Semiotic Objectifications of the Compensation Strategy: En Route to the Reification of Integers

Andreas Koukkoufis [1]
Julian Williams [1]

RESUMEN

Reportamos aquí el análisis de una experiencia que reproduce el trabajo de investigación "Object-Process Linking and Embedding" (OPLE) en el caso de la enseñanza de la aritmética de los enteros, desarrollada por Linchevski y Williams (1999) en la tradición de la Educación Matemática Realista (realistic mathematics education (RME)). Nuestro análisis aplica la teoría de la objetivación de Radford, con el propósito de aportar nuevas pistas sobre la forma en que la reificación tiene lugar. En particular, el método de análisis muestra cómo la generalización factual de la estrategia llamada de compensación encapsula la noción de "agregar de un lado es lo mismo que quitar del otro lado"; una base fundamental de esto que será, más tarde, las operaciones con enteros. Discutimos, de igual modo, otros aspectos de la objetivación susceptibles de llegar a ser importantes en la cadena semiótica que los alumnos ejecutan en la secuencia OPLE, secuencia que puede llevar a un fundamento intuitivo de las operaciones con los enteros. Sostenemos que es necesario elaborar teorías semióticas para comprender el papel vital de los modelos y de la modelación en la implementación de las reificaciones en el seno de la Educación Matemática Realista (RME).

- **PALABRAS CLAVE:** Enteros, semiótica, teorías del aprendizaje.

ABSTRACT

We report an analysis of data from an experimental replication of "Object-Process Linking and Embedding" (OPLE) in the case of integer arithmetic instruction originally developed by Linchevski and Williams (1999) in the realistic mathematics education (RME) tradition. Our analysis applies Radford's theory of semiotic objectification to reveal new insights into how reification is achieved. In particular the method of analysis shows how the factual generalization of the so-called compensation strategy encapsulates the notion that "adding to one side is the same as subtracting from the other side": a vital grounding for symbolic integer operations later. Other aspects of objectification are discussed that are considered likely to be important to the semiotic chaining that students achieve in the OPLE sequence that can lead to an intuitive grounding of integer operations. We

Fecha de recepción: Febrero de 2006/ Fecha de aceptación: Abril de 2006

[1] School of Education, the University of Manchester.

argue that semiotic theory needs to be elaborated to understand the vital role of models and modelling in leveraging reifications in RME.

• **KEY WORDS:** Integers, Semiotics, Theories of Learning.

RESUMO

Reportamos aqui o análise de uma experiencia que reproduce o trabalho de investigação “Object-Process Linking and Embedding” (OPLE) en o caso da ensino da aritmética dos inteiros, desenvolvida por Linchevski e Williams (1999) na tradição da Educação Matemática Realista (realistic mathematics education (RME)). Nossa análise aplica a teoria da objetivação de Radford, com o propósito de surgir novas pistas sobre a forma em que a reificação tem lugar. Em particular, o método de análise mostra como a generalização factual da estratégia chamada de compensação encapsula a noção de “agregar de um lado é o mesmo que quitar do outro lado”; uma base fundamental disso que será, mais tarde, as operações com inteiros. Discutimos, de igual modo, outros aspectos da objetivação suscetíveis de chegar a ser importante na cadeia semiótica que os alunos executam na seqüência OPLE, seqüência que pode levar a um fundamento intuitivo das operações com os inteiros. Sustentamos que é necessário elaborar teorias semióticas para compreender o papel vital dos modelos e da modelação na implementação das reificações no seio da Educação Matemática Realista (RME).

• **PALAVRAS CHAVE:** Inteiros, Semióticos, Teoria de Aprendizagem.

RÉSUMÉ

Nous rapportons ici l’analyse d’une expérience qui vise à reproduire le travail de recherche “Object-Process Linking and Embedding” (OPLE) dans le cas de l’enseignement de l’arithmétique des entiers développé par Linchevski et Williams (1999) dans la tradition de l’Éducation Mathématique Réaliste (realistic mathematics education (RME)). Notre analyse applique la théorie de l’objectivation sémiotique de Radford afin d’apporter de nouveaux éclairages sur la façon dont la réification est accomplie. La méthode d’analyse montre, en particulier, comment la généralisation factuelle de la stratégie appelée de compensation encapsule la notion que « ajouter d’un côté, c’est la même chose qu’enlever de l’autre côté » : une base fondamentale de ce que sera plus tard les opérations avec des entiers. Nous discutons également d’autres aspects de l’objectivation susceptibles de devenir importants dans la chaine sémiotique que les élèves accomplissent dans la séquence OPLE, séquence qui peut mener à un fondement intuitif des opérations sur des entiers. Nous soutenons qu’il est nécessaire d’élaborer des théorisations sémiotiques pour comprendre le rôle vital des modèles et de la modélisation dans l’implémentation des réifications au sein de l’Éducation Mathématique Réaliste (RME).

• **MOTS CLÉS:** Entiers, sémiotique, théories de l´apprentissage.

The Need for a Semiotic Analysis

Based on the instructional methodology of *Object-Process Linking and Embedding (OPLE)* (Linchevski & Williams, 1999; Williams & Linchevski, 1997), the *dice games instruction method* for integer addition and subtraction showed how students could intuitively construct integer operations. This methodology, underpinned by the theory of *reification* (Sfard, 1991; Sfard & Linchevski, 1994), was developed within the Realistic Mathematics Education (RME) instructional framework. Until very recently, the dice games method had not been analysed semiotically. We believe a semiotic analysis of students' activities in the dice games will illuminate students' meaning-making processes. It will also provide some further understanding of the reification of integers in the dice games in particular and more generally of the theory of reification, which does not explain "what spur[s] the students to make the transitions between stages" (Goodson-Espy, 1998, p. 234). Finally, it will contribute to the discussion of the semiotic processes involved in RME, which are currently insufficiently investigated (Cobb, Gravemeijer, Yackel, McClain, & Whitenack, 1997; Gravemeijer, Cobb, Bowers, & Whitenack, 2000). In this paper we focus on the *compensation strategy* (Linchevski & Williams, 1999), a dice game strategy on which integer addition and subtraction are grounded, and begin to address the following questions:

1. What are the students' semiotic processes of the compensation strategy in the reification of integers through the OPLE teaching of integers in the dice games method?

2. What is the semiotic role of the abacus in the OPLE teaching of integers through the dice games and what can we generally hypothesise about the significance of models and modelling in the RME tradition?

We found Radford's semiotic theory of *objectification* (Radford, 2002, 2003) to be a particularly useful theoretical framework for analysing students' semiotic processes in the dice games, despite the very different context in which it was developed.

The Object-Process Linking and Embedding Methodology

Sfard (1991) reported as follows:

> But here is a vicious circle: on the one hand, without an attempt at the higher-level interiorization, the reification will not occur; on the other hand, existence of objects on which the higher-level processes are performed seems indispensable for the interiorization – without such objects the processes must appear quite meaningless. In other words: *the lower-level reification and the higher-level interiorization are prerequisites of each other!* (p. 31)

In order to overcome this '*vicious circle*', the *Object-Process Linking and Embedding (OPLE)* pedagogy (Linchevski & Williams, 1999) was developed: "children a) build strategies in the situation, b) attach these to the new numbers to be discovered, and finally c) embed them in mathematics by introducing the mathematical voice and signs" (Linchevski & Williams, 1999, p. 144). The pedagogy can be best understood through the dice games context in which it was developed (Linchevski & Williams, 1999), which aimed at overcoming the paradox of reification described above for the case of arithmetic of the integers.

The *dice games instruction method* (Linchevski & Williams, 1999) is an intuitive instruction of integer addition and subtraction in the RME instructional framework aiming at the reification of integers. The transition from the narrower domain of natural numbers to the broader domain of integers in the method is achieved through *emergent modelling* (Gravemeijer, 1997a, 1997b, 1997c; Gravemeijer et al., 2000) and takes advantage of students' *intuition of fairness* (Liebeck, 1990) for the cancellation of negative amounts by equal positive amounts (Dirks, 1984; Linchevski & Williams, 1999; Lytle, 1994). Practically, the model – the double abacus (see figure 1) – affords the representation and manipulation of integers as objects before they are abstracted and symbolised as such by the students (Linchevski & Williams, 1999):

> The integer is identifiable in the children's activity first as a process on the numbers already understood by the children, then as a 'report' or score recorded (concretised by the abacus). The operations on the integers arise as actions on their abacus representations, then recorded in mathematical signs. Finally, the operations on the mathematical signs are encountered in themselves, and justified by the abacus manipulations and games they represent. Thus the integers are encountered as objects in social activity, *before* they are symbolised mathematically, thus intuitively filling the gap formerly considered a major obstacle to reification. (Linchevski & Williams, 1999, p. 144)

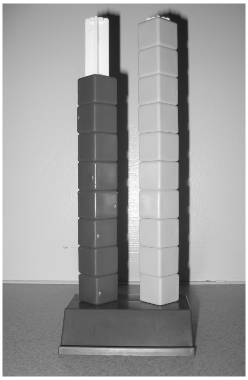


Figure 1: *The abacus*

Therefore, in the games the situated strategies are constructed in a realistic context which allows intuitions to arise. In this process the abacus model is utilized which "affords representation of the two kinds of numbers, and allows addition and subtraction (though clearly not multiplication and division) of the integers to be based on an extension of the children's existing cardinal schemes" (Linchevski & Williams, 1999, p. 135). These strategies are linked to objects (yellow and red team points, see next section), thus allowing object-process linking. Later, the formal mathematical language and symbols enter the games. In the following section we present the games more analytically.

The Dice Games Instruction

The method involves 4 games in each of which two teams of two children are throwing dice (e.g. a yellow and a red die in game 1) and recording team points on abacuses: the points for the yellow team are recorded by yellow cubes on the abacuses and those for the red team are red cubes on the abacuses. The students sit in two pairs, each having a member of each team and an abacus (see figure 2). On each pair's abacus, points for both teams are being recorded and the team points on the two abacuses add up. The students in turn throw the pair of dice, recording each time the points for the two teams on their abacus. When the two abacuses combine to give one team a score of 5 points ahead of their opponents, that team wins the game. For instance in game 1, if the yellow team at a certain point is 2 ahead and they get a score on the pair of dice, say 4 yellows and one red, then they can add 3 yellows to their existing score of 2 and so get 5 ahead, and they win. But note the complication that because we have two abacuses for the two pairs, a 'combined score of 2 yellows' might involve, say 1 red ahead on the one abacus and 3 yellows ahead on the other abacus: so there are multiple 'compensations' of reds and yellows going on in various combinations. Therefore, the important thing in the games is not how many points a team has, but how many points *ahead* of the opponent: hence the nascent directivity of the numbers.

In the first game (game 1) two dice are used, a yellow and a red one, giving points to the yellow and red team in each throw. Shortly after the beginning of game 1, often with the urging of the researcher, the students intuitively understand that they can cancel the team points on the dice, thus introducing an important game strategy, the *cancellation strategy* (not examined in this paper). For example, according to this strategy, if a throw of the pair of dice shows 3 points for the yellow team and 1 for the red team, this is equivalent to just giving 2 points for the yellow team. The rationale is that the *directed difference* of the points of the yellow and red team (i.e. the amount of points that the yellows are ahead or behind the reds) will be the same anyway. As the abacus columns have only space for 10 points for each team, a team column will often be full before a team gets 5 points ahead of the opponent. In order for the game to go on, the *compensation strategy* is formulated, that is, if you can't add points to one team, subtract the same amount of points from the other, so as to maintain the correct directed difference of team points.

Figure 2: *Students playing one of the dice games*

This strategy is the second important game strategy and it is the one focused upon in this paper. By the end of the games, this strategy will lead to the intuitive construction of equivalences like: $+(+2) \equiv -(-2)$ and $+(-2) \equiv -(+2)$.

Game 2 is similar to game 1, and is introduced as soon as (and not before) the children are able to cancel the pair of dice into ONE score quite fluently. In this game an extra die is now thrown whose faces are marked '*add'* and '*sub'* (subtract). From now on this will be called the *add/sub die*. The introduction of this die allows for subtraction to come into play, instead of just addition, as in game 1. In analogy to game 1, according to the compensation strategy, if you have to subtract points from a team but there are none on the abacus to subtract, you can add points to the other team instead.

In game 3, formal mathematical symbols for integers are introduced. The *add/sub die* is not used and the yellow and red die are replaced with an *integer die* giving one of the following results on each throw: -1, -2, -3, $+1$, $+2$, $+3$. Positive integers are points for the yellow team and negative integers are points taken from the yellows, thus they are points for the reds (for more details see Linchevski & Williams, 1999). Here the mathematical voice is encouraged, so that the children say "minus 3" and "plus 2" etc.

In the final game (game 4), the *add/sub die* is back into the game, allowing again for subtraction to be concerned. In these two games the cancellation strategy is no longer needed and the compensation strategy is transformed into a formal symbolic, though still verbal, form: "add minus 3" etc. Once the students become fluent in game 4, they begin recording the games for a transition from verbal to written use of formal mathematical symbols, but we are not going to discuss this transition further in this paper.

Some Earlier Analyses: Reification in the Dice Games

Linchevski and Williams (1999) have analysed the dice games in terms of reification. Through the instructional methodology of Object-Process Linking and Embedding, they achieved the intuitive reification of integers and the construction of processes related to integer addition and subtraction through the manipulation of objects on a model (i.e. the yellow and red team points). However, they did not provide a semiotic-analytical account of the reification processes – their main concern was to show that reification of integers was possible through their method. We will discuss here the reifications taking place in the dice games, as we understand them, so that we can better appreciate the need for a semiotic analysis of students' processes.

In relation to the reification of integers, according to Linchevski & Williams (1999), the object-process linking allows the intuitive manipulation of integers as objects from the very beginning of the dice games. As a result of this methodological innovation, some elementary processes are obvious from the beginning. These are, that if a team gets points (or points are subtracted from it), the new points add-up to (or are subtracted from) the points the team already has. These processes are intuitively obvious from the introduction of game 1 (and game 2 respectively). However, one may argue that the students still operate at the level of natural numbers, not integers.

Integer processes begin to be constructed, though integers are not yet introduced explicitly, once the students focus on the *score* of the game, that is, which team is ahead and by how many points. The calculation of the *score* as the *directed difference* of the piles of cubes of the points of the two teams is the first *object-process link* to be constructed. The second *object-process link* to be achieved in the games is the cancellation of the team points on the dice: i.e. if in a throw the yellows get 2 points and the reds 1 point, you might as well just give 1 point to the yellows. Thus this link is possible through the establishment of the so called *cancellation strategy*. Further, the *compensation strategy* – according to which *adding to one side of the abacus is the same as subtracting from the other side* – needs to be introduced as an *object-process link*. Up to this point, all the necessary object-process links are in place. Next, at the beginning of game 3, integers are introduced into the games: the formal mathematical voice enters the games. Through the manipulation of the formal mathematical symbols of integers in the above object-process links, integers are being reified and the addition and subtraction of integers are being established.

However, in the above analysis the following significant question arises: What are the meaning-making processes (semiotic) involved in students' integer reification in the dice games? We certainly do not claim that we will exhaust this issue here, but we will begin to address it through the vital component of the compensation strategy.

Semiotics are Needed to Complement Reification Analyses

The theory of reification, drawing support from a cognitivist/constructivist view of learning, is mainly interested in the internal processes of students' abstraction of mathematical objects. It does not generally refer to the social semiotic means students used to achieve the abstraction of these objects, (e.g. in the dice games, the integers). The analysis of Linchevski & Williams (1999) did in fact go some way in providing a social analysis of the context as a resource for construction of the compensation strategy: they were excited mainly here by the accessing of the socio-cultural resource of 'fairness' in the games as a basis for an intuitive construction of compensation. Semiotic chaining was adduced to explain the significance of the transition to the 'mathematical voice', so that "two points from you is the same as two points to us" slides under a new formulation like "subtract minus two is the same as adding two… plus two". However, we will complement Linchevski & Williams' (1999) study with a more detailed semiotic analysis of the way that the abacus, gesture and deictics mediate children's generalisations (after Radford's, 2003, 2005 methodology).

We wish to clarify at this point that we do not reject the reification analyses. Instead, we agree with Cobb (1994) who takes an approach of *theoretical pragmatism*, suggesting that we should focus on "what various perspectives might have to offer relative to the problems or issues at hand" (p. 18). We propose that in this sense semiotic social theories can be complementary to constructivist ones. More precisely, we propose that Radford's theory of objectification (Radford, 2002, 2003) can be seen as complementary to the theory of reification (Sfard, 1991; Sfard & Linchevski, 1994): while Sfard (1991) provides a model for the cognitive changes taking place, Radford (2002, 2003) provides the means to analyse these changes on the social, 'intermental' plane.

Radford addresses the issue of semiotic mediation through his theory of objectification (Radford, 2002, 2003). This theory, presented in some detail in the following section, analyses students' dependence on the available *semiotic means of objectification (SMO)* (Radford, 2002, 2003) to achieve increasingly socially-distanced levels of generality. Radford explains this reliance on SMO through reference to Frege's triad: the *reference* (the object of knowledge), the *sense* and the *sign* (Radford, 2002). The SMO refer to Frege's *sense*, that is, they mediate the transition from the *reference* to the *sign*. Moreover, Radford extended the Piagetian *schema* concept to include a *sensual dimension*, as Piaget's emphasis on the process of reflective abstraction can lead to an inadequate analysis of the role of signs and symbols (Radford, 2005).

> The schema …is … both a sensual and an intellectual action or a complex of actions. In its intellectual dimension it is embedded in the theoretical categories of the culture. In its sensual dimension, it is executed or carried out in accordance to the technology of *semiotic activity…* (Radford, 2005, p. 7)

Given this extended schema definition, the process of abstraction of a new mathematical object needs to be investigated in relation to the semiotic activity mediating it. This investigation should expose students' meaning making processes in the objectifications taking place in the dice games, which allow the construction of integers as new mathematical objects, i.e. their reification in Sfard's sense.

In the next section we present analytically Radford's *theory of objectification* (Radford, 2002, 2003), which will then be applied in the section following it to some of our data from the instruction through the dice games.

