance from it to the object in question. When it connects to a graphic calculator (for example, the TI-83+®, shown on the right), it is possible to get graphics such as space-time, velocity-time, etc.

The students had begun using the CBR in the ninth grade. The problem given to the students was the following:

Two students, Pierre and Marthe, move apart so that there is one meter's distance seperating them and they start to walk in a straight line. Marthe, who is behind Pierre, is carrying a calculator connected to a CBR. The graph obtained can be found below. Describe how Pierre and Marthe were able to obtain this graph.
According to the design of the activity in the problem that followed the students had to verify their hypotheses by reproducing the walk in one of the school's corridors.

As usual, the students worked in small groups of 3. In the previous problems, the students had been confronted with situations involving movement in which either the CBR or the object had remained in a fixed place. In this case, both were moving. As we had expected, the conceptual difficulties were significant. In general, the students transformed the problem into one that they could solve: the students assumed that Marthe did not move.





This is illustrated by the discussion that took place between Samuel, Carla and Jenny the reproduction of certain parts of which will follow:

1. Samuel: Ok, Pierre moves slowly from A to B … He stopped for a few seconds (see figure 5, photo 1), then he ran to D (figure 5, photo 2).
2. Carla: Ah, Yes! He walked, he stopped, he ran.
3. Jenny: Mm-hmm.
4. Samuel: Wait, wait a second… [Pierre] he went back really quickly.
5. Carla: That's true. He started off slowly *(inaudible)* then he stopped, then he ran.
6. Samuel: Yes, backwards.
7. Jenny: (addressing herself to Carla) Yes, backwards, because [the segment] goes down (see figure 5, photo 3).

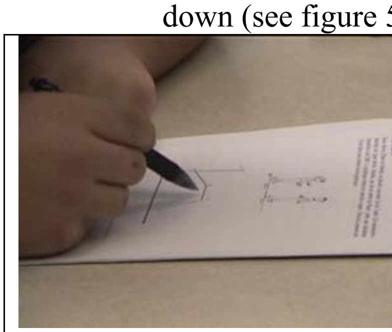
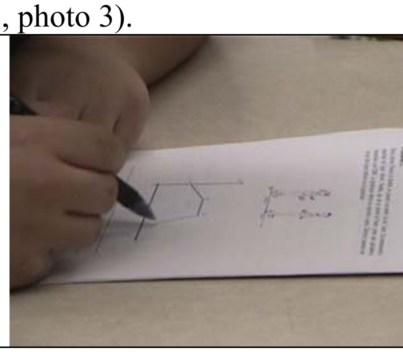
	

Photo 1:…He stopped for a few Seconds …(line 1).	Photo 2: … then he ran to D (line 1).	Photo 3: because [the segment] goes down.

Figure 5. Photos 1 to 3.

Our interest here does not lie in entering into an analysis of the students' errors, but rather in demonstrating some elements of the social process of the objectification of knowledge. It is worth noting, in this respect, that we do not consider Carla's interpolation in line 2 to be a simple replica or imitation of the preposition enunciated by Samuel in line 1. From the perspective of objectification, Carla *appropriates* the interpretation of the phenomenon that is offered by Samuel. The appropriation passes through a verbalization that Carla reformulates with more brevity (for example, there is no allusion made to the letters A, B. etc.). Samuel's preposition and the gesture with the pen on the graph are for Carla the raw matter she uses to be able to see something that she could not see before.

If Samuel gives Carla access to an initial interpretation of the problem (however rudimentary it may be), in her turn, Carla's reformulation allows Samuel to realize that there is something important that has gone without being noticed: namely, that in order to account for the difference in the inclination of the segments of the graph, in the history of the problem, Pierre had to go back "really quickly" (line 4). Carla reformulates the idea over again in line 6; Samuel insists that Pierre not only had to run more quickly, but also in a certain direction ("backwards"). In line 7, making a hand gesture (see figure 5, photo 3) Jenny proposes a reason for this.

The students continued discussing for quite a while. The interpretation that they have obtained is not convincing for Carla and Jenny, for it assumes that Marthe does not walk.

The discussion continues between them:

8. Jenny: No … uhhh… (both of them) have to walk!
9. Samuel: If she was doing that [that is, walking] exactly at the same distance [from Pierre] … like if she was doing *that* (see the gesture in figure 6), it would be a plain straight line [that is, a horizontal one] (…) So, she has to stay still and he has to move!
10. Jenny: But this [the problem] says that both of them are walking! (…)

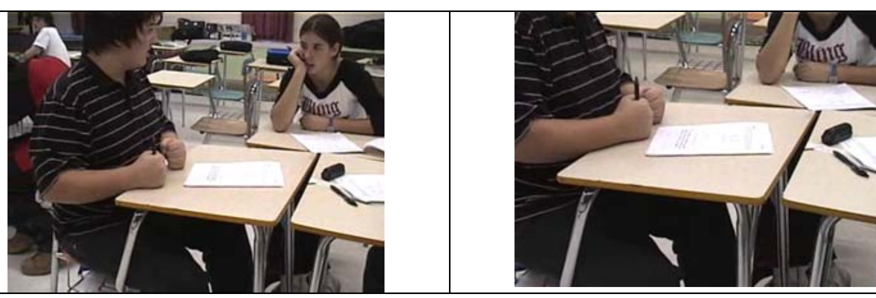


Figure 6. In order to simulate the case where Pierre and Marthe are walking, Samuel continuously moves his hands from right to left, keeping them the same distance apart.

11. Samuel: (after a moment of keeping silent) Maybe she is walking, but he is walking a little faster than she is.

At this moment, the description of the movement is no longer a description regarding a fixed point and reaches a description of relative movement. Through this exchange, the students managed to get a little closer to the kind of cultural reflection that was the goal of the activity envisioned by the teacher and our research team. The corporal expression of movement with the CBR and the symbolic expression of movement (through the physical experience in the school corridor and later in the calculation of the equation for the segments) will still be necessary for the students to achieve a greater level of objectification.

Exchanges between small groups

The reflections produced by the small groups are, eventually, objects to be exchanged. One group can exchange its solutions with those of another group in order to understand other points of view or to improve upon its own ideas. Figure 7 shows the encounter between two groups of students regarding the problem involving Pierre and Marthe. The groups reached a point where mutual agreement was impossible. Marc and his group approached the explanation in terms of changes in speed. On the contrary, Dona and her group maintained that Pierre's speed remained constant in relation to that of Marthe. Faced with the incapacity of reaching a consensus, the students opted to call the teacher. In figure 7, Marc (to the left) explains his reasoning to the teacher (standing, behind the students):

Figure 7. Above, the discussion between the Groups. Below, Marc explains his group's solution to the teacher, who is seen standing between Marc and Dona in the photo.

1. Marc: And if the two begin at the same speed, later on he starts running faster? (Marc supports his argument with hand gestures)
2. Teacher: You think that the boy [Pierre] keeps walking faster and faster?
3. Dona: (objecting to the idea) The speed is constant! (…) There aren't any curves! This means that he [Pierre] walks the same distance per second.

The teacher suggests that the students think of the situation in terms of two moving objects travelling at 80 km/hr and 100 km/hr. Marc realizes that the increase in distance does not necessarily mean an increase in speed. The teacher makes sure that the other students in Marc's group have understood the difference (she says, for example: "Edgar, what do you think now?") and takes advantage of the circumstances to make the students think about the effect that a movement increasing in speed would have on the graph, as Marc suggested in line 1.

In this case, the students noticed the difference between arguments and interpretations. Nevertheless, many times students do not realize that the arguments being presented are different or they tend to minimize the differences. One of the difficulties in the acquisition of ways of reflecting mathematically has to do with being able to note the differences between arguments. Naturally, in one case just as in another, the teacher plays a crucial role. In both cases, the teacher enters the group's zone of proximal development. What is most important is that the teacher does not enter this zone in a neutral way, but rather with a precise conceptual plan.

General discussions

General discussion is another way of exchanging ideas and talking about them. It is another chance for the teacher to open up the discussion to points that need to be looked at in more depth, in accordance with curricular standards. For example, during the general discussion regarding the problem involving Pierre and Marthe, the teacher took advantage of the situation to underline some points that were not grasped by all groups: namely, that the position of the segment BC did not necessarily indicate that Pierre and Marthe were held up, nor did the position of the segment CD necessarily indicate that Pierre was walking in Marthe's direction. In figure 8, two students execute a walk in front of the whole class, while Susan, the third student in this group (not visible in the photo), explains to the rest of the class:

1. Susan : Hem, the person who was in front was walking faster than the one who was in the back, this made a bigger distance between the CBR and the object point. Then … hem … next B and C in our diagram [Pierre and Marthe] were walking at the same speed, so there was the same distance between them. Then, … you?
2. Teacher : Yes, continue!
3. Susan: then … hem … in the end, the person who was behind walks faster in order to catch up to the person who was in front (see figure 6).

Figure 8. The student who is walking behind gets closer to the other student.

Synthesis

Some theories in mathematics education have intentionally excluded the psychological aspects of learning and have occupied themselves with mathematical situations that can favour the emergence of precise mathematical reasoning. Such is the case for the theory of situations. On the contrary, other theories have fixated themselves on the mechanisms of the negotiation of meaning in the classroom and the way in which this negotiation explains the construction of representations that the student makes of the world. Such is the case of socio-constructivism. The intellectual debt that the theory of objectification owes to these two theories is immense and our reference to them should not be seen in a negative light. These theories are sustained by fundamental principles and clear modes of operation that confer upon them an impeccable solidity. Nevertheless, the theory of objectification takes off from other principles. On the one hand, it bases itself on the idea that the psychological dimension of learning has to be an object of study in mathematics education. On the other hand, it suggests that the meanings circulating in the classroom cannot be confined to the interactive dimension that takes place in the class itself; rather, they have to be conceptualized according to the context of the historical-cultural dimension.