Radford's Semiotic Theory of Objectification

Objectification is "a process aimed at bringing something in front of someone's attention or view" (Radford, 2002, p.15). It appears in three modes of generalization: generalization through actions, through language and through mathematical symbols. These are *factual, contextual and symbolic generalization* (Radford, 2003). Objectification during these generalizations is carried out gradually through the use of *semiotic means of objectification* (Radford, 2002):

> …objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities, I call *semiotic means of objectification*. (Radford, 2003, p. 41)

Factual generalization, a generalization of actions (but not of objects), is described as follows:

> … A factual generalization is a generalization of actions in the form of an operational scheme (in a neo-Piagetian sense). This operational scheme remains bound to the concrete level (e.g., "1 plus 2, 2 plus 3" …). In addition, this scheme enables the students to tackle virtually any particular case successfully. (Radford, 2003, p. 47)

The formulation of the operational scheme of factual generalization is based on deictic semiotic activity, e.g. deictic gestures, deictic linguistic terms and rhythm. The students rely on the signification power provided by deictics to refer to actions on non-generic physical objects. These are perceivable, non-abstract objects which can be manipulated accordingly. In the example from Radford (2003) below, the students had to find the number of toothpicks for any figure in the following pattern.

The elaboration of the operational scheme in this case can be seen in the following section of an episode provided by Radford (2003).

1. Josh: It's always the next. Look! [and pointing to the figures with the pencil he says the following] 1 plus 2, 2 plus 3 […]. (Radford, 2003, p. 46-47)

Josh constructed the operational scheme for the calculation of the toothpicks of any figure in the form "1 plus 2, 2 plus 3", while pointing to the figures. Moreover, he used the linguistic term *always* to show the general applicability of this calculation method for any specific figure and the term *next* which "emphasizes the ordered position of objects in the space and shapes a perception relating the number of toothpicks of the next figure to the number of toothpicks in the previous figure" (p. 48). Hence, in factual generalization:

> …the students' construction of meaning has been grounded in a type of social understanding based on implicit agreements and mutual comprehension that would be impossible in a nonface-to-face interaction. … Naturally, some means of objectification may be powerful enough to reveal the individuals' intentions and to carry them through the course of achieving a certain goal. (Radford, 2003, p. 50)

In contextual generalization the previously constructed operational scheme is generalised through language. Its generative capacity lies in allowing the emergence of new abstract objects to replace the previously used specific concrete objects. This is the first difference between contextual and factual generalization: new abstract objects are introduced (Radford, 2003). Its second difference is that students' explanations should be comprehensible to a "generic addressee" (Radford, 2003, p. 50): reliance on face-to-face communication is excluded. Consequently, contextual generalization reaches a higher level of generality. More specifically, in Radford (2003) the operational scheme "1 plus 2, 2 plus 3" presented above becomes "You add the figure and the next figure" (p. 52). Therefore, the pairs of specific succeeding figures *1, 2* or *2, 3* become *the figure* and *the next figure*. These two linguistic terms allow for the emergence of two new abstract objects, still situated, spatial and temporal (Radford, 2003). Reliance on face-to-face communication is eliminated, and deictic means subside. However, the personal voice, reflected through the word *you*, still remains.

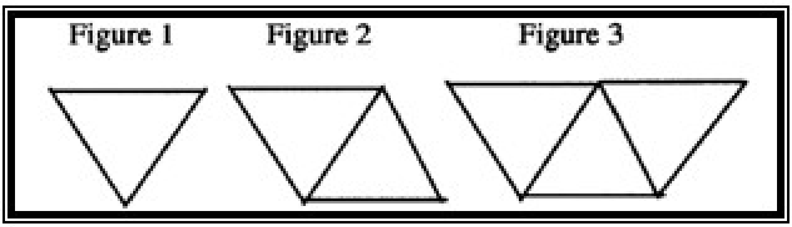


Figure 3: *First three 'Figures' of the 'toothpick pattern', labelled 'Figure1', 'Figure 2', 'Figure 3' by Radford (the picture in the box was taken from Radford, 2003, p. 45)*

In symbolic generalization, the spatial and temporal limitations of the objects of contextual generalization have to be withdrawn. Symbolic mathematical objects (in Radford's case algebraic ones) should become "nonsituated and nontemporal" (Radford, 2003, p.55) and the students lose any reference point to the objects. To accomplish these changes, Radford's (2003) students excluded the personal voice (such us *you*) from their generalization and replaced the generic linguistic terms *the figure* and *the next figure* with the symbolic expressions n and $(n+1)$ correspondingly. Hence, the expression *you add the figure and the next figure* became $n+(n+1)$. Still, Radford (2003) points out that for the students the symbolic expressions n and $(n+1)$ remained indexed to the situated objects they substituted. This is evident in students' persistent use of brackets and their refusal to see the equivalence of the expressions $n+(n+1)$ and $(n+n)+1$. Summarising, the mathematical symbols of symbolic generalization were indexes of the linguistic objects of contextual generalization, which in turn were indexes of the actions on concrete physical objects enclosed in the factual generalization operational scheme.

The Compensation Strategy – Factual Generalization

In this section we analyse the objectification of the compensation strategy in terms of factual, contextual and symbolic generalization. We present excerpts of the discourse contained in the games, which we analyse in terms of their contribution to the progressive abstraction of integers through the means of objectification. We also discuss the SMO involved in students' processes. The analyses of factual, contextual and symbolic generalization are presented separately, but first we provide some information about the students and the episodes in this paper.

The study, part of an ongoing PhD research, involves year 5 students in Greater Manchester, who had not yet been taught integer addition and subtraction. The PhD involves two experimental methods (respectively containing 5 and 6 groups of 4 students) from 2 separate classes and a control group from a third class. In each experimental method class the students were arranged by their teacher in mixed gender and ability groups, which were taught for three one-hour lessons. In this paper we focused on a microanalysis of one group of one of the methods – the dice games as originally applied by Linchevski and Williams (1999).

Radford's factual generalization is quite a clear-cut process based on action on physical objects formulated into an operational scheme through deictic activity. However, in our investigation of the compensation strategy, we find a multi-step process of semiotic contraction happening inside it. The three following episodes co-constitute in our view the factual generalization. In these episodes, occurring during game 1 (in lesson 1), the students were faced with a situation where they had to add cubes/points to one of the two teams, but there was no space on the abacus. As a result, a breakthrough was needed for the scoring to continue.

<u>Episode 1 (Minutes 14:30-14:50, lesson 1):</u> Umar had to add 1 yellow cube on the abacus but, as there was no space in the relevant column, he got stuck. Fay proposed taking away 1 red cube instead. ""…" indicates a pause of 3 sec or more, and "." or "," indicate a pause of less than 3 sec" (Radford, 2003, p. 46).

Fay: You take 1 off the reds [*pointing to the red column on her abacus*]. […] Because then you still got the same, because you're going back down [*showing with both her hands going down at the same level*] 'cause instead of the yellows getting one [*raising the right hand at a higher level than her left hand*] the red have one taken off [*raising her left hand and immediately moving it down, to show that this time the reds decrease*].

Fay's proposal for the subtraction of a red cube instead of the addition of a yellow one is the first articulation of the compensation strategy in the games for this group of students. We especially noticed the analytical explanation of the proposed action, which allows the process of compensation to be introduced for the first time. Deictic activity was associated both with the proposed action of taking away a red cube and with the justification following it. Fay used pointing to the red cubes on the abacus, as well as a gesture with both her hands indicating the increase/decrease of the pile of cubes in each team's column. Moreover, the names "the yellows" and "the red" have a deictic role. We also notice the phrase "you still got the same", stressing that something (obviously important) remains unaltered: either we add a yellow point/cube or subtract a red point/cube. This significant unaltered game characteristic, which we call the *directed difference* of the points of the yellow and red team, still cannot be articulated as it has not yet acquired a name.

<u>Episode 2 (Minutes 20:15-20:43, lesson 1):</u> The yellows' column was full and the reds' only had space for 1 cube. Compensation was needed and as Zenon could not understand, Jackie explained as follows.

Jackie: It's still the same, like … [*a very characteristic gesture (see figure 4): she brings her hands to the same level and then she begins to move them up and down in opposite directions, indicating the different resulting heights of the cubes of the two columns of the abacus*] because it's still 2, the yellows are still 2 ahead [*she does the same gesture while she talks*] and the reds are still 2 below, so it's still the same… [*again the gesture*] … em like… [*closing her eyes, frowning hard*] … I don't know what it's called but it's still the same… score [*the gesture 'same' again before and while articulating the word "score" – indicating 'same' score on her abacus*].

Figure 4: *Jackie's gesture (this sequence of action performed fast and repeated several times)*

In episode 2, we noticed the repeated use of the phrase "it's still the same", the word "still" followed by the difference in team points (i.e. "still 2", "still 2 ahead"), as well as the accompanying characteristic gesture. The gesture, too, emphasized the importance of the unaltered directed difference of the cubes of the two teams. We also noticed Jackie's difficulty in finding a proper word for this important unaltered game characteristic: "em like… [closing her eyes, frowning hard] … I don't know what it's called but it's still the same… score" (extract from episode 2 above). We believe the articulation of the word *score*, meaning what we call the directed difference of team points, as well as Jackie's gesture were very important for the factual generalization process, because they achieved the *semiotic contraction* (Radford, 2002) of the process originally established in episode 1. From this point onward, the students do not need to provide an analytical semiotic justification of the proposed action, as Fay needed to in episode 1. Just saying that the score will be the same is enough. A similar effect was accomplished by the word *difference* in a different group (Koukkoufis & Williams, 2005).

<u>Episode 3 (Minutes 21:27-21:57, lesson 1):</u> There's only space for 2 yellow cubes, but Fay has to add 3 yellows and 1 red.

Fay: Add 2 on [*she adds 2 yellow cubes*] and then take 1 of theirs off [*she takes off a red cube*] and then for the reds [*pointing to the red dice*] you add 1, so you add the red back on [*she adds 1 red cube*].

Researcher: […] Does everybody agree? (Jackie and Umar say "Yeah").

Finally, in the above episode further semiotic contraction took place. In fact, no justification of the proposed action was provided, as it seemed to be unnecessary – indeed Jackie and Umar agreed with Fay without further explanation. We argue that the further semiotic contraction happening in episode 3 completed the factual generalization of the compensation strategy.

To sum up, we see in the three episodes provided up to this point a continuum as follows: in episode 1 Fay presented a proper action and an analytical process to justify it; in episode 2 again a proper action was presented but the process justifying it was contracted; finally in episode 3 the presentation of the proposed action was sufficient, therefore further semiotic contraction took place and the process for resulting in this proposed action disappeared.

The Compensation Strategy – Contextual Generalization

Contextual generalization, in which abstraction of new objects through *language* takes place, has not yet been completed in this case. If we had had a contextual generalization of the compensation strategy, we would have a generalization like this: if you can't add a number of yellow/red points, you can subtract the same number of red/yellow points instead. Similarly for subtraction, the generalization would be similar to this: if you can't subtract a number of yellow points/red points, you can add the same number of red/yellow points instead. However, our students did not spontaneously produce such a generalization, neither does the instructional method demand it; therefore we did not insist that the students produce it. We believe that the lack of articulation of the compensation strategy through generic linguistic terms, and thus the incompleteness of the production of a contextual generalization, has to do with the compensation strategy being too intuitively obvious. On the contrary, in the case of the *cancellation strategy* (Linchevski & Williams, 1999) which was not so obvious, the same students produced a contextual generalization as follows (Fay, minutes 38:17-38:40, lesson 1, 5 reds and 2 yellows): "you find the biggest number, then you take off the smaller number". In the case of the contextual generalization of the cancellation strategy, we notice that new abstract objects ("the biggest number", "the smaller number") enter the discourse, as in Radford (2003). However, we will not discuss the contextual objectification of the cancellation strategy here.

The Compensation Strategy – Symbolic Generalization

Despite the incompleteness of the contextual generalization, we found that symbolic generalization was not obstructed! In this section we discuss the symbolic generalization of the compensation strategy, which presents some differences from that of the case presented by Radford (2003).

To begin with, in Radford (2003) symbolic generalization remained indexical throughout the instruction. In our case, the students began using symbolic generalization non-indexically. For convenience, we present indexical and non-indexical symbolic generalization separately.

Indexical Symbolic Generalization

The elaboration of a symbolic generalization for the compensation strategy demands the replacement of pre-symbolic signs with symbolic ones. Therefore, the reference to yellow and red team points has to be substituted by reference to positive and negative integers. According to the dice games method, this is achieved in the beginning of game 3, when *the red and the yellow die* are replaced by the *integer die*. Analytically, the numbers +1, +2 and +3 (on the integer die) are points for the yellow team. Further, –1, –2 and –3 (on the integer die) are points taken away from the yellow team, thus they are points for the red team. Of course, similarly one can say that +1, +2 and +3 are point taken away from the red team. Conclusively, when it is "+" it is yellow points, while when it is "–" it is red points. In the following episode we witness the transition from the pre-symbolic signs of "yellow team points" and "red team points" to the symbolic signs of "+" and "–" (positive and negative integers).

Episode 4: Minutes: 20:45-21:55, lesson 3.

Researcher:	+1. Who is getting points?
Jackie:	The yellows
Researcher:	[…] Who is losing points?
Jackie, Umar:	The reds
Fay:	[…] reds are becoming called minuses and then the yellows are becoming called plus.

As a result of the above introduction of the formal mathematical symbols for integers, positive integers are used to indicate yellow team points and negative integers are used to indicate red team points. Here lies the first difference from Radford's symbolic generalization, which is soon to become evident.

In Radford (2003), the symbolic signs/ expressions used in symbolic generalization were indexes of the contextual abstract objects of contextual generalization. Hence, the expressions n

and $n+1$ indicated the generic linguistic terms *the figure* and *the next figure*. Instead, in the dice games the formal mathematical symbols of integers were indexes not of the generic linguistic terms of contextual generalization (which was never completed), but of the concrete objects of factual generalization. For example, +2 is an index of "2 more for yellows" as well as of "2 yellow points", as in episode 5.

Episode 5 (Minutes: 33:15-33:53, lesson 3)

Researcher: […] you get –2. What would you do? (Fay takes 2 yellow cubes off) […] What if you had +3?
Umar: You take away 3 of the reds.
Zenon: … or you could add 3 to the yellow.
Fay, Jackie: … add 3 to the yellow.
Researcher: Oh, 3 off the reds or 3 to the yellows. (All the students agree)

Indeed, the students read +2 on the die, the researcher articulates it as "plus 2", but then the students' discussion is in terms of reds and yellows. If symbolic signs were being used non-indexically at that point, Umar would have said "minus 3" instead of saying "3 of the reds" (as in the phrase "take away 3 of the reds"). Also the others would have said "plus 3" instead of "3 to the yellow" (as in the phrase "add 3 to the yellow"). It becomes clear that in our case, we witnessed a *direct* transition from *factual* to *indexical symbolic generalization*, without the completion of contextual generalization being necessary. This transition was afforded due to the RME context and the abacus model.

In indexical symbolic generalization, though the operational scheme of factual generalization is reconstructed through the use of symbolic signs instead of concrete physical objects, it is not a simple repetition of factual generalization in symbolic terms that takes place. No semiotic contraction needs to take place for the establishment of the compensation strategy in symbolic terms. The students know right away that instead of adding +2 (2 yellow points) they can subtract 2 red points.

Non-indexical Symbolic Generalization

Up to now the formal symbolic signs for integers are being used indexically, but the intended instructional outcome is that students will eventually be using these symbols non-indexically. We do not imply that the symbols should drop their connection to the context though. Indeed it is essential that students can go back to the contextual meanings of these symbols in the dice games, so as to draw intuitive support regarding integers. We just emphasize that the students should become flexible in using the formal symbols of integers either indexically or non-indexically. A non-indexical use of integer symbols would mean explicit reference solely to *pluses* and *minuses* (i.e. +2, –3 etc). Therefore, the compensation strategy should be constructed only based on the formal symbols of integers, excluding the pre-symbolic signs of yellow and red team points.

In order to target *non-indexical symbolic generalization*, we encouraged students to articulate the symbols on the dice as "+" (*plus*) and "–" (*minus*), in an attempt to facilitate the connection of the verbalization *plus/minus* to the symbolic signs +/–. Though in the beginning most students needed to be reminded to use the "proper" names of the signs, by the time the students had played game 4 for a while they were able to refer to integers in a formal manner, as can be seen in the following examples of student verbalizations. We believe that the introduction of the *add/sub die* in game 4 obliged the students to refer correctly to the integers with their formal names, so as to be able to perform the actions of addition and subtraction on these symbols. For example (brackets added), Fay said: *add [minus* 3*]*, *subtract [*2 *of the minuses];* Zenon said: *add [*2 *to the pluses]*; Jackie said: *add [minus* 2*]*. Umar was still struggling with the verbalization and sometimes said *[minus* 1*] add* or *add [subtract* 2*]* etc.

Finally, we checked if students had spontaneously produced a more general verbalization in a form like "if you can't add pluses/minuses, you can subtract minuses/ pluses" or the other way around. In this group, such a generalization did not take place. We believe, however, that this will not necessarily be the case for other groups of students, and indeed that it may be desirable to encourage this in the teaching.

The Semiotic Role of the Abacus Model

As may be clear by now, the abacus model and the RME context of the dice games are very significant for the reification of integers and the instruction of integer addition and subtraction through the dice games method. Up to now we have referred to the semiotic processes, but we have not referred to the abacus model: though Radford's theory of objectification has been crucial in the analyses so far, we contend it needs to be complemented by an analysis of the role of the abacus in affording these semiotics. We claim that analysing the contribution of the model in students' semiosis will afford some primary discussion of phenomena such as (i) the embodiment of semiotic activity, (ii) the incompleteness of the contextual generalization and (iii) the direct transition from factual to symbolic generalization.

The abacus model in the games seems in many ways to be the centre of the activity: the abacus is in the centre of a 'circle of attention', as we are all sitting around the abacuses (see figure 2 again); it affords the representation of the yellow and red team points through their red and yellow cubes; it is the constant point of reference about which team is ahead. It was only natural that the abacus, being in the centre of the spatial arrangement and credited with allowing the students to keep the score, became the focus of semiotic activity. What is even more important: the abacus mediated in some cases the semiotic activity.

This can be seen in several features of the games. To begin with, the team points referred to the above episodes as "points for the yellow/red" (or as "yellow/red points") were concretized or 'objectified' from the start: they were yellow and red cubes. That is, *the points were embodied into the cubes*. This allows, as Linchevski and Williams (1999) point out, for integers to be introduced in the discourse as objects from the very beginning: the students speak about the general categories of yellow and red points from the beginning. Additionally, the *directed difference was embodied on the abacus*, as the difference of yellow and red points can be seen with a glance at the abacus, and the sign is evidently that of the larger pile of cubes: i.e. in figure 1 the yellows on that abacus are 2 points ahead. This convenient reference to the directed difference in the two piles of cubes afforded the association of semiotic activity to it, which made the establishment of the compensation strategy possible. Such semiotic activity is Fay's gesture in episode 1 in which the movements of her hands were matched with a verbal manipulation of the difference of the two piles of cubes (i.e. "you're going back down") to show that the directed difference remained the same. Also Jackie's quick movement of her hands up and down in episode 2 again indicates the difference in the two piles of cubes, in other words it points to the directed difference as it is embodied on the abacus. We suggest that this embodiment is crucial because it mediates the emergence of deictic semiotic activity such as that of Fay and Jackie in episodes 1 and 2 and hence allows the objectification to take place. We may even consider the toothpick figures in Radford (2003) to afford the same role.