Therefore, the theory of objectification proposes a didactic anchored on principles according to which learning is viewed as a social activity (*praxis cogitans*) deeply rooted in a cultural tradition that precedes it. Its fundamental principles are articulated according to five interrelated concepts.

The first of these is a concept of a psychological order: the concept of *thinking*, elaborated in non-mentalist terms. We have suggested that thinking is above-all a form of active *re-flection* about the world, mediated by artefacts, the body (through perception, gestures, movements, etc.), language, signs, etc. This concept of *re-flection* differs from idealist and rationalist conceptualizations where reflection "is nothing else than attention to what is in us" (Leibniz, 1949, p. 45), and which contemporary cognitive psychology usually calls meta-cognition. For the theory of objectification, *re-flection* is a dialectical movement between a historically and culturally constituted reality and an individual who refracts it (as well as modifies it) according to his/her own subjective interpretations and feelings. The concept of thinking as re-*flection* is inscribed in a

> peculiar form of cognition in which the act of knowing alters what it contemplates. In trying to understand myself and my condition, I can never remain quite identical with myself, since the self which is doing the understanding, as well as the self understood, are now different from what they were before. And if I wanted to understand all *this*, then just the same process would set in (…). Since such knowledge also moves people to change their condition in a practical way, it becomes itself a kind of social or political force, part of the material situation it examines rather than mere 'reflection' of or upon it. (Eagleton, 1997, p. 4)

The second concept of the theory is of a socio-cultural order. This is the concept of learning. Learning is seen as an activity through which individuals enter into relationships not only with the world of cultural objects (subject-object plane) but also with other individuals (subject-subject plane or interaction plane) and acquire, in the joint pursuit of the objective and in the social use of signs and artefacts, human experience (Leontiev, 1993).

This socio-cultural perspective immediately overlaps with another concept –the third concept of the theory– one of an epistemological nature. Like all activity, learning is framed by *semiotic systems of cultural signification* that "naturalize" the ways that one questions and investigates the world. Aristotle would probably have incited our students to study the problem of Pierre and Marthe in different terms given that, within the Aristotelian frame of reference, it is not time and space which describe movement but, on the contrary, it is time which is derived from movement.[15] Our students belong to a culture where the measure of

[15] "We take cognizance of time, when we have defined the movement by defining the before and after; and only then we say that time has been (has elapsed) when we perceive the before and after in the movement…for, when we think [noesomen] that the extremities are other than the middle, and the soul pronounces the present/instants [nun] to be two, the one before, the other after, it is only then that we say that this is time" (Physics IV, 11, 219a 22-25; 26-29).

time has become omnipresent, measuring not only movement but human labour, the growth of money (interest rates), etc., a culture where

> This *temporalization* of experience –this notion of time as the framework within which life forms are embedded and carry on their existence –is the defining quality of the modern world. (Bender and Wellbery, 1991, p.1)

The aforementioned concepts allow one to reformulate, in general terms, the learning of mathematics as the communal acquisition of a form of reflecting upon the world guided by historically formed epistemic-cultural modes of knowing.

Furthermore, just as learning is always about something (a certain content), the aforementioned concepts come to be completed by a fourth concept of an ontological nature—that of mathematical objects, which we have defined as *fixed patterns of reflexive activity incrusted in the ever-changing world of social practice mediated by artefacts*.

To render the theory operational in its ontogentic aspect, it was necessary to introduce a fifth concept of a semiotic-cognitive nature—that of objectification, or a subjective awareness of the cultural object. In this context, and in light of the previous fundamental concepts, learning is defined as the social process of *objectification* of those external patterns of action fixed in the culture.

From a methodological point of view, our non-mentalist and non-rationalist concept of thinking leads us to pay attention to the semiotic means of objectification that the student uses in an effort that is, at the same time, an elaboration of meanings and an awareness of conceptual objects. The photos that we have included are not there with an end of "embellishing the text," but rather are there precisely to show some of the semiotic means of objectification, such as gestures, language, symbols. Gestures, language, symbols thus become the very constituents of the cognitive act that positions the conceptual object, not inside the head, but on the social plane. The brief classroom excerpts included at the beginning and at the end of the article give an idea of the way in which the theory investigates this objectification of knowledge that moves across the planes of interaction and mediated action (the territory of the artefact).[16]

Finally, our theoretical position with respect to learning entails a reconsideration of the concept of the individual who learns. As we have mentioned, the concept of the individual in the modern era that appears with the emergence of Capitalism in the 15^{th} and 16^{th} centuries is founded on the concepts of autonomy and liberty. The theory of objectification takes off from another point and offers a different concept: the individual is an individual inasmuch as he/she *is-with-others*.

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[16] Detailed examples can be found in Radford, 2000, 2003c; Radford, Bardini and Sabena, 2005; Sabena, Radford and Bardini, 2005.

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