As we noted above, the embodiment of the yellow and red team points through the cubes allowed the introduction of points for the yellow and points for the red as general abstract categories. We mentioned earlier that the students did not complete the contextual generalization of the compensation strategy to produce a generalization like "if you can't add a number of yellow/red points, you can subtract the same number of red/yellow points instead". However, the embodiment of the yellow and red team points of the abacus had already introduced generic situated objects into the discourse, even though this was not achieved through language. Consequently, the students could obviously see that the operational scheme of the factual generalization can be applied for any number of points for a team. This is an additional reason to the one presented earlier for the incompleteness of contextual generalization. Hence, this embodiment of the team points in a sense shapes the semiotic activity in the compensation strategy, providing one more reason why the completeness of contextual generalization was unnecessary in this case.

Further, the semiotic role of the abacus was crucial in the direct transition from factual to symbolic generalization. As we have seen in episode 4, the yellow points became "plus" and the red points are now "minuses". We say that this direct transition was afforded through the construction of a *chain of signification* (Gravemeijer et al., 2000; Walkerdine, 1988), in the form of a transition from the embodiment of yellow and red points through the abacus cubes to the embodiment of positive and negative integers. As a consequence of this transition, the formal symbols could be embedded into the operational scheme for the compensation strategy established through the factual generalization. Quite naturally then, the embedding of the formal symbols in the operational scheme performed on the abacus produced the symbolic generalization directly from factual generalization.

Conclusion

Beginning with a presentation of the OPLE methodology and the dice games instruction, we argued the need for a finer grained, semiotic analysis of objectifications to explain how reification is accomplished.

We have applied Radford's theory of objectification to fill this gap in understanding the case of the compensation strategy, a vital link in the chain of significations necessary to OPLE's success: thus we were able to identify relevant objectifications applying Radford's semiotic categories of generalisation. This work began to reveal the significance of the abacus itself, which affords, and indeed shapes the semiosis in essential ways. We have also shown how the effectiveness of the pedagogy based on OPLE can be explained as semiotic chaining using multiple semiotic objectifications and begun to discuss the significance of models and modelling in the dice games, and hence in the OPLE methodology. Finally, we suggest that our discussion over the semiotics of the abacus model might be the route to understanding the significance of models and modelling in the RME tradition more generally. We suggest that the role of the abacus as a model in this case might be typical of other models in RME. Indeed, Williams & Wake (in press) provide an analysis of the role of the number line in a similar vein.

In applying Radford's theory in a very different context we are bound to point out certain differences in the two cases: for instance, the differing roles of contextual generalisation in the two cases. Though the adaptation of the theory was necessary at some points, we have shown that this theory can be a powerful tool of analysis. The question arises as to whether the instruction method adopted here gains or perhaps loses something by eliding contextual generalisation: thus we suggest that Radford's categories might in fact be regarded as raising design-related issues as well as providing tools of analysis.

References

Cobb, P. (1994). Where Is the Mind? Constructivist and Sociocultural Perspectives on Mathematical Development. *Educational Researcher*, *23*(7), 13-20.

Cobb, P., Gravemeijer, K., Yackel, E., McClain, K., & Whitenack, J. (1997). Mathematizing and Symbolizing: The Emergence of Chains of Signification in One First-Grade Classroom. In D. Kirshner & J. A. Whitson (Eds.), *Situated Cognition. Social, Semiotic, and Psychological Perspectives* (pp. 151-233). Mahwah, N.J.: Lawrence Erlbaum Associates.

Dirks, M. K. (1984). The integer abacus. *Arithmetic Teacher*, *31*(7), 50–54.

Goodson-Espy, T. (1998). The roles of reification and reflective abstraction in the development of abstract thought: transitions from arithmetic to algebra. *Educational Studies in Mathematics*, *36*(3), 219-245.

Gravemeijer, K. (1997a). Instructional design for reform in mathematics education. In K. P. E. G. M. Beishuizen, and E.C.D.M. van Lieshout (Ed.), *The Role of Contexts and Models in the Development of Mathematical Strategies and Procedures* (pp. 13-34). Utrecht: CD-ß Press.

Gravemeijer, K. (1997b). Mediating Between Concrete and Abstract. In T. Nunez & P. Bryant (Eds.), *Learning and Teaching in Mathematics. An International Perspective* (pp. 315-345). Sussex, UK: Psychology Press.

Gravemeijer, K. (1997c). Solving word problems: A case of modelling? *Learning and Instruction*, *7*(4), 389-397.

Gravemeijer, K., Cobb, P., Bowers, J., & Whitenack, J. (2000). Symbolising, Modelling and Instructional Design. Perspectives on Discourse, Tools and Instructional Design. In

P. Cobb, E. Yackel & K. McClain (Eds.), *Symbolizing and Communicating in Mathematics Classrooms* (pp. 225-273). Mahwah, NJ: Lawrence Erlbaum Associates.

Koukkoufis, A., & Williams, J. (2005). *Integer Operations in the Primary School: A Semiotic Analysis of a «Factual Generalization»*. Paper presented at the British Society for Research in to the Learning of Mathematics Day Conference November 2005, St. Martin's College, Lancaster.

Liebeck, P. (1990). Scores and forfeits – an intuitive model for integer arithmetic. *Educational Studies in Mathematics*, *21*(3), 221–239.

Linchevski, L., & Williams, J. (1999). Using intuition from everyday life in 'filling' the gap in children's extension of their number concept to include the negative numbers. *Educational Studies in Mathematics, 39*(1-3), 131-147.

Lytle, P. A. (1994). *Investigation of a model based on neutralization of opposites to teach integers*. Paper presented at the Nineteenth International Group for the Psychology of Mathematics Education, Universidade Federal de Pernambuco, Recife, Brazil.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the Learning of Mathematics*, *22*(2), 14-23.

Radford, L. (2003). Gestures, speech, and the sprouting of signs. *Mathematical Thinking and Learning*, *5*(1), 37-70.

Radford, L. (2005). The semiotics of the schema. Kant, Piaget, and the Calculator. In M. H. G. Hoffmann, J. Lenhard & F. Seeger (Eds.), *Activity and Sign. Grounding Mathematics Education* (pp. 137-152). New York: Springer.

Sfard, A. (1991). On the dual nature of mathematical conceptions: Reflections on processes and objects as different sides of the same coin. *Educational Studies in Mathematics*, *22*, 1-36.

Sfard, A., & Linchevski, L. (1994). The gains and pitfalls of reification: the case of algebra. *Educational Studies in Mathematics*, *26*, 87–124.

Walkerdine, V. (1988). *The mastery of reason*. London: Routledge.

Williams, J. & Wake, G. (in press). Metaphors and models in translation between College and workplace mathematics. *Educational Studies in Mathematics.*

Andreas Koukkoufis
School of Education
The University of Manchester
UK

E-mail: andreas_koukkoufis@hotmail.com

Julian Williams
School of Education
The University of Manchester
UK

E-mail: julian.williams@manchester.ac.uk

Objetos, significados, representaciones semióticas y sentido

Bruno D'Amore [1]

RESUMEN

En este artículo intento mostrar una consecuencia que algunas veces se evidencia en las transformaciones semióticas de tratamiento y conversión de una representación semiótica a otra, cuyo sentido deriva de una práctica compartida. El pasaje de la representación de un objeto matemático a otra, por medio de transformaciones, de una parte conserva el significado del objeto mismo, pero, en ocasiones, puede cambiar su sentido. Este hecho está aquí detalladamente evidenciado por medio de un ejemplo, pero insertándolo en el seno de un amplio marco teórico que pone en juego los objetos matemáticos, sus significados y sus representaciones.

- **PALABRAS CLAVE**: Registros semióticos, sentido de un objeto matemático, objeto matemático, cambio de sentido.

ABSTRACT

In this paper, I want to illustrate a phenomenon related to the treatment and conversion of semiotic representations whose sense derives from a shared practice. On the one hand, the passage from one representation of a mathematical object to another, through transformations, maintains the meaning of the object itself, but on the other hand, sometimes can change its sense. This is shown in detail through an example, inserted within a wide theoretical framework that takes into account mathematical objects, their meanings and their representations.

- **KEY WORDS:** Semiotic registers, sense of a mathematical object, mathematical object, change of sense.

Fecha de recepción: Enero de 2006/ Fecha de aceptación: Marzo de 2006

[1] NRD, Departamento de Matemática, Universidad de Bolonia, Italia. Facultad de Ciencia de la Formación, Universidad de Bolzano, Italia. Alta Escuela Pedagógica, Locarno, Suiza. MESCUD, Universidad Distrital "F. José de Caldas", Bogotá, Colombia

RESUMO

Neste artigo intento mostrar uma conseqüência que algumas vezes se evidencia nas transformações semióticas de tratamento e conversão de uma representação semiótica a outra, cujo sentido deriva de uma pratica compartida. A passagem da representação de um objeto matemático a outra, por meio de transformações, de uma parte conserva o significado do objeto mesmo, mas, em ocasiões, pode mudar seu sentido. Este fato está aqui detalhadamente evidenciado por meio de um exemplo, pero inserindo-o em um amplo marco teórico que trabalha os objetos matemáticos, seus significados e suas representações.

- **PALAVRAS CHAVES:** Registros semióticos, sentido de um objeto matemático, objeto matemático, mudança de sentido.

RÉSUMÉ

Dans cet article, je montre un phénomène relié au traitement et à la conversion des représentations sémiotiques dont le sens provient de pratiques partagées. D'une part, le passage de la représentation d'un objet mathématique à une autre représentation, à travers des transformations, conserve le sens de l'objet lui-même. D'autre part, ce passage peut entraîner quelquefois une modification du sens. Ce phénomène est ici mis en évidence à travers un exemple inséré dans un cadre théorique ample qui met en jeu les objets mathématiques, leurs significations et leurs représentations.

- **MOTS CLÉS:** Registre sémiotique, sens d´ un objet mathématique,objet mathématique, changement de sens.

Este trabajo está dividido en dos partes. En la primera parte se discuten aspectos de carácter epistemológico, ontológico y semiótico desarrollados en algunos marcos teóricos de investigación en didáctica de la matemática.

En la segunda, a través de la narración de un episodio de sala de clase, se propone una discusión sobre la atribución de sentidos diversos de varias representaciones semióticas en torno a un mismo objeto matemático.

Primera parte

1. Un recorrido

1.1. *Ontología y conocimiento*

En diversos trabajos de finales de los años 80 y 90 se declaraba que, mientras el matemático puede no interrogarse sobre el *sentido* de los *objetos matemáticos* que usa o sobre el sentido que tiene el *conocimiento* *matemático*, la didáctica de la matemática no puede obviar dichas cuestiones (ver D'Amore, 1999, pp. 23-28). En un trabajo reciente, Radford resume la situación de la manera siguiente:

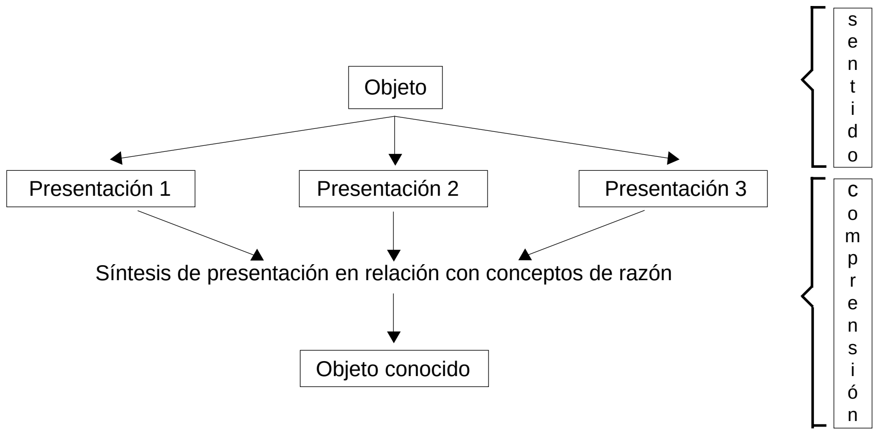
> Se puede sobrevivir muy bien haciendo matemática sin adoptar una ontología explícita, esto es, una teoría sobre la naturaleza de los objetos matemáticos. Es por eso que es casi imposible inferir de un artículo técnico en matemáticas la posición ontológica de su autor. (...) La situación es profundamente diferente cuando hablamos del *saber matemático*. (…) Cuestiones teóricas acerca del contenido de ese saber y de la manera como dicho contenido es transmitido, adquirido o construido nos ha llevado a un punto en el que no podemos seguir evitando hablar seriamente de ontología. (Radford, 2004, p. 6)

El debate es antiguo y se puede señalar como punto de partida la Grecia clásica. Como he señalado en trabajos anteriores, dicho debate está enmarcado por una creencia ontológica que parte del *modo* que tienen los seres humanos de *conocer* los conceptos (D'Amore, 2001a,b; 2003a,b). Radford retoma el debate y se detiene, en particular, en el trabajo de Kant quien dice que los individuos tienen un conocimiento conceptual *a priori* gracias a una actividad autónoma de la mente, independiente del mundo concreto (Radford, 2004, pp. 5-7).

Como Radford pone en evidencia, el apriorismo kantiano tiene raíces en la interpretación de la filosofía griega hecha por San Agustín y su influencia en los pensadores del Renacimiento. Refiriéndose al matemático Pietro Catena (1501-1576), por mucho tiempo profesor de la Universidad de Padua y autor de la obra *Universa Loca* (Catena, 1992), Radford afirma que, para Catena, "los objetos matemáticos eran entidades ideales e innatos" (Radford, 2004, p. 10). El debate se vuelve moderno, en todo el sentido de la palabra, cuando, con Kant, se logra hacer la distinción entre los "conceptos del intelecto" (humano) y los "conceptos de objetos". Como Radford observa:

> [Estos] conceptos del intelecto puro no son conceptos de objetos; son más bien esquemas lógicos sin contenido; su función es hacer posible un reagrupamiento o *síntesis* de las intuiciones. La síntesis es llevada a cabo por aquello que Kant identificó como una de nuestras facultades cognitivas: el entendimiento. (Radford, 2004, p. 15)

El siguiente gráfico presenta las ideas de *sentido* y de *comprensión* en el lugar adecuado:



La relación entre los sentidos y la razón en la epistemología Kantiana (tomado de Radford, 2004, p. 15)

1.2. *Aproximación antropológica*

La línea de investigación antropológica parece fundamental en la comprensión del pensamiento matemático (D'Amore, 2003b). Dicha línea de investigación debe atacar ciertos problemas, entre ellos el del uso de signos y artefactos en la cultura. En la aproximación antropológica al pensamiento matemático que propone Radford, el autor sugiere que

> una aproximación antropológica no puede evitar tomar en cuenta el hecho de que el empleo que hacemos de las diversas clases de signos y artefactos cuando intentamos llegar a conocer algo está subsumido en prototipos culturales de uso de signos y artefactos. (...) Lo que es relevante en este contexto es que el uso de signos y artefactos alteran la manera en que los objetos conceptuales nos son dados a través de nuestros sentidos (…) Resumiendo, desde el punto de vista de una epistemología antropológica, la manera en que me parece que puede resolverse el misterio de los objetos matemáticos es considerando dichos objetos como patrones (*patterns*) fijados de actividad humana; incrustados en el dominio continuamente sujeto a cambio de la práctica social reflexiva mediatizada. (Radford, 2004, p. 21).

En esta línea de pensamiento, existe una aceptación general de consenso:

> Los objetos matemáticos deben ser considerados como símbolos de unidades culturales, emergentes de un sistema de usos ligados a las actividades de resolución de problemas que realizan ciertos grupos de personas y que van evolucionando con el tiempo. En nuestra concepción, es el hecho de que en el seno de ciertas instituciones se realizan determinados tipos de prácticas lo que determina la emergencia progresiva de los "objetos matemáticos" y que el "significado" de estos objetos esté íntimamente ligado con los problemas y a la actividad realizada para su resolución, no pudiéndose reducir este significado del objeto a su mera definición matemática. (D'Amore & Godino, 2006, p. 14).

1.3. *Sistema de prácticas*

Tal acuerdo viene ulteriormente clarificado por proposiciones explícitas:

> La noción de "significado institucional y personal de los objetos matemáticos" implica a las de "práctica personal", "sistema de prácticas personales", "objeto personal (o mental)", herramientas útiles para el estudio de la "cognición matemática individual" (Godino & Batanero, 1994; 1998). Cada una de tales nociones tiene su correspondiente versión institucional. Es necesario aclarar que con estas nociones se trata de precisar y hacer operativa la noción de "relación personal e institucional al objeto" introducida por Chevallard (1992). (D'Amore & Godino, 2006, p. 28)

Aquello que nosotros entendemos por "sistema de prácticas personales" está en la misma línea de la aproximación semiótica antropológica (ASA) de Radford:

> En la aproximación semiótica antropológica (ASA) a la que estamos haciendo referencia, la

idealidad del objeto conceptual está directamente ligada al contexto histórico-cultural. La idealidad de los objetos matemáticos es decir de aquello que los vuelve generales es completamente tributaria de la actividad humana. (Radford, 2005, p. 200).

Los aspectos sociológicos de esta adhesión a la actividad humana y a la práctica social son así confirmados:

Considero que el aprendizaje matemático de un objeto O por parte de un individuo I en el seno de la sociedad S no sea más que la adhesión de I a las prácticas que los otros miembros de S desarrollan alrededor del objeto dado O. (D'Amore, en D'Amore, Radford & Bagni, 2006, p. 21)

De igual manera, "la práctica de sala de clase puede considerarse como un sistema de adaptación del alumno a la sociedad" (Radford, en D'Amore, Radford & Bagni, 2006, p. 27).

1.4. *Objeto y objeto matemático*

Se necesita, sin embargo, dar una definición de este "objeto matemático". Para lograrla preferimos recurrir a una generalización de la idea de Blumer sugerida por (Godino, 2002): *Objeto matemático* es todo lo que es indicado, señalado, nombrado cuando se construye, se comunica o se aprende matemáticas. Esta idea es tomada de Blumer (Blumer 1969, ed. 1982, p. 8): un objeto es "cualquier entidad o cosa a la cual nos referimos, o de la cual hablamos, sea real, imaginaria o de cualquier otro tipo.

En un trabajo anterior hemos sugerido considerar los siguientes tipos de objetos matemáticos:

- "lenguaje" (términos, expresiones, notaciones, gráficos, ...) en sus diversos registros (escrito, oral, gestual, ...)
- "situaciones" (problemas, aplicaciones extra-matemáticas, ejercicios, ...)
- "acciones" (operaciones, algoritmos, técnicas de cálculo, procedimientos, ...)
- "conceptos" (introducidos mediante definiciones o descripciones) (recta, punto, número, media, función, ...)
- "propiedad o atributo de los objetos" (enunciados sobre conceptos, ...)
- "argumentos" (por ejemplo, los que se usan para validar o explicar los enunciados, por deducción o de otro tipo, ...).

A su vez estos objetos se organizan en entidades más complejas: sistemas conceptuales, teorías,... (D'Amore & Godino, 2006, p. 28-29).

En el trabajo citado, se aprovecha la idea de función semiótica:

se dice que se establece entre dos objetos matemáticos (ostensivos o no ostensivos) una función semiótica cuando entre dichos objetos se establece una dependencia representacional o instrumental, esto es, uno de ellos se pone en el lugar del otro o uno es usado por otro. (D'Amore & Godino, 2006, p. 30).

Y, más allá:

Los objetos matemáticos que intervienen en las prácticas matemáticas y los emergentes de las mismas, según el juego de lenguaje en que participan, pueden ser considerados desde las siguientes facetas o dimensiones duales:

- personal – institucional: como ya hemos indicado, si los sistemas de prácticas son compartidos en el seno de una institución, los objetos emergentes se consideran "objetos institucionales"; mientras que si estos sistemas son específicos de una persona los consideramos como "objetos personales";
- ostensivos (gráficos, símbolos, ...), no ostensivos (entidades que se evocan al hacer matemáticas, representados en forma textual, oral, gráfica, gestual, ...);
- extensivo – intensivo: esta dualidad responde a la relación que se establece entre un objeto que interviene en un juego de lenguaje como un caso particular (un ejemplo *concreto*: la función $y=2x+1$) y una clase más general o *abstracta* (la familia de funciones, $y = mx+n$);
- elemental – sistémico: en algunas circunstancias los objetos matemáticos participan como entidades unitarias (que se suponen son conocidas previamente), mientras que otras intervienen como sistemas que se deben descomponer para su estudio;
- expresión – contenido: antecedente y consecuente (significante, significado) de cualquier función semiótica.

Estas facetas se presentan agrupadas en parejas que se complementan de manera dual y dialéctica. Se consideran como atributos aplicables a los distintos objetos primarios y secundarios, dando lugar a distintas "versiones" de dichos objetos. (D'Amore & Godino, 2006, p. 31).

Pero, si se hace referencia a la práctica de representación lingüística: "Creo que se deben distinguir dos tipologías de objetos en el ámbito de la creación de la competencia matemática (aprendizaje matemático): el objeto matemático mismo y el objeto lingüístico que lo expresa" (D'Amore, en D'Amore, Radford & Bagni, 2006, p. 21).

En los siguientes partes de este articulo, será discutido lo referente a la representación, de forma especifica.

1.5. *Aprendizaje de objetos*

En los intentos hechos por sintetizar las dificultades en el aprendizaje de conceptos (D'Amore, 2001a, b, 2003a) he recurrido en varias ocasiones a la idea que se encuentra en la *paradoja de* Duval (1993):

> de una parte, el aprendizaje de los objetos matemáticos no puede ser más que un aprendizaje conceptual y, de otra, es sólo por medio de representaciones semióticas que es posible una actividad sobre los objetos matemáticos. Esta paradoja puede constituir un verdadero círculo vicioso para el aprendizaje. ¿Cómo sujetos en fase de aprendizaje no podrían no confundir los objetos matemáticos con sus representaciones semióticas si ellos sólo pueden tener relación con las representaciones semióticas? La imposibilidad de un acceso directo a los objetos matemáticos, fuera de toda representación semiótica, vuelve la confusión casi inevitable. Y, por el contrario, ¿cómo pueden ellos adquirir el dominio de los tratamientos matemáticos, necesariamente ligados con las representaciones semióticas, si no tienen ya un aprendizaje conceptual de los objetos representados? Esta paradoja es aún más fuerte si se identifican actividades matemáticas y actividades conceptuales y si se consideran las representaciones semióticas como secundarias o extrínsecas. (Duval, 1993, p. 38)

Estas frases reclaman fuertemente no solamente un cierto modo de concebir la idea de semiótica sino también su relación con la epistemología. Como apunta Radford: "El problema epistemológico puede resumirse en la siguiente pregunta: ¿cómo llegamos a conocer los objetos generales, dado que no tenemos acceso a éstos sino a través de representaciones que nosotros mismos nos hacemos de ellos?" (Radford, 2005, p. 195).

1.6. *La representación de los objetos*

A propósito de la representación de los objetos, Radford menciona que

> En una célebre carta escrita el 21 de febrero de 1772, Kant pone en duda el poder de nuestras representaciones. En esta carta, enviada a Herz, Kant dice: "¿sobre qué fundamento reposa la relación de lo que llamamos representación y objeto correspondiente?". En esa carta, Kant cuestiona la legitimidad que tienen nuestras representaciones para representar fielmente al objeto. En términos semióticos, Kant cuestiona la adecuación del signo. (…) La duda kantiana es de orden epistemológico. (Radford, 2005, p. 195)

Todo esto pone en juego, de forma particular, la idea de signo, dado que para la matemática esta forma de representación es específica; el signo es de por sí especificación de lo particular, pero esto puede ser interpretado dando sentido a lo general; al respecto Radford nota que: "Si el matemático tiene derecho a ver lo general en lo particular, es, como observa Daval (1951, p. 110) 'porque está seguro de la fidelidad del signo. El signo es la representación adecuada del significado (*signifié*)' ". (Radford, 2005, p. 199).

Pero los signos son artefactos, objetos a su vez "lingüísticos" (en sentido amplio), términos que tienen el objetivo de representar para indicar:

> [La] objetivación es un proceso cuyo objetivo es mostrar algo (un objeto) a alguien. Ahora bien, ¿cuáles son los medios para mostrar el objeto? Esos medios son los que llamo *medios semióticos de objetivación*. Estos son objetos, artefactos, términos lingüísticos y signos en general que se utilizan con el fin de volver aparente una intención y de llevar a cabo una acción. (Radford, 2005, p. 203)

Estos signos tienen múltiples papeles, sobre los cuales no entro en detalle para evitar grandes tareas que ligan signo - cultura - humanidad: "la entera cultura es considerada como un sistema de signos en los cuales el significado de un significante se vuelve a su vez significante de otro significado o de hecho el significante del propio significado". (Eco, 1973, p. 156)

No último en importancia, es el "papel cognitivo del signo" (Wertsch, 1991; Kozoulin, 1990; Zinchenko, 1985) sobre el cual no profundizo con el fin de abreviar, pero, no sin antes reconocerlo, en las bases mismas de la semiótica general: "*todo proceso de significación entre seres humanos (…) supone un sistema de significaciones como propia condición necesaria*" (Eco, 1975, p. 20; el cursivo es del Autor), lo que quiere decir un acuerdo cultural que codifica e interpreta; es decir, produce conocimiento.

La elección de los signos, también y básicamente cuando se componen en lenguajes, no es neutra o independiente; esta elección señala el destino en el cual se expresa el pensamiento, el destino de la comunicación; por ejemplo:

> El lenguaje algebraico impone una sobriedad al que piensa y se expresa, una sobriedad en los modos de significación que fue impensable antes del Renacimiento. Impone lo que hemos llamado en otro trabajo una *contracción semiótica*. Presupone también la pérdida del *origo*. (Radford, 2005, p. 210)

La pérdida del *origo* (es decir del origen, del inicio) fue discutida por Radford también en otros trabajos (2000, 2002, 2003).

Y es propio sobre *este punto* que se cierra mi larga premisa, que es también el punto de partida para lo que sigue.

Segunda parte

2. Objeto, su significado compartido, sus representaciones semióticas: la narración de un episodio

2.1. *El episodio*

Estamos en quinto de primaria y el docente ha desarrollado una lección en situación a-didáctica sobre los primeros elementos de la probabilidad, haciendo construir a los alumnos, por lo menos a través de unos ejemplos, la idea de "evento" y de "probabilidad de un evento simple". Como ejemplo, el docente ha hecho uso de un dado normal de seis caras, estudiando los resultados casuales desde un punto de vista estadístico. Emerge una probabilidad frecuencial, pero que es interpretada en sentido clásico. En este punto el docente propone el siguiente ejercicio:

Calcular la probabilidad del siguiente evento: lanzando un dado se obtenga un número par.

Los alumnos, discutiendo en grupo y básicamente compartiendo prácticas bajo la dirección del docente, alcanzan a decidir que la respuesta se expresa con la fracción $\frac{3}{6}$ porque "los resultados posibles al lanzar un dado son 6 (el denominador) mientras que los resultados que hacen verdadero el evento son 3 (el numerador)".

Después de haber institucionalizado la construcción de este saber, satisfecho de la eficaz experiencia, contando con que este resultado fue obtenido más bien rápidamente y con el hecho de que los alumnos han demostrado gran habilidad en el manejo de las fracciones, el docente propone que, dada la equivalencia de $\frac{3}{6}$ y $\frac{50}{100}$, se puede expresar esta probabilidad también con la escritura 50%, que es mucho más expresiva: significa que se tiene la mitad de la probabilidad de verificarse el evento respecto al conjunto de los eventos posibles, tomado como 100. Alguno de los alumnos nota que "entonces es válida también [la fracción] $\frac{1}{2}$"; la propuesta es validada a través de las declaraciones de quien hace la propuesta, rápidamente es acogida por todos y, una vez más, institucionalizada por el docente.

2.2. *Análisis semiótico*

Si se analizan las representaciones semióticas diferentes que han emergido en esta actividad, relativas al mismo evento: "obtener un número par al lanzar un dado", son encontradas, por lo menos, las siguientes:

- registro semiótico lengua natural: probabilidad de obtener un número par al lanzar un dado

- registro semiótico lenguaje de las fracciones: $\frac{3}{6}$, $\frac{50}{100}$, $\frac{1}{2}$
- registro semiótico lenguaje del porcentaje: 50%.

2.3. *El sentido compartido por diversas representaciones semióticas*

Cada una de las representaciones semióticas precedentes es el significante "aguas abajo" del mismo significado "aguas arriba" (Duval, 2003). El "sentido" compartido a propósito de aquello que se estaba construyendo estaba presente idénticamente y por tanto la práctica matemática efectuada y así descrita ha llevado a transformaciones semióticas cuyos resultados finales fueron fácilmente aceptados:

- conversión: entre la representación semiótica expresada en el registro lenguaje natural y $\frac{3}{6}$
- tratamiento: entre $\frac{3}{6}$, $\frac{50}{100}$ y $\frac{1}{2}$
- conversión: entre $\frac{50}{100}$ y 50%.

2.4. *Conocimientos previos necesarios*

Entran en juego diversos conocimientos, aparentemente cada uno de estos bien construido, que interactúan entre ellos:

- conocimiento y uso de las fracciones
- conocimiento y uso de los porcentajes
- conocimiento y uso del evento: obtener un número par lanzando un dado.

Cada uno de estos conocimientos se manifiesta a través de la articulación en un todo unitario y la aceptación de las prácticas en el grupo clase.

2.5. *Continuación del episodio: la pérdida del sentido compartido a causa de transformaciones semióticas*

Terminada la sesión, se propone a los alumnos la fracción $\frac{4}{8}$ y se pide si, siendo equivalente a $\frac{3}{6}$, también esta fracción representa el evento explorado poco antes. *La respuesta unánime y convencida fue negativa.* El mismo docente, que antes había dirigido con seguridad la situación, afirma que " $\frac{4}{8}$ *no puede* representar el evento porque las caras de un dado son 6 y no 8". El investigador pide al docente de explicar bien su pensamiento al respecto; el docente declara entonces que "existen no sólo dados de 6 caras, sino también dados de 8 caras; en tal caso, y sólo así, la fracción $\frac{4}{8}$ representa el resultado obtener un número par al lanzar un dado".

Examinaré lo que está sucediendo en el aula desde un punto de vista semiótico; pero me veo obligado a generalizar la situación.

3. Un simbolismo para las bases de la semiótica

En esta parte, son utilizadas las definiciones usuales y de la simbología introducida en otros trabajos (D'Amore, 2001a, 2003a,b):

semiótica $=_{df}$ representación realizada por medio de signos noética $=_{df}$ adquisición conceptual de un objeto[2].

Se indica, de ahora en adelante:

r^m $=_{df}$ registro semiótico m-ésimo

$R^m_i(A)$ $=_{df}$ representación semiótica i-ésima de un concepto A en el registro semiótico r^m $(m = 1, 2, 3, \ldots; i = 1, 2, 3, \ldots)$.

Se puede notar que, si cambia el registro semiótico, cambia necesariamente la representación semiótica, mientras que no es posible asegurar lo contrario; es decir, puede cambiar la representación semiótica manteniéndose aún el mismo registro semiotico.

Uso un gráfico para ilustrar la situación, porque me parece mucho más eficaz[3]:

características de la semiótica $\left\{ \begin{array}{l} \textit{representación} \\ \textit{tratamiento} \\ \textit{conversión} \end{array} \right.$ (implican actividades cognitivas diversas)

concepto A para representar

$\Downarrow$

elección de los rasgos distintivos de A

$\Downarrow$

REPRESENTACIÓN de A: $R^m_i(A)$ en un registro semiótico dado r^m

$\Downarrow$ transformaciones de representación *TRATAMIENTO*

nueva representación $(i \neq j): R^m{}_j(A)$ en el *mismo* registro semiótico r^m

$\Downarrow$ transformación de registro *CONVERSIÓN*

nueva representación $(h \neq i, h \neq j): R^n{}_h(A)$ en *otro* registro semiótico r^n $(n \neq m)$ $(m, n, i, j, h = 1, 2, 3, \ldots)$

4. Volvamos al episodio

- Existe un objeto (significado) matemático O_1 por representar: probabilidad de obtener un número par al lanzar un dado;
- se le da un *sentido* derivado de la experiencia que se piensa aceptada, en una práctica social construida en cuanto compartida en el aula;
- se elige un registro semiótico r^m y en éste se representa O_1: $R^m_i(O_1)$;
- se realiza un tratamiento: $R^m{}_i(O_1) \rightarrow R^m{}_j(O_1)$;
- se realiza una conversión: $R^m{}_i(O_1) \rightarrow R^n{}_h(O_1)$;
- se interpreta $R^m_j(O_1)$ reconociendo en esto el objeto (significado) matemático O_2;
- se interpreta $R^n_h(O_1)$ reconociendo en esto el objeto (significado) matemático O_3.

¿Qué relación existe entre O_2, O_3 y O_1?

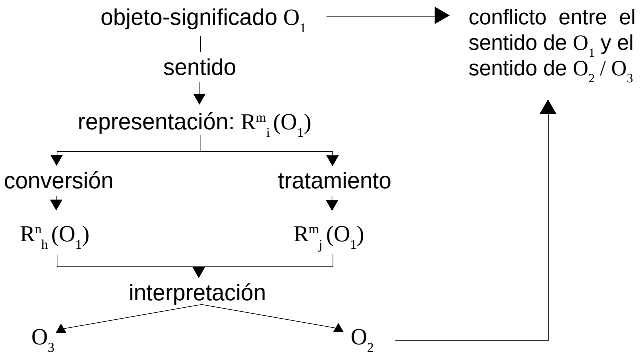
Se puede reconocer identidad; y esto significa entonces que existe un conocimiento previo, en la base sobre la cual la identidad puede ser establecida.

2 Para Platón, la noética es el acto de concebir a través del pensamiento; para Aristóteles, es el acto mismo de comprensión conceptual.

3 Hago referencia a Duval (1993).

De hecho, se puede no reconocer la identidad, en el sentido que la "interpretación" es o parece ser diferente, y entonces se pierde el *sentido* del objeto (significado) de partida O_1.

Un esquema como el siguiente puede resumir lo que ha sucedido en el aula desde un punto de vista complejo, que pone en juego los elementos que se desea poner en conexión entre ellos: objetos, significados, representaciones semióticas y sentido:



En el ejemplo aquí discutido:

- objeto - significado O_1: "probabilidad de obtener un número par al lanzar un dado";
- sentido: la experiencia compartida como práctica de aula en situación a-didáctica y bajo la dirección del docente, lleva a considerar que el sentido de O_1 sea el descrito por los alumnos y deseado por el docente: tantos resultados posibles y, respecto a estos, tantos resultados favorables al verificarse el evento;
- elección de registro semiótico r^m: números racionales Q expresados bajo forma de fracción ; representación: $R^m_i(O_1)$: $\frac{3}{6}$;
- tratamiento: $R^m{}_i(O_1) \rightarrow R^m{}_j(O_1)$, es decir, de $\frac{3}{6}$ a $\frac{1}{2}$;
- tratamiento: $R^m{}_i(O_1) \rightarrow R^m{}_k(O_1)$, es decir, de $\frac{3}{6}$ a $\frac{4}{8}$;
- conversión: $R^m{}_i(O_1) \rightarrow R^n{}_k(O_1)$,es decir, de $\frac{3}{6}$ a 50%;
- se interpreta $R^m_j(O_1)$ reconociendo en esto el objeto (significado) matemático O_2;
- se interpreta $R^m_k(O_1)$ reconociendo en esto el objeto (significado) matemático O_3;
- se interpreta $R^n_h(O_1)$ reconociendo en esto el objeto (significado) matemático O_4.

¿Qué relación existe entre O_2, O_3, O_4 y O_1?

En algunos casos (O_2, O_4), se reconoce identidad de significantes; y esto significa que existe de base un conocimiento ya construido que permite reconocer el mismo objeto; el *sentido* está compartido, es único;en otra situación (O_3), no se le reconoce la identidad de significante, en el sentido que la "interpretación" es o parece ser diferente, y entonces se pierde el *sentido* del objeto (significado) O_1.

La temática relativa a más representaciones del mismo objeto está presente en Duval (2005).

No está dicho que la pérdida de sentido se presente sólo a causa de la conversión; en el ejemplo aquí dado, tal como ya fue discutido, se presentó a causa de un tratamiento (el pasaje de $\frac{3}{6}$ a $\frac{4}{8}$).

La interpretación de $\frac{4}{8}$ dada por el docente no admitía como objeto plausible el mismo O_1 que había tomado origen del sentido compartido que había llevado a la interpretación $\frac{3}{6}$.

5. Otros episodios

En seguida, son propuestos algunos ejemplos de interpretación solicitados a estudiantes que están cursando los últimos semestres en la universidad, programa de matemática; aquellos indicados como "sentidos" son mayormente compartidos entre los estudiantes entrevistados:

1)

$$x^2 + y^2 + 2xy - 1 = 0 \xrightarrow{\text{TRATAMIENTO}} x + y = \frac{1}{x+y}$$

sentido: de "una circunferencia" a "una suma que tiene el mismo valor de su recíproca"; Investigador: "Pero, ¿es o no es una circunferencia?"; A: "Absolutamente no, una circunferencia debe tener $x^2 + y^2$ "; B: "Si se simplifica, ¡si!" [es decir, es la transformación semiótica de tratamiento que da o no cierto *sentido*]*;*

2)

$$n + (n+1) + (n+2) \Longrightarrow 3n + 3$$

sentido: de "la suma de tres naturales consecutivos" a "el triple de un número más 3"; Investigador: "Pero, ¿se puede pensar como suma de tres naturales consecutivos?"; C: "No, ¡no entra nada!";

3)

$$(n-1) + n + (n+1) \Longrightarrow 3n$$

sentido: de "la suma de tres enteros consecutivos" a "el triple de un número natural"; Investigador: "Pero, ¿se puede pensar como suma de tres enteros consecutivos?"; D: "No, así no, así es la suma de tres números iguales, es decir n ".

6. Representaciones de un mismo objeto dado por el docente de primaria, consideradas apropiadas para sus alumnos

En un curso de actualización para docentes de primaria, fue discutido el tema: *Primeros elementos de probabilidad*. Al final de la unidad, se pidió a los docentes representar el objeto matemático: "obtener un número par al lanzar un dado", usando un simbolismo oportuno que fuese el más apropiado, según ellos, a los alumnos de primaria. Fueron dadas a conocer todas las representaciones propuestas y se sometieron a votación. En seguida se muestran los resultados obtenidos en orden de preferencia (del mayor al menor):

$\frac{3}{6}$ 50% $\frac{1}{2}$ (tres y tres) $\frac{\bullet\bullet\bullet}{\text{ooo}}$

(tres sobre seis) $\frac{\text{ooo}}{6}$ (tres sobre seis) $\frac{\text{ooo}}{\bullet\bullet\bullet\bullet\bullet\bullet}$

(2, 4, 6 respecto de 1, 2, 3, 4, 5, 6) $^{2\cdot 4\cdot 6}/_{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$

(figural-operativa) $\boxed{1}$ $\underset{\text{OK}}{\boxed{2}\downarrow}$ $\boxed{3}$ $\underset{\text{OK}}{\boxed{4}\downarrow}$ $\boxed{5}$ $\underset{\text{OK}}{\boxed{6}\downarrow}$

La importancia de tomar en consideración el análisis de la producción de los alumnos es subrayada así por Duval (2003):

> No se puede subrayar la importancia de las descripciones, en la adquisición de conocimientos científicos así como en las primeras etapas de los aprendizajes matemáticos, sin afrontar otra cuestión fundamental tanto para la investigación como para los docentes: el análisis de las producciones de los alumnos. Pues es en el cuadro del desarrollo de la descripción, que se obtienen las producciones más personales y más diversificadas, dado que éstas pueden ser hechas verbalmente o con la ayuda de diseños, de esquemas ... En este caso se trata, para la investigación, de una cuestión metodológica y, para los docentes, de una cuestión diagnóstica. Veremos que cada análisis de las producciones de los alumnos requiere que se distinga con atención en cada producción semiótica, discursiva o no discursiva, *diversos niveles de articulación del sentido,* que no revelan las mismas operaciones. (p. 16)

En el ejemplo discutido en este parágrafo 6, los "alumnos" son docentes de escuela primaria que frecuentan el curso, mientras los "docentes" son los profesores universitarios que impartían las lecciones .[4]

Diversos pueden ser los análisis de las precedentes producciones de los alumnos - docentes, evidenciadas al inicio de este parágrafo, pero se prefiere seguir la bipartición que, de nuevo, se encuentra en Duval (2003):

> [N]o se debe confundir aquello que llamaremos una tarea "real" de descripción y una tarea "puramente formal" de descripción. (...). Una tarea de descripción es real cuando requiere una observación del objeto de la situación que se desea describir (...). Aquí, el alumno tiene acceso a cada uno de los dos elementos de la pareja {objeto, representación del objeto}, independientemente uno del otro. Al contrario, una tarea de descripción es puramente formal cuando se limita a un simple cambio de registro de representación: descripción verbal a partir de un diseño o de una "imagen" o viceversa. El alumno sólo tiene un acceso independiente al objeto representado. Las descripciones formales son entonces tareas de conversión que buscan respetar la invarianza de aquello que representan. (p. 19)

Creo que esta distinción de Duval ayuda a explicar, por lo menos en parte, el episodio narrado en los parágrafos 2 y 5 de este artículo:

4 Que este "cambio de rol" pueda ser concebido como plausible es ampliamente demostrado por la literatura internacional; por brevedad me limito a citar sólo el amplio panorama propuesto en el ámbito PME por Llinares & Krainer (2006), con abundante bibliografía específica.

> Respecto a un objeto matemático observable, conocido sobre la base de prácticas compartidas, la "descripción real" responde plenamente a las características del objeto, es decir de la práctica realizada alrededor de éste y con éste, y por tanto del *sentido* que todo esto adquiere por parte de quien dicha práctica explica. Pero el uso de transformaciones semióticas a veces lleva a cambios sustanciales de dichas descripciones, convirtiéndose en una "descripción puramente formal", obtenida con prácticas semióticas sí compartidas, pero que niegan un acceso al objeto representado o, mejor, le niegan la conservación del *sentido.* (Duval, 2003, p. 18)

7. Otros episodios semióticos tomados de la práctica matemática compartida en aula

7.1. *Probabilidad y fracciones*

He repetido el experimento descrito en el parágrafo 2, con estudiantes que han aprobado cursos más avanzados de matemática y con estudiantes en formación como futuros docentes de escuela primaria y de secundaria. Si la conversión que hace perder el sentido en el pasaje de tratamiento de $\frac{3}{6}$ a $\frac{4}{8}$ es un ejemplo fuerte de pérdida de sentido, lo es aún más el de pasar de $\frac{3}{6}$ a $\frac{7}{14}$; mientras lo es en menor medida la conversión de $\frac{3}{6}$ a 0.5.

7.2. *Un ejemplo en el primer año de escuela secundaria superior*

Objeto matemático: El gasto total de y\$ para el alquiler de algún instrumento por x horas a a\$ cada hora, más el costo fijo de b\$; los alumnos y el docente llegan a la representación semiótica: $y = ax + b$; se sigue la transformación de tratamiento que lleva a $x - \frac{y}{a} + \frac{b}{a} = 0$, que se representa como:

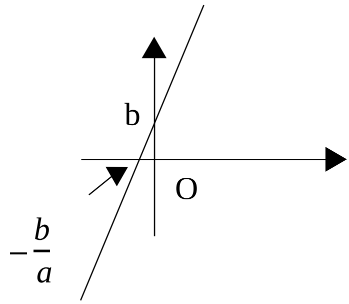


Figura 1

y es interpretada universalmente como "una recta".

Dicha representación semiótica obtenida por tratamiento y conversión, a partir de la representación inicial, no se le reconoce como el mismo objeto matemático de partida; ésta asume otro *sentido.*

7.3. *Un ejemplo en un curso para docentes de escuela primaria en formación*

Objeto matemático: La suma (de Gauss) de los primeros 100 números naturales positivos; resultado semiótico final después de sucesivos cambios operativos con algunas transformaciones de conversión y tratamiento: 101•50; esta representación no se reconoce como representación del objeto de partida; la presencia del signo de multiplicación dirige a los futuros docentes a buscar un sentido en objetos matemáticos en los cuales aparezca el término "multiplicación (o términos similares).

7.4. *Un primer ejemplo en un curso (postgrado) de formación para futuros docentes de escuela secundaria*

Objeto matemático: La suma de dos cuadrados es menor que 1; representación semiótica universalmente aceptada: $x^2+y^2<1$; después de cambios de representación semiótica, siguiendo operaciones de tratamiento: $(x+iy)(x-iy)<1$ y de conversión:

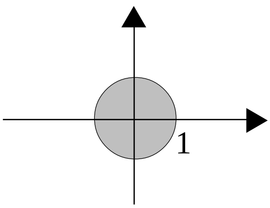


Figura 2

hasta llegar a: $\rho^2+i^2<0$.

No obstante que las diversas transformaciones se efectúen con total evidencia y en forma explícita, discutiendo cada uno de los cambios de registro semiótico, ninguno de los estudiantes futuros docentes, está dispuesto a admitir la unicidad del objeto matemático en juego. La última representación es interpretada como "desigualdad paramétrica en C"; el *sentido* fue modificado.

7.5. *Un segundo ejemplo en un curso (postgrado) de formación para futuros docentes de escuela secundaria*

A) Objeto matemático: Sucesión de los números triangulares; interpretación y conversión: 1, 3, 6, 10, ...; cambio de representación por tratamiento: 1, 1+2, 1+2+3, 1+2+3+4,....; esta representación es reconocida como "sucesión de las sumas parciales de los naturales sucesivos".

B) Objeto matemático: Sucesión de los números cuadrados; interpretación y conversión: 0, 1, 4, 9,...; cambio de representación por tratamiento: 0, (0)+1, (0+1)+3, (0+1+3)+5,...; esta representación es reconocida como "suma de las sumas parciales de los impares sucesivos".

En ninguno de los casos precedentes descritos brevemente, los alumnos pudieron aceptar que el *sentido* de la representación semiótica obtenida finalmente, *después* de transformaciones semióticas evidenciadas, coincide con el *sentido* del objeto matemático de partida.

8. Conclusiones

No parecen necesarias largas conclusiones. Urge sólo evidenciar cómo el *sentido* de un objeto matemático sea algo mucho más complejo respecto a la pareja usual (objeto, representaciones del objeto); existen relaciones semióticas entre las parejas de este tipo:

(objeto, representación del objeto) –
(objeto, otra representación del objeto),

relaciones derivadas de transformaciones semióticas entre las representaciones del mismo objeto, pero que tienen el resultado de hacer perder el sentido del objeto de partida. Si bien, tanto el objeto como las transformaciones semióticas son el resultado de prácticas compartidas, los resultados de las transformaciones pueden necesitar de otras atribuciones de sentido gracias a otras prácticas compartidas. Lo que enriquece de mayor interés todo estudio sobre ontología y conocimiento.

Los fenómenos descritos en la primera parte de este artículo pueden ser usados para completar la visión que Duval ofrece del papel de las múltiples representaciones de un objeto en la comprensión de dicho objeto, y también para "romper el círculo vicioso" de su paradoja. En realidad cada representación lleva asociado un "subsistema de prácticas" diferentes, de donde emergen objetos diferentes (en el parágrafo anterior denominados: O_1, O_2, O_3 y O_4). Pero la articulación de estos objetos en otro más general requiere un cambio de perspectiva, el paso a otro contexto en el que se plantee la búsqueda de la *estructura común* en el sistema de prácticas global en el que intervienen los distintos "objetos parciales".

Sin duda, el uso de distintas representaciones y su progresiva articulación enriquecen el significado, el conocimiento, la comprensión del objeto, pero también su complejidad. El objeto matemático se presenta, en cierto sentido, como único, pero en otro sentido, como múltiple. Entonces, ¿cuál es la naturaleza del objeto matemático? No parece que haya otra respuesta que no sea la estructural, formal, gramatical (en sentido epistemológico), y al mismo tiempo la estructural, mental, global, (en sentido psicológico) que los sujetos construimos en nuestros cerebros a medida que se enriquecen nuestras experiencias.

Es obvio que estas observaciones abren las puertas a futuros desarrollos en los cuales las ideas que parecen diversas, confluyen por el contrario en el intento de dar una explicación a los fenómenos de atribución de sentido.

Reconocimientos

Este trabajo fue desarrollado dentro del programa estratégico de investigación: *Aspetti metodologici (teorici ed empirici) della formazione iniziale ed in servizio degli insegnanti di matematica di ogni livello scolastico*, con fondos de la Universidad de Bologna. Traducción de Martha Isabel Fandiño Pinilla, con la colaboración de Juan Díaz Godino.

Referencias

Blumer, H. (1969). *Symbolic interactionism. Perspective and method.* Englewood Cliffs NJ: Prentice Hall.

Catena P. (1992). *Universa loca in logicam Aristetolis in mathematicas disciplinas*. (Editor G. Dell'Anna). Galatina (Le): Congedo.

Chevallard, Y. (1992). Concepts fondamentaux de la didactique: perspectives apportées par une approche anthropologique. *Recherches en Didactique des Mathématiques, 12* (1), 73-112.

D'Amore, B. (1999). *Elementi di didattica della matematica*. Prefazione di Colette Laborde. Bologna: Pitagora. [Versión en idioma español: D'Amore, B. (2006). *Didáctica de la Matemática*. Con una carta de Guy Brousseau. Prefacio a la edición en idioma español de Luis Rico. Bogotá: Editorial Magisterio].

D'Amore, B. (2001a). Concettualizzazione, registri di rappresentazioni semiotiche e noetica. *La matematica e la sua didattica*, *2*, 150-173. [Versión en idioma español: D'Amore, B. (2004). Conceptualización, registros de representaciones semióticas y noética: interacciones constructivistas en el aprendizaje de los conceptos matemáticos e hipótesis sobre algunos factores que inhiben la devolución. *Uno*, *35*, 90-106].

D'Amore, B. (2001b). Un contributo al dibattito su concetti e oggetti matematici: la posizione "ingenua" in una teoria "realista" vs il modello "antropologico" in una teoria "pragmatica". *La matematica e la sua didattica*, *1*, 4-30. [Versión en idioma español: D'Amore, B. (2001). Una contribución al debate sobre conceptos y objetos matemáticos. *Uno*, *27*, 51-76].

D'Amore, B. (2003a). La complexité de la noétique en mathématiques ou les raisons de la dévolution manquée. *For the learning of mathematics*, *23*(1), 47-51. [Versión preliminar reducida en idioma español: D'Amore, B. (2002). La complejidad de la noética en matemáticas como causa de la falta de devolución. *TED*. Bogotá, Universidad Pedagógica Nacional, *11*, 63-71].

D'Amore, B. (2003b). *Le basi filosofiche, pedagogiche, epistemologiche e concettuali della Didattica della Matematica*. Bologna: Pitagora. [Versión en idioma español: D'Amore, B. (2005). *Bases filosóficas, pedagógicas, epistemológicas y conceptuales de la Didáctica de la Matemática*. Prefacio de Guy Brousseau. Prefacio a la edición en idioma español de Ricardo Cantoral. Traducción de Martha Isabel Fandiño Pinilla. México DF, México: Reverté-Relime.]. [Versión en idioma portugués: D'Amore, B. (2005). *As bases filosóficas, pedagógicas, epistemológicas e conceituais da didáctica da matematica*. Prefácio da edição italiana: Guy Brousseau. Prefácio: Ubiratan D'Ambrosio Tradução: Maria Cristina Bonomi Barufi. Escrituras: São Paulo].

D'Amore, B., & Godino, D. J. (2006). Punti di vista antropologico ed ontosemiotico in Didattica della Matematica. *La matematica e la sua didattica*, *1*, 7-36.

D'Amore, B., Radford, L., & Bagni, G.T. (2006). Ostacoli epistemologici e prospettiva socio-culturale. *L'insegnamento della matematica e delle scienze integrate*, 29B, 1, 11-40.

Daval, R. (1951). *La métaphysique de Kant*. París: PUF.

Duval, R. (1993). Registres de représentations sémiotiques et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives*, *5*, 37-65.

Duval, R. (2003). Décrire, visualiser ou raisonner: quels 'apprentissages premiers' de l'activité mathématique? *Annales de Didactique et de Sciences Cognitives*, *8*, 13-62.

Duval, R. (2005). Transformations de représentations sémiotiques et démarche de pensée en mathématiques. Colloque COPIRELEM, Strasbourg, 30 mayo - 1 junio 2006. Actas en curso de impresión.

Eco, U. (1973). *Segno*. Milano: ISEDI.

Eco, U. (1975). *Trattato di semiotica generale*. Milano: Bompiani.

Godino, J. D. (2002). Un enfoque ontológico y semiótico de la cognición matemática. *Recherches en Didactique des Mathématiques*, *22*(2/3), 237-284.

Godino, J. D., & Batanero, C. (1994). Significado institucional y personal de los objetos matemáticos. *Recherches en Didactique des Mathématiques*, *14* (3), 325-355.

Godino, J. D., & Batanero, C. (1998). Clarifying the meaning of mathematical objects as a priority area of research in mathematics education. En A. Sierpinska, & J. Kilpatrick (Eds.), *Mathematics Education as a Research Domain: A Search for Identity* (pp. 177-195). Dordrecht: Kluwer A. P.

Kozoulin, A. (1990). *Vygotsky's psychology*. Cambridge, MA: Harvard Univ. Press.

Llinares, S., & Krainer, K. (2006). Mathematics (student) teachers and teacher educators as learners. In A. Gutierrez, & P. Boero (Eds.), *Handbook of Research on the Psychology of Mathematics Education. Past, Present and Future.* Rotterdam /Taipei: Sense Publishers 429-460.

Radford, L. (2000). Signs and meanings in students' emergent algebraic thinking: a semiotic analysis. *Educational studies in mathematics*, *42*(3), 237-268.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge. *For the learning of mathematics*, *22*(2) 14-23.

Radford, L. (2003). Gestures, speech and the sprouting of signs. *Mathematical thinking and learning*, *5*(1), 37-70.

Radford, L. (2004). Cose sensibili, essenze, oggetti matematici ed altre ambiguità. *La matematica e la sua didattica*, *1*, 4-23.

Radford, L. (2005). La generalizzazione matematica come processo semiotico. *La matematica e la sua didattica*, *2*, 191-213.

Wertsch, JV. (1991). *Voices in the mind. A sociocultural approach to mediate action*. Cambridge, MA: Harvard Univ. Press.

Zinchenko, VP. (1985). Vygotsky's ideas about units for the analysis of mind. In J.V.Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives*. Cambridge, MA: Harvard Univ. Press.

Bruno D'Amore
Dipartimento di Matematica
Università di Bologna
Italia

E-mail: damore@dm.unibo.it

Are registers of representations and problem solving processes on functions compartmentalized in students' thinking?

Athanasios Gagatsis [1]
Iliada Elia [1]
Nikos Mousoulides [1]

RESUMEN

El objetivo de este artículo es doble. En primer lugar, se hace un resumen superficial de investigaciones sobre la compartimentación de diferentes registros de representación, así como de las aproximaciones de resolución de problemas, relacionadas con el concepto de función. En segundo lugar, se aportan elementos que clarifican las posibles maneras que permiten superar el fenómeno de la compartimentación. Investigaciones precedentes muestran que la mayoría de los alumnos de secundaria e, incluso de universidad, tienen dificultades para cambiar, de forma flexible, los sistemas de representación de funciones, de seleccionar y de utilizar aproximaciones apropiadas de resolución de problemas. Los resultados de dos estudios experimentales previos, llevados a cabo por miembros de nuestro equipo de investigación, centrados sobre la utilización de aproximaciones no tradicionales de enseñanza y sobre el empleo de software matemático, proveen pistas preliminares, en cuanto a la manera de cómo puede superarse con éxito el fenómeno de la compartimentación.

- **PALABRAS CLAVE:** Aproximación algebraica, compartimentalización, función, aproximación geométrica, resolución de problemas, registros de representación, transformación de representaciones.

ABSTRACT

The purpose of the present study is twofold: first, to review and summarize previous research on the compartmentalization of different registers of representations and problem solving approaches related to the concept of function; second, to provide insights into possible ways to overcome the phenomenon of compartmentalization. To this extent, previous research shows that the majority of high school and university students experience difficulties in flexibly changing systems of representations of function and in selecting and employing appropriate approaches to problem solving. Two previous experimental efforts, by the authors, focusing on the use of non-traditional teaching approaches and on the use of mathematical software respectively, provided some initial strategies for successfully overcoming the phenomenon of compartmentalization.

Fecha de recepción: Febrero de 2006/ Fecha de aceptación: Mayo de 2006

1 Department of Education, University of Cyprus. Nicosia, Cyprus.

- **KEY WORDS:** Algebraic approach, compartmentalization, function, geometric approach, problem solving, registers of representation, transformation of representations.

RESUMO

O objetivo deste de artigo é duplo. Primeiro, é feito um resumo superficial de investigações sobre a compartimentação de diferentes registros de representação, e aproximações de resolução de problemas, apostas em relação ao de conceito de função. Em segundo lugar, traz elementos que clarificam as possíveis maneiras que permitem superar o fenômeno da compartimentarão. Investigações precedentes mostram que a maioria dos alunos do ensino médio e, mesmo de universidade, tem dificuldades para alterar, de maneira flexível, os sistemas de representação de funções, de escolher e utilizar aproximações adequadas à resolução de problemas. Os resultados de dois estudos experimentais prévios, levados a efeito por membros do nosso grupo de pesquisa, centrados no utilização de aproximações não tradicionais de ensino e sobre ou emprego de «software» matemático, fornecem pistas preliminares, quanto à maneira como pode ser superar com sucesso o fenômeno da compartimentação.

- **PALAVRAS CHAVE:** Aproximação algébrica, compartimentação, função, geométrica aproximação, solução de problema, registros de representação, transformação de representações.

RÉSUMÉ

Le but de cet article est double. En premier lieu, il s'agit de faire un survol et une synthèse des recherches sur la compartimentation de différents registres de représentation et des approches de résolution de problèmes reliées au concept de fonction. En deuxième lieu, il s'agit d'apporter un éclairage sur les manières possibles de surmonter le phénomène de compartimentation. Des recherches antérieures montrent que la majorité des élèves de l'école secondaire et de l'université ont de la difficulté à changer de façon flexible les systèmes de représentation des fonctions ainsi qu'à sélectionner et à utiliser des approches appropriées de résolution de problèmes. Deux efforts expérimentaux préalables, menés par les auteurs, centrés sur l'utilisation des approches non-traditionnelles d'enseignement et sur l'emploi de logiciels mathématiques, fournissent des indications préliminaires quant à la manière de surmonter avec succès le phénomène de compartimentation.

- **MOTS CLÉS:** Approche algébrique, compartimentation, fonction, approche géométrique, résolution de problèmes, registres de représentation, transformation de représentations.

1.INTRODUCTION

During the last decades, a great deal of attention has been given to the concept of representation and its role in the learning of mathematics. Nowadays, the centrality of multiple representations in teaching, learning and doing mathematics seems to have become widely acknowledged (D' Amore, 1998). Representational systems are fundamental for conceptual learning and determine, to a significant extent, what is learnt (Cheng, 2000). A basic reason for this emphasis is that representations are considered to be "integrated" with mathematics (Kaput, 1987). Mathematical concepts are accessible only through their semiotic representations (Duval, 2002). In certain cases, representations, such as graphs, are so closely connected with a mathematical concept, that it is difficult for the concept to be understood and acquired without the use of the corresponding representation. Any given representation, however, cannot describe thoroughly a mathematical concept, since it provides information regarding merely a part of its aspects (Gagatsis & Shiakalli, 2004). Given that each representation of a concept offers information about particular aspects of it without being able to describe it completely, the ability to use various semiotic representations for the same mathematical object (Duval, 2002) is an important component of understanding. Different representations referring to the same concept complement each other and all these together contribute to a global understanding of it (Gagatsis & Shiakalli, 2004). The use of different modes of representation and connections between them represents an initial point in mathematics education at which pupils use one symbolic system to expand and understand another (Leinhardt, Zaslavsky, & Stain, 1990). Thus, the ability to identify and represent the same concept through different representations is considered as a prerequisite for the understanding of the particular concept (Duval, 2002; Even, 1998). Besides recognizing the same concept in multiple systems of representation, the ability to manipulate the concept with flexibility within these representations as well as the ability to "translate" the concept from one system of representation to another are necessary for the acquisition of the concept (Lesh, Post, & Behr, 1987) and allow students to see rich relationships (Even, 1998).

Duval (2002) assigns the term "registers" of representation to the diverse spaces of representation in mathematics and identifies four different types of registers: natural language, geometric figures, notation systems and graphic representations. Mathematical activity can be analyzed based on two types of transformations of semiotic representations, i.e. treatments and conversions. Treatments are transformations of representations, which take place within the same register that they have been formed in. Conversions are transformations of representations that consist in changing the register in which the totality or a part of the meaning of the initial representation is conserved, without changing the objects being denoted. The conversion of representations is considered as a fundamental process leading to mathematical understanding and successful problem solving (Duval, 2002). A person who can easily transfer her knowledge from one structural system of the mind to another is more likely to be successful in problem solving by using a plurality of solution strategies and regulation processes of the system for handling cognitive difficulties.

2. THE ROLE OF REPRESENTATIONS IN MATHEMATICS LEARNING: EMPIRICAL BACKGROUND

Students experience a wide range of representations from their early childhood years onward. A main reason for this is that most mathematics textbooks today make use of a variety of representations more extensively than ever before in order to promote understanding. However, a reasonable question can arise regarding the actual role of the use of representations in mathematics learning. A considerable number of recent research studies in the area of mathematics education in Cyprus and Greece investigated this question from different perspectives. In an attempt to explore more systematically the nature and the contribution of different modes of representation (i.e., pictures, number line, verbal and symbolic representations) on mathematics learning, Gagatsis and Elia (2005a) carried out a review of a number of these studies, which examined the effects of various representations on the understanding of mathematical concepts and mathematical problem solving in primary and secondary education. Many of these studies identified the difficulties that arise in the conversion from one mode of representation of a mathematical concept to another. They also revealed students' inconsistencies when dealing with relative tasks that differ in a certain feature, i.e. mode of representation. This incoherent behaviour was addressed as one of the basic features of the phenomenon of compartmentalization, which may affect mathematics learning in a negative way.

The research of Gagatsis, Shiakalli and Panaoura (2003) examined the role of the number line in second grade Cypriot students' performance in executing simple addition and subtraction operations with natural numbers. By employing implicative statistical analysis (Gras, 1996), they detected a complete compartmentalization between the students' ability to carry out addition and subtraction tasks in the symbolic form of representation and their ability to perform the same tasks by using the number line. A replication of the study by Gagatsis, Kyriakides and Panaoura (2004) with students of the same age in Cyprus, Greece and Italy, and this time using a different statistical method, namely structural equation modelling, resulted in congruent findings. This uncovers the strength of the phenomenon of compartmentalization despite differences in curricula, teaching methods, mathematics textbooks and even culture.

Michaelidou, Gagatsis and Pitta-Pantazi (2004) have examined 12-year-old students' understanding of the concept of decimal numbers based on the threefold model of the understanding of an idea, proposed by Lesh et al. (1987). To carry out the study, three tests on decimal numbers were developed. These tests aimed at investigating students' abilities to recognize and represent decimal numbers with a variety of different representations and their ability to transfer decimal numbers from the symbolic form to the number line and vice versa. The application of the implicative statistical method demonstrated a compartmentalization of students' abilities in the different tasks and this signifies that there was a lack of coordination between recognition, manipulation within a representation and conversion among different representations of decimal numbers. This finding means that some students who can recognize decimal numbers in different representations cannot use the representations to represent the decimal numbers by themselves and, what is more important, fail to transfer from one representation of decimal numbers to another. In other words, students have not developed a unified cognitive structure concerning the concept of decimals since their ideas seemed to be partial and isolated. Given the three aspects of the understanding of mathematical concepts related to representations, namely, recognition, flexible use and conversion, it can be suggested that in this study students did not understand the concept of decimal numbers.

Finally, Marcou and Gagatsis (2003) examined 12-year-old students' understanding of the concept of fractions and more specifically the equivalence and the addition of fractions. The researchers designed three types of tests on fractions, which involved conversions among the symbolic expressions, verbal expressions and the diagrammatic representations of fractions (area of rectangles). Students' responses to the tasks were compartmentalized with respect to the starting representation of the conversions, as indicated by the implicative analysis of the data. In line with the afore mentioned studies' results, this finding means that students had a fragmentary understanding of fractions.

In the present paper, four recent studies are combined and discussed to explore secondary school and university students' abilities to use multiple modes of representation for one of the most important unifying ideas in mathematics (Romberg, Carpenter, & Fennema, 1993; Mousoulides & Gagatsis, 2004), namely functions, and to flexibly move from one representation of the concept to another. The main concern of this paper is twofold; first to identify and further clarify the appearance of the phenomenon of compartmentalization in students' thinking about the particular concept and second to examine possible ways for succeeding at de-compartmentalization in registers of representations and problem solving processes in functions.

3. REPRESENTATIONS AND THE CONCEPT OF FUNCTION

The concept of function is central to mathematics and its applications. It emerges from the general inclination of humans to connect two quantities, which is as ancient as mathematics itself. The didactical metaphor of this concept seems difficult, since it involves three different aspects: the epistemological dimension as expressed in the historical texts; the mathematics teachers' views and beliefs about function; and the didactical dimension which concerns students' knowledge and the restrictions imposed by the educational system (Evangelidou, Spyrou, Elia, & Gagatsis, 2004). On this basis, it seems natural for students of secondary or even tertiary education, in any country, to have difficulties in conceptualizing the notion of function. The complexity of the didactical metaphor and the understanding of the concept of function have been a main concern of mathematics educators and a major focus of attention for the mathematics education research community (Dubinsky & Harel, 1992; Sierpinska, 1992). An additional factor that influences the learning of functions is the diversity of representations related to this concept (Hitt, 1998). An important educational objective in mathematics is for pupils to identify and use efficiently various forms of representation for the same mathematical concept and to move flexibly from one system of representation of the concept to another. The influence of different representations on the understanding and interpretation of functions has been examined by a substantial number of research studies (Hitt, 1998; Markovits, Eylon, & Bruckheimer, 1986).

Several researchers (Evangelidou et al., 2004; Gagatsis, Elia & Mougi, 2002; Gagatsis & Shiakalli 2004; Mousoulides & Gagatsis, 2004; Sfard 1992; Sierpinska 1992) indicated the significant role of different representations of function and the conversion from one representation to another on the understanding of the concept itself. Thus, the standard representational forms of the concept of function are not enough for students to be able to construct the whole meaning and grasp the whole range of its applications. Mathematics instructors, at the secondary level, have traditionally focused their instruction on the use of algebraic representations of functions. Eisenberg and Dreyfus (1991) pointed out that the way knowledge is constructed in schools mostly favours the analytic elaboration of the notion to the detriment of approaching function from the graphical point of view. Kaldrimidou and Iconomou (1998) showed that teachers and students pay much more attention to algebraic symbols and problems than to pictures and graphs. A reason for this is that, in many cases, the iconic (visual) representations cause cognitive difficulties because the perceptual analysis and synthesis of mathematical information presented implicitly in a diagram often make greater demands on a student than any other aspect of a problem (Aspinwall, Shaw, & Presmeg, 1997).

In addition, most of the aforementioned studies have shown that students tend to have difficulties in transferring information gained in one context to another (Gagatsis & Shiakalli, 2004). Sfard (1992) showed that students were unable to bridge the algebraic and graphical representations of functions, while Markovits et al. (1986) observed that the translation from graphical to algebraic form was more difficult than the reverse. Sierpinska (1992) maintains that students have difficulties in making the connection between different representations of functions, in interpreting graphs and manipulating symbols related to functions. A possible reason for this kind of behaviour is that most instructional practices limit the representation of functions to the translation of the algebraic form of a function to its graphic form.

Lack of competence in coordinating multiple representations of the same concept can be seen as an indication of the existence of compartmentalization, which may result in inconsistencies and delays in mathematics learning at school. This particular phenomenon reveals a cognitive difficulty that arises from the need to accomplish flexible and competent translation back and forth between different modes of mathematical representations (Duval, 2002). Making use of a more general meaning of compartmentalization which does not refer necessarily to representations, Vinner and Dreyfus (1989) suggested that compartmentalization arises when an individual has two divergent, potentially contradictory schemes in her cognitive structure and pointed out that inconsistent behaviour is an indication of this phenomenon.

The first objective of this study is to identify the phenomenon of compartmentalization in secondary school students and university students' strategies for dealing with various tasks using functions on the basis of the findings of four recent research studies (Elia, Gagatsis & Gras, 2005; Gagatsis & Elia, 2005b; Mousoulides & Gagatsis, 2006; Mousoulides & Gagatsis, 2004). Although these studies explored the students' ability to handle different modes of the representation of function and move flexibly from one representation to another, there is a fundamental difference between the mathematical activities they proposed. The study of Elia et al., (2005) investigated students' understanding of function based on their performance in mathematical activities that integrated both types of the transformation of representations proposed by Duval (2002), i.e. treatment and conversion. The study of Mousoulides and Gagatsis (2004) investigated students' performance in mathematical activities that principally involved the second type of transformations, that is, the conversion between systems of representation of the same function, and concentrated on students' approaches to the use of representations of functions and the connection with students' problem solving processes. The studies of Gagatsis et al., (2004) and Mousoulides and Gagatsis (2006) introduced two approaches that might succeed at de-compartmentalization, namely a differentiated instruction and the use of a computerized environment for solving problems in functions. Thus, what is new in this review is that students' understanding of function is explored from two distinct perspectives (which will be further clarified in the next section), but nevertheless based on the same rationale, that is, Duval's semiotic theory of representations. The second objective of the review is to discuss strategies for overcoming compartmentalization in functions.

4. CAN WE "TRACE" THE PHENOMENON OF COMPARTMENTALIZATION BY USING THE IMPLICATIVE STATISTICAL METHOD OF ANALYSIS?

Previous empirical studies have not clarified compartmentalization in a comprehensive or systematic way. Thus, we theorize that the implicative relations between students' responses in the administered tasks, uncovered by Gras's implicative statistical method (Gras, 1996), as well as their connections (Lerman, 1981) can be beneficial for identifying the appearance of compartmentalization in students' behaviour. To analyze the collected data of both studies, a computer software called C.H.I.C. (Classification Hiérarchique Implicative et Cohésitive) (Bodin, Coutourier, & Gras, 2000) was used.

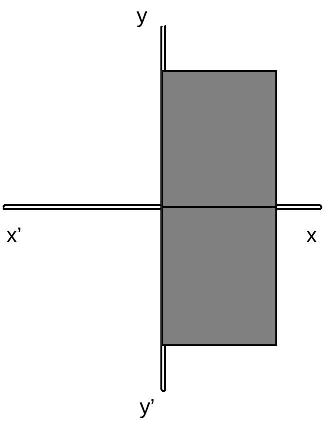
We assume that the phenomenon of compartmentalization in the understanding of function as indicated by students' performance in tasks integrating treatment and conversion (Gagatsis & Elia, 2005b) appears when at least one of the following conditions emerges: first, when students deal inconsistently or incoherently with tasks involving the different types of representation (i.e., graphic, symbolic, verbal) of functions or conversions from one mode of representation to another; and/or second, when success in using one mode of representation or one type of conversion of function does not entail success in using another mode of representation or in another type of conversion of the same concept. As regards students' ways of approaching tasks requiring only conversions among representations of the same function (Mousoulides & Gagatsis, 2004), our conjecture is that compartmentalization appears when students deal with all of the tasks using the same approach, even though a different approach is more suitable for some of them.

4.1. Secondary school students' abilities in the transformation of representations of function (Study 1)

Recent studies (Gagatsis & Elia, 2005b; Elia et al., 2005) investigated secondary school students' ability to transfer mathematical relations from one representation to another. In particular, the sample of the study consisted of 183 ninth grade students (14 years of age). Two tests, namely A and B, were developed and administered to the participants. The tasks of both tests involved conversions of the same algebraic relations, but with different starting modes of representation. Test A consisted of six tasks in which students were given the graphic representation of an algebraic relation and were asked to translate it to its verbal and symbolic forms respectively. Test B consisted of six tasks (involving the same algebraic relations as test A) in which students were asked to translate each relation from its verbal representation to its graphical and symbolic mode. For each type of translation , the following types of algebraic relations were examined: $y<0,\ xy>0,\ y>x,\ y=-x,\ y=3/2,\ y=x-2$ based on a relevant study by Duval (1993).

The former three tasks corresponded to regions of points, while the latter three tasks corresponded to functions. Each test included an example of an algebraic relation in graphic, verbal and symbolic forms to facilitate students' understanding of what they were asked to do, as follows:

Table 1: *An example of the tasks included in the test*

Graphic representation	Verbal representation	Symbolic representation
	It represents the region of the points having positive abscissa.	$x>0$

It is apparent that the tasks involved conversions, which were employed either as complex coding activities or as point-to-point translations and were designed to correspond to school mathematics. However, a general use of the processes of treatment and conversion was required for the solution of these tasks. For instance, the conversion of the function $y=x-2$ from the algebraic expression to the graphical one could be accomplished by carrying out various kinds of treatment, such as calculations in the same notation system. It is evident that in this kind of task the process of treatment cannot be easily distinguished from the process of conversion. According to this perspective, these tasks differ from the tasks proposed by Duval (1993).

The results of the study revealed that students achieved better outcomes in the conversions starting with verbal representations relative to the conversions of the corresponding relations starting with graphic representations. In addition, all of the conversions from the graphic form of representation to the symbolic form of representation appeared to be more difficult than the conversions of the corresponding relations from the graphic form of representation to the verbal form of representation. Students perceived the latter type of conversion more easily at a level of meta-mathematical expression rather than at a level of mathematical expression. In fact, students were asked to describe verbally (in a text) a property perceived by the graph. On the contrary, the conversions from the graphical form to the symbolic form entailed mastering algebraic concepts concerning equality or order relations as well as using the algebraic symbolism efficiently.

Figure 1 presents the similarity diagram of the tasks of Test A and Test B based on the responses of the students.

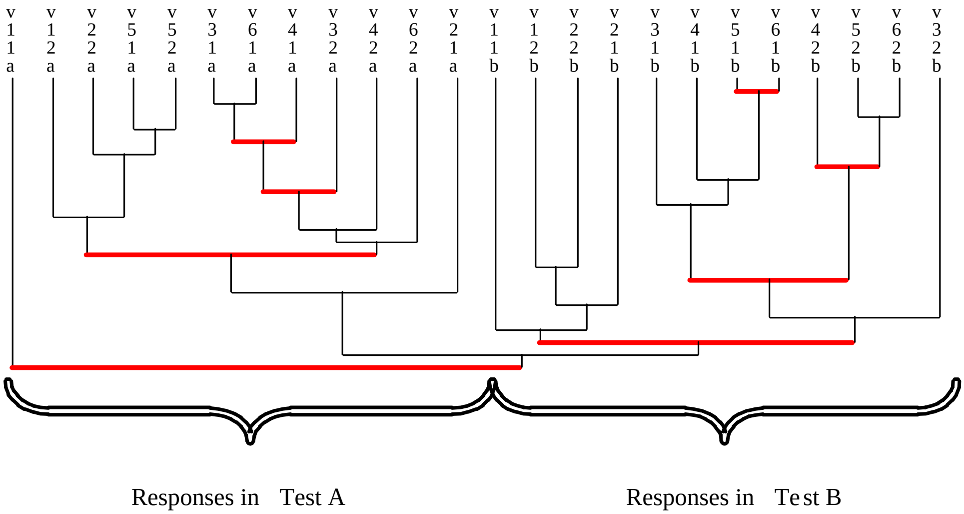


Figure 1: Similarity diagram of the tasks of Test A and Test B according to Grade 9 students' responses

Note: The symbolism used for the variables of this diagram (and the diagram that follows) is explained below.

1. "a" stands for Test A, and "b" stands for Test B

2. The first number after "v" stands for the number of the task in the test
i.e., 1: $y<0$, 2: $xy>0$, 3: $y>x$, 4: $y=-x$, 5: $y=3/2$, 6: $y=x-2$

3. The second number stands for the type of conversion in each test, i.e., for Test A, 1: graphic to verbal representation, 2: graphic to symbolic representation; for Test B, 1: verbal to graphic representation, 2: verbal to symbolic representation.

The similarity diagram allows for the grouping of students' responses to the tasks based on their homogeneity. Two distinct similarity groups of tasks are identified. The first group involves similarity relations among the tasks of Test A, while the second group involves similarity relations among the tasks of Test B. This finding reveals that different types of conversions among representations of the same mathematical content were approached in a completely distinct way. The starting representation of a conversion, i.e., graphic or verbal representation, seems to have influenced the students' performance, even though the tasks involved the same algebraic relations. Thus, we observe a complete separation of students' responses to the two tests even in tasks that were similar and rather "easy" for this grade of students. The similarity relations within the group of variables of the tests are also of great interest since they provide some indications of the students' way of understanding the particular algebraic relations and further support the likelihood that the phenomenon of compartmentalization was present.

For example, the similarity group of Test B is comprised of three subgroups. The **first subgroup** contains students' responses to the tasks v11b and v12b ($y<0$) and the tasks v21b and v22b ($xy>0$), that is, the two conversions from verbal to graphic representation and from verbal to symbolic representation of the first two tasks of Test B. These two tasks involve relations that represent "regions of points" and they are the easiest tasks of the test. The **second subgroup** is formed by the variables v31b ($y>x$), v41b($y=-x$), v51b ($y=3/2$) and v61b ($y = x-2$) that is the conversion from verbal to graphic representation of four relations of "functional character," as the relation of task 3 corresponds to a region of points related to the function $y=x$, while the relations of tasks 4, 5 and 6 are functions. The **third subgroup** is comprised by the variables v42b ($y=-x$), v52b ($y=3/2$) and v62b ($y=x-2$), that is, the conversion from verbal to algebraic representation of the tasks that involve functions.

To sum up, the formation of the first subgroup separately from the other two is of a "conceptual nature," since it is due to the conceptual characteristics of the relations involved, whereas, the distinction between the third subgroup and the forth subgroup is of a "representational character," since it is a consequence of the target of the conversion. To summarize, one can observe two kinds of compartmentalization in the similarity diagram: one "first order" compartmentalization (between the tasks of the two tests) and one "second order" compartmentalization (between the tasks of the same test).

The implicative diagram in Figure 2 was derived from the implicative analysis of the data and contains implicative relations, indicating whether success at a specific task implies success at another task related to the former one. The implicative relations are in line with the connections in the similarity diagram and the above remarks. In particular, one can observe the formation of two groups of implicative relations. The first group involves implicative relations among the responses to the tasks of Test B and the second group involves implicative relations among the responses to the tasks of Test A.

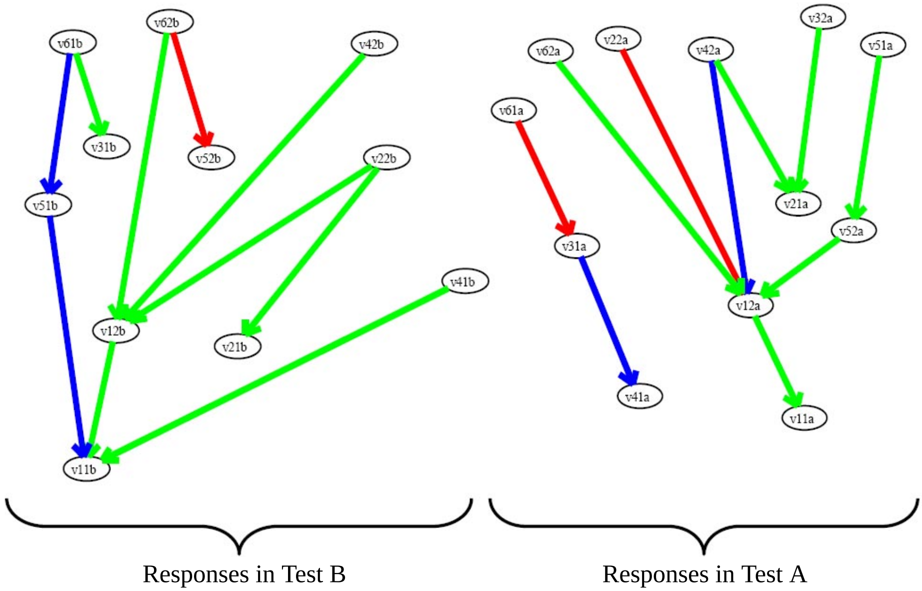


Figure 2: *Implicative diagram of 14-year-old students' responses to the tasks of Test A and Test B*

The fact that implicative relations appear only between students' responses to the tasks of the same test indicates that success at one type of conversion of an algebraic relation did not necessarily imply success at another type of conversion of the same relation. For example, students who accomplished the conversion from a graphical representation of a mathematical relation to its verbal representation were not automatically in a position to translate the same relation from its verbal representation to its graphical form successfully. This is the first order compartmentalization that appears between students' responses to the tasks of the two tests. Additionally, evidence is provided for the appearance of the second order compartmentalization, that is, between students' responses to the tasks of the same test. The implicative chain "v61a-v31a-v41a" of Test A and the implicative chain "v61b-v51b-v11b" of Test B can be taken as examples of the second order compartmentalization, probably due to the same "target" representations of the conversions.

Other useful information can also be obtained by this implicative diagram. For example the simplest tasks in both tests are the tasks which involve the relation $y<0$ (v11), corresponding to a region of points. The students' failure in the tasks involving the particular relation (v11a or v11b) also implies failure at most of the other tasks in both tests. This inference is tenable as the implicative diagram was constructed by using the concept of "entropy." This means that for every implication where "a implies b" the counter-inverse "no a implies no b" is also valid.

Overall, based on the relations included in the similarity and the implicative diagrams for secondary school students, it can be inferred that there was a compartmentalization between students' responses to the tasks of the first test and the tasks of the second test, which involved conversions of the same algebraic relations but different starting modes of representation (i.e., graphic and verbal respectively). Students' higher success rates at the tasks of Test B, i.e., conversions starting with graphic representations, relative to the tasks of Test A, i.e., conversions starting with verbal representations, provide further evidence for their inconsistent behaviour in the two types conversions. Another kind of compartmentalization was also uncovered within the same test, indicating students' distinct ways of carrying out conversion tasks with reference to their conceptual (kind of mathematical relation) or representational (target of the conversion) discrepancies.

4.2. Student teachers' approaches to the conversion of functions from the algebraic to the graphical representation (Study 2)

In this section, we present some elements from a study of Mousoulides and Gagatsis (2004) that used a different approach to explore the idea of the conversion between representations and the phenomenon of compartmentalization. The researchers investigated student teachers' approaches to solving tasks of functions and the connection of these approaches with complex geometric problem solving. The theoretical perspective used in their study is related to a dimension of the framework developed by Moschkovich, Schoenfeld and Arcavi (1993). According to this dimension, there are two fundamentally different perspectives from which a function is viewed, i.e., the *process perspective* and the *object perspective*. From the *process perspective,* a function is perceived of as linking x and y values: For each value of x, the function has a corresponding y value. Students who view functions under this perspective can substitute a value for x into an equation and calculate the resulting value for y or find pairs of values for x and y to draw a graph. In contrast, from the *object perspective,* a function or relation and any of its representations are thought of as entities - for example, algebraically as members of parameterized classes, or in the plane, as graphs that are thought of as being "picked up whole" and rotated or translated (Moschkovich et al., 1993). Students who view functions under this perspective can recognize that equations of lines of the form $y = 3x + b$ are parallel or can draw these lines without calculations if they have already drawn one line or they can fill a table of values for two functions (e.g., $f(x) = 2x$, $g(x) = 2x + 2$) using the relationship between them *(e.g.* $g(x) = f(x) + 2$) (Knuth, 2000).

Mousoulides and Gagatsis (2004) have adopted the terms "algebraic approach" and "geometric approach" in order to emphasize the use of the algebraic expression or the graphical representations by the students in the conversion tasks and in problem solving. The algebraic approach is relatively more effective in making salient the nature of the function as a process, while the geometric approach is relatively more effective in making salient the nature of function as an object (Yerushalmy & Schwartz, 1993).

Data were obtained from 95 sophomore pre-service teachers enrolled in a basic algebra course at the University of Cyprus. A questionnaire, which consisted of four tasks and two problems, was administered at the beginning of the course. Each task involved two linear or quadratic functions. Both functions were in algebraic form and one of them was also in graphical representation. Functions in each task were related in a way such as $f(x)$, $g(x)=f(x)+c$, or $h(x)=-f(x)$, etc. The four particular tasks were as follows:

1. $y=2x$ and $y=-2x$ (T1)
2. $y=x^2$ and $y=x^2+3$ (T2)
3. $y=x^2+3x-2$ and $y=x^2-3x-2$(T3)
4. $y=x^2+x$ and $y=x^2+2x+1$(T4)

Students were asked to sketch the graph of the second function. An example of the form in which the four tasks were proposed is as follows:

The following diagram presents a graph of the function $y=x^2+x$*. Sketch the graph of the function* $y=x^2+2x+1$*.*

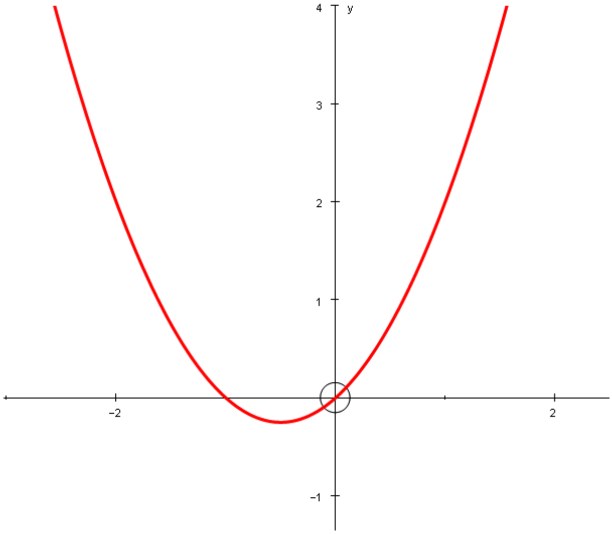


Figure 3: *The graph of the function* $y=x^2+x$ *(Task 4)*

It is obvious that obtaining the correct solution of the tasks did not necessarily require carrying out a treatment in the same system of representation. What was required was the conversion of the algebraic representation of a function to the graphical one, on the basis of its relation with the corresponding representations of a given function.

Additionally, students were asked to solve two problems. One of the problems consisted of textual information about a tank containing an initial amount of petrol and a tank car filling the tank with petrol. Students were asked to use the information to draw the graphs of the two linear functions, i.e. the graph of the amount of petrol in the tank with respect to time and the graph of the amount of petrol in the tank car with respect to time and to find the time at which the amounts of petrol in the tank and in the car would be equal. The other problem involved a function in a general form $f(x) = ax^2+bx +c$. Numbers a, b and c were real numbers and the $f(x)$ was equal to 4 when $x=2$ and $f(x)$ was equal to -6 when $x=7$. Students were asked to find how many real solutions the equation $ax^2+bx +c$ had and explain their answer.

In light of the above, this study differs from the previous one in the following two basic characteristics:

- First the proposed conversions can be carried out geometrically by paying attention to the graphical representation of a given function in order to construct the representation of a second function or algebraically.

- Second, the study attempts to investigate how students' approaches to the conversions between different registers of functions are associated with their processes in problem solving on functions.

The results of this study indicated that the majority of students responded correctly in the first two tasks (T1: 73.2% and T2: 80%). Their rate of success was radically reduced in tasks involving quadratic functions involving complex transformations (T3: 41.1% and T4: 45.3%) and especially in solving complex geometric problems. More specifically, only 27.4% and 11.6% of the 95 participants provided appropriate solutions.

As regards students' approaches, more than 60% of the students that provided a correct solution followed a process perspective or the algebraic approach, which involved the construction of the function graph by finding pairs of values x and y. The other students used an object perspective or the geometric approach by observing and using the relation between the two functions. It is noteworthy that students who chose the algebraic approach applied it even in situations in which a geometric approach seemed easier and more efficient than the algebraic. Furthermore, in the second problem, most of the students (88.4%) failed to recognize or suggest a graphical solution as an option at all, even though the problem could not be solved algebraically.

For the similarity diagram and the implicative analysis of the data, students' answers to the tasks were codified as follow: (a) «A» was used to represent "algebraic approach – function as a process" to tasks and problems; (b) «G» stands for students who adopted a "geometric approach – function as an entity." The similarity diagram of students' responses to the tasks in Figure 4 involved two distinct clusters with reference to students' approaches. The first cluster represents the use of the algebraic approach (process perspective), while the second cluster refers to the use of the geometric approach (object perspective) and solving geometric problems. It is thus demonstrated that students who used the geometric approach in one task were likely to employ the same approach in all the other tasks. Similarly, students who used the algebraic approach employed it consistently in the tasks of the test. It can also be observed that the second cluster includes the variables corresponding to the solution of the complex geometric problems along with the variables representing the geometric approach. This means that students who effectively used the geometric approach for simple tasks on functions also succeeded in solving complex geometric problems on function. In line with the similarity diagram, success rates indicated that students who were able to use a geometric approach achieved better outcomes in solving complex function problems, probably because they were able to observe and use the connections and the relations in the problems flexibly. The formation of the two clusters reveals that students tended to solve tasks and problems in functions using the same approach, even in tasks where a different approach was more suitable, providing support for the emergence of the phenomenon of compartmentalization in students' processes.

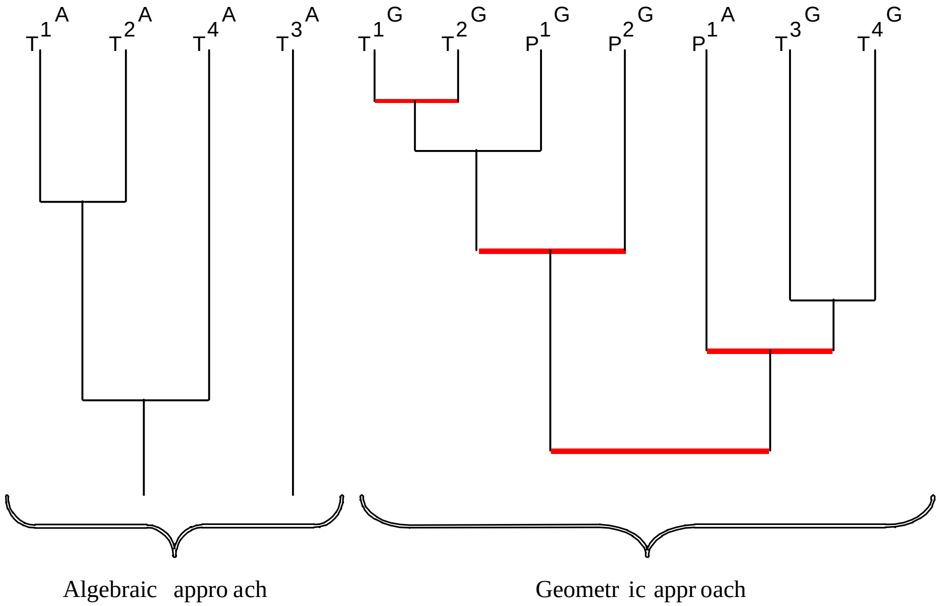


Figure 4: *Similarity diagram of student teachers' approaches to the tasks*

Note: The symbolization of the variables that were used to represent students' responses to the tasks are presented below.

1. Symbols "T1A", "T2A", "T3A" and "T4A" represent a correct algebraic approach to the tasks and "P1A" to the first problem (second problem could not be solved algebraically)

2. Symbols "T1G", "T2G", "T3G" and "T4G" represent a correct geometric approach to the tasks and "P1G" and "P2G", correct graphical solutions to the two problems

The hierarchical tree in Figure 5 involves the implicative relationships between the variables. Three groups of implicative relationships can be identified. The first group and the third group of implicative relationships include variables concerning the use of the geometric approach – object perspective and variables concerning the solution of the geometric problems. The second group involves links among variables standing for the use of the algebraic solution-process perspective. These relations are in line with the findings derived from the similarity diagram. The establishment of these groups of links provides support once again for the consistency that characterizes students' provided solutions towards the function tasks and problems. Furthermore, the implicative relationships of the third group indicate that students who solved the second problem by applying the correct graphical solution have followed the object perspective – graphical representation for the other problem and the other two simple tasks. A possible explanation is that students who have a solid and coherent understanding of functions can recognize relations in complex geometric problems and thus can flexibly connect pairs of equations with their graphs and then easily apply the geometric approach in solving simple tasks on functions.

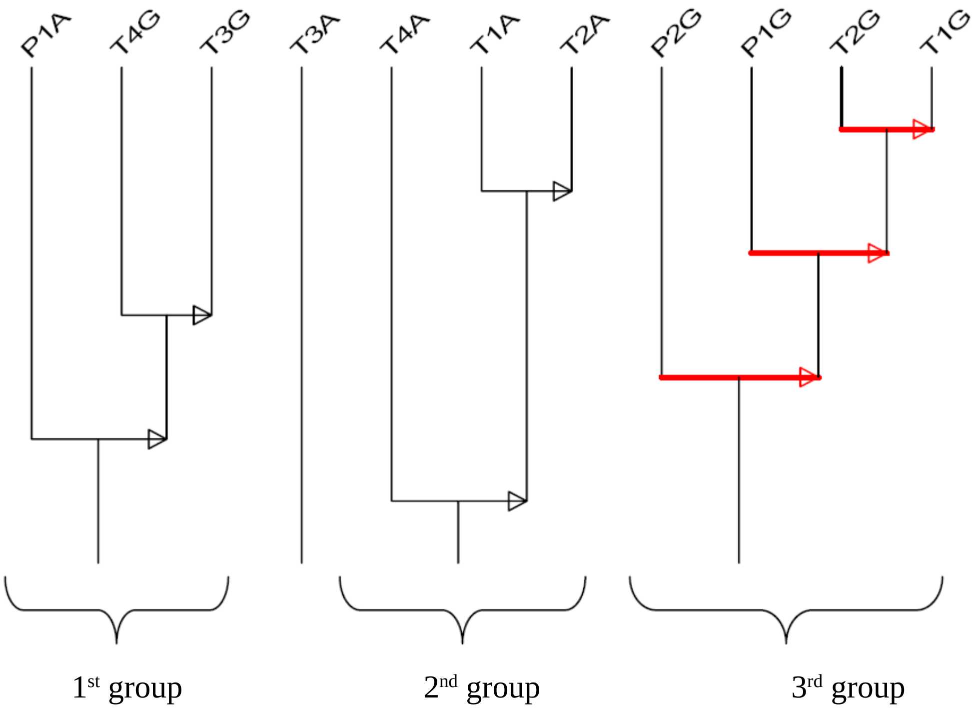


Figure 5: *Hierarchical tree illustrating implicative relations among student teachers' approaches to the tasks*

Note: The implicative relationships in bold colour are significant at a level of 99%

5. CAN WE SUCCEED AT DE-COMPARTMENTALIZATION?

Since an important aspect of this paper is to examine whether the registers of representations and the problem solving cognitive processes in functions are compartmentalized in students' thinking, we will present data from two current investigations. These studies (study 3 and study 4) are related to the previously presented studies, with their objective being to replicate previous results and support further findings for accomplishing de-compartmentalization in functions.

5.1. First effort to succeed at de-compartmentalization (Study 3)

In an attempt to accomplish de-compartmentalization, an experimental study was designed by Gagatsis, Spyrou, Evangelidou and Elia (2004). The researchers developed two experimental programs for teaching functions to university students based on two different perspectives, which are presented below. Two similar tests were administered pre- and post- the intervention in order to investigate students' understanding of functions and to compare the effectiveness of each experimental design.

One hundred fifty-seven university students participated in this study. The participants were second year students of the Department of Education (prospective teachers) who attended the course "Contemporary Mathematics" at the University of Cyprus. The students were randomly assigned to two groups which were taught by two different professors. Experimental Group 2 was comprised of 68 students and Experimental Group 2 was comprised of 81 students. The students in both groups differed in the level and length of the mathematics courses that they had attended in school. Nevertheless, all of the students who participated in this study had received a similar curriculum on functions during the last three grades of high school.

The study was carried out in three stages. In the first stage, a pre-test was administered to both groups of students in order to investigate their initial understanding of the construct of function before the instruction. In the second stage, the two groups received instructional sessions spread over a period of the same duration for both groups. To compare the two groups, in the third stage, a post-test similar to the pre-test was used to assess students' understanding of functions.

The two experimental programs, conducted by two different university professors (Professors A and B), approached the teaching of the notion of function from two different perspectives.

Experimental Program 1 started by providing a revision of some of the functions that were already known to the students from school mathematics, physics and economics. Professor A reminded students about the difference between an equation and a function, which typically appear in a similar symbolic form. Different types of functions were presented next, starting from the simple ones and proceeding to the more complicated ones. At first, the program introduced different kinds of linear functions and described the various representations of functions in the form: $y=ax+b$. Functions with a disconnected domain were also presented. Discrete functions described by discrete types of range and the characteristic function of a set were also presented. Arrow diagrams were also introduced in order to demonstrate to the students a way to examine the ideas of one-to-one and many-to-one types of correspondence as a condition for the definition to be held. Next, the quadratic polynomial function of the form ax^2+bx+c was taught. Special attention was given to the main features of the graph of the polynomial function (e.g., maximum and minimum points, possible roots, symmetry axis, possible qualitative manipulation of functions in the form ax^2). Various special cases and the general form of the rational function $y=\frac{c}{x}$ were also examined.

Trigonometric functions and their composition were studied next. The basic features and properties of the exponential functions were also discussed as well as the ill-defined functions of Weierstrass or Dirichlet without any reference to the geometrical representation. Reference was made to the inverse functions and to which functions can be inversed. The program ended by giving the set-theoretical definition of a function. The definition was then applied in order to identify whether each of the aforementioned types of relations as well as others, such as the formula of the circle, were functions or not.

Experimental Program 2 encouraged the interplay between the different modes of the representation of function in a systematic way. The instruction that was developed by Professor B on functions was based on two dimensions. The first dimension involved the intuitive approach and the definition of function. The second dimension emphasized the different representations of function. The instruction began with issues that are related to sets, the elements of a set and the operations of sets. The coordinate pairs and the Cartesian product were also discussed. The concept of correspondence was introduced, and equivalence and arrangement relations were defined. Then the activities for the study of the concept of function were based on the different relations between two sets, namely A and B, and examples of arrow diagrams, coordinate pairs and graphs were presented.

The second dimension of the instruction concerned representations. It included the following elements: theoretical models and interesting empirical studies on the connection of representations with mathematics learning, theories on the use of semiotic representations in the teaching of mathematics and the pedagogical implications as well as the concept of function. Then the solution of tasks in graphical and algebraic representations and examples of conversion of functions from one representation to another were presented.

In the light of the above, an essential epistemological difference can be identified between the two experimental programs. Experimental Program 1 involved instruction of a classic nature, widely used at the university level. In contrast, Experimental Program 2 was based on a continuous interplay between different representations of various functions.

The pre- and the post-tests involved conversion tasks that were similar to the tasks of the test used in the study 1 described above (Gagatsis & Elia, 2005b). In addition, another two questions asked what a function is and requested two examples of functions from their application in real life situations. The tests also included tasks asking students to identify, by applying the definition of the concept, whether mathematical relations in different modes of representation (verbal expressions, graphs, arrow diagrams and algebraic expressions) were presenting functions.

Comparing the success percentages of the students before and after instruction indicated great improvement with regards to the definition of function. In particular, while only 19% of the students gave an approximately correct definition (i.e. (i) accurate set-theoretical definition, (ii) correct reference to the relation between variables but without the definition of the domain and range, (iii) definition of a special kind of function, e.g. real, bijective, injective or continuous function) before the instruction, 69% of the students gave the corresponding definition after instruction. Students' success rates after instruction were also radically improved in most of the recognition and conversion tasks of the tests. For instance, the graph of the straight line $y=4/3$ was recognized as a function only by 26% of the students before instruction, while the graph of the line $y=-3$ was identified as a function by 82% of the students after instruction.

Analysis of the data gave four similarity diagrams. Two of the similarity diagrams involved the answers of the two experimental groups of students separately to the tasks of the test before instruction. The other two similarity diagrams included the answers of the two experimental groups of students separately, after instruction. Within the former two similarity diagrams distinct groups or subgroups of variables of students' responses in recognition tasks involving the same mode of representation of functions, i.e., in verbal form, in graphical form, in an algebraic form, in an arrow diagram, were formed separately. The particular finding revealed the consistency with which students dealt with tasks in the same representational format, but with different mathematical relations. However, lack of direct connections between variables of similar content, but different representational format, indicated that students were able to identify a function in a particular mode of representation (e.g., algebraic form), but not necessarily in another mode of representation (e.g., graphical). This inconsistent behaviour among different modes of representation was an indication of the existence of compartmentalization. This phenomenon also appeared in the similarity diagram referring to the students of Experimental Group 1, especially in the cases of the graphical representations and arrow diagrams. The compartmentalization was limited to a great extent, though, in the similarity diagram involving the responses of students of Experimental Group 2. Similarity connections were formed between students' performance in recognizing functions in different forms of representation, indicating that students dealt similarly with tasks irrespective of their mode of representation. In other words, success was independent from the mode of representation of the mathematical relation. This finding revealed that Experimental Program 2 was successful in developing students' abilities to flexibly use various modes of representations of functions and thus accomplished the breach of compartmentalization, i.e. de-compartmentalization, in their behaviour. The research in this direction, described briefly above, is still in progress.

5.2. Second effort to succeed at de-compartmentalization (Study 4)

Mousoulides and Gagatsis (2006) conducted a study exploring the effectiveness of computer based activities in de-compartmentalized registers of representations and problem solving processes in functions. A considerable number of research studies have examined the effects of technology usage on many aspects of students' mathematical achievement and attitudes, their understanding of mathematical concepts, and the instructional approaches in teaching mathematics. Despite this, only a limited number of researchers focused on the effects of using appropriately different modes of representations and making the necessary connections between them by using technological tools (Mousoulides & Gagatsis, 2006). The investigation presented here follows the investigation presented in Section 4.2. Researchers in the aforementioned study examined whether students' work with the aid of a mathematical software package could assist students in adopting and implementing effectively the "geometric approach" to solving problems in functions and therefore promote the de-compartmentalization of registers of representations in students' thinking.

The participants were ninety sophomore students in the Department of Education. Students were attending an undergraduate course on introductory calculus. Of these, 18% were males and 82% were females. The study was conducted in three phases. In the first phase, a questionnaire similar to the one that was developed in the second study, reported here, was administered at the beginning of the course. The second phase of the study was conducted over the course of the subsequent two weeks. During this period, forty of the 90 students were randomly selected to participate in four two-hour sessions. During these sessions students, working individually or in pairs, were asked to solve problems in functions using Autograph and to present and discuss their results in discussions with the whole class. Autograph (www.autograph-math.com), a visually compelling mathematical software, was used for the purposes of the study. Autograph and other similar software packages have various features which can facilitate a constructive approach to learning mathematics (Mousoulides, Philippou & Hoyles, 2005). Autograph allows the user to "grab and move" graphs, lines and points on the screen whilst observing changes in parameters, and vice versa. Additionally, with its multiple representation capabilities, it allows the user to switch easily between numeric, symbolic and visual representations of information. A sample problem that was discussed during the second phase is presented below:

The following is the graph of the function $f(x) = ax^2+bx+c$. Suggest possible values for a,b,c and explain your answers. Pose a related problem for the other students of your class that could be solved using your worksheet in Autograph.

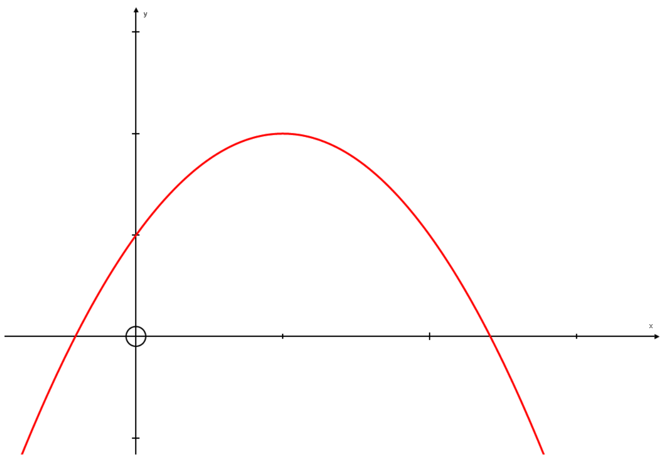


Figure 6: *The graph of the function $f(x) = ax^2+bx+c$ presented in one problem*

A second test, involving a second set of four tasks and two problems in functions was administered ten days after the completion of the second phase. All items in the second test were similar to the ones of the first test administered in the first phase.

Similar to the study presented in Section 4.2, researchers proposed that conversions could be carried out geometrically by focusing their attention and efforts on the relation of the symbolic representations of the two functions in order to construct the second graph or, algebraically, by selecting pairs of points to construct the new graph by "*ignoring*" its relation to the other one. Additionally, the study attempted to investigate how students' approaches in the conversions between different registers of functions were associated with their processes in problem solving. The main focus of Mousoulides and Gagatsis (2006) investigation was to examine whether student work on problems on functions with the aid of the appropriate mathematical software could result in the de-compartmentalization of the different registers of representations and their use in problem solving in functions.

The results of the study duplicate earlier findings (Mousoulides & Gagatsis, 2004), indicating that most of the students can correctly answer tasks on graphing linear (with success percentages being higher than 80%) and quadratic functions (with success percentages being higher than 65%). At the same time, their successful performance in solving related problems was limited to less than 25%. An important finding related to students' approaches showed that, in all tasks, more students preferred using the algebraic than the geometric approach. It is noteworthy that students who chose the algebraic approach applied it even in situations in which a geometric approach seemed easier and more efficient than the algebraic. Of interest is the second problem, for which the great majority of students failed to recognize or suggest a graphical solution as an option at all, even though the problem could not be solved algebraically.

Analysis of the data from the second test showed that both groups of students improved their percentages in solving both simple tasks and problems in functions. Of interest, is the finding that students who participated in the intervention phase (Group 1) outperformed their counterparts (Group 2) in all tasks and problems. In detail, Group 1 students' percentages were higher than those of Group 2 students with percentage differences varying from 4 % to 12 % in solving tasks and from 10% to 12% in problems. Furthermore, Group 1 students significantly improved their selection of geometric approach in solving tasks and problems in functions, indicating that the exploration and discovery of open ended problems in the environment of mathematical software like Autograph might have an influence on students' selection and use of the geometric approach in functions.

The findings from the two similarity diagrams were also quite impressive. One of the similarity diagrams involved Group 2 student responses, while the second one presented the results from Group 1 students. The similarity diagram for Group 2 students involved two *distinct* clusters with reference to students' approach. In keeping with previous findings, students who used the algebraic approach employed it consistently in the tasks and problems of the test, even in cases where the use of the geometric approach was more suitable. The similarity diagram for Group 1 students showed that their responses again formed two clusters, but these clusters were not compartmentalized into algebraic and geometric approaches. Indeed, one of the clusters showed that students were flexible in selecting the most appropriate approach for solving tasks on functions. Additionally, students were eager to switch their approach in solving a problem, especially in a problem which could not be solved using an algebraic approach. This was not the case for students in Group 2.

6. DISCUSSION

6.1. Identifying the phenomenon of compartmentalization and seeking ways to breach it

A main concern of the present paper was to investigate students' understanding of the concept of function via two perspectives. The first point of view concentrates on students' ability to handle different modes of the representation of function in tasks involving treatment and conversion and the second perspective refers to students' approaches in conversion tasks and complex function problems. Furthermore, this paper entailed some considerations with regards to the difficulties confronted by the students when dealing with different modes of mathematical representations and more specifically the phenomenon of compartmentalization. Another aim of this paper was to present two on-going investigations which attempted to design and implement different intervention programs having a common objective, i.e. to help students develop flexibility in working with various representations of function and thus accomplish de-compartmentalization of the different registers of representations in students' thinking.

The first study reported in this paper examined student performance in the conversions of algebraic relations (including functions) from one mode of representation to another. It was revealed that success in one type of conversion of an algebraic relation did not necessarily imply success in another type of conversion of the same relation. Lack of implications or connections among different types of conversion (i.e., with different starting representations or even with different target representations) of the same mathematical content indicated the difficulty in handling two or more representations in mathematical tasks. This incompetence provided a strong case for the existence of the phenomenon of compartmentalization among different registers of representation in students' thinking, even in tasks involving the same relations or functions. The differences among students' scores in the various conversions from one representation to another, referring to the same algebraic relation or function provided support for the different cognitive demands and distinctive characteristics of different modes of representation. This inconsistent behaviour was also seen as an indication of students' conception that different representations of the same concept are completely distinct and autonomous mathematical objects and not just different ways of expressing the meaning of a particular notion. Inconsistencies were also observed in students' responses with reference to the different conceptual features of the mathematical relations involved in the conversions, i.e. functions or not.

The most important finding of the second study was that two distinct groups were formatted with consistency, that is the algebraic and the geometric approach groups. The majority of student work with functions was restricted to the domain of the algebraic approach. This method, which is a point to point approach giving a local image of the concept of function, was followed with consistency in all of the tasks carried out by the students. Many students have not mastered even the fundamentals of the geometric approach in the domain of functions. Most of the students' understanding was limited to the use of algebraic representations and the algebraic approach, while the use of graphical representations was fundamental in solving geometric problems. Only a few students used an object perspective and approached the function holistically, as an entity, by observing and using the association of it with the closely related function that was given. Only these students developed the ability to employ and select graphical representations, thus the geometric approach. The second study's findings are in line with the results of previous studies indicating that students cannot use the geometric approach effectively (Knuth, 2000). The fact that most of the students chose an algebraic approach (process perspective) and also demonstrated consistency in their selection of this approach, even in tasks and problems in which the geometric approach (object perspective) seemed more efficient, or the fact that they failed to suggest a graphical approach at all, is a strong indication of the phenomenon of compartmentalization in the students' processes in tasks and problems on functions involving graphical and algebraic representations.

Moreover, an important finding of the second study involved the relation between the graphical approach and geometric problem solving. This finding is consistent with the results of previous studies (Knuth, 2000; Moschkovich et al., 1993), indicating that the geometric approach enables students to manipulate functions as an entity, and thus students are capable of finding the connections and relations between the different representations involved in problems. The data presented in the second study suggested that students who had a coherent understanding of the concept of functions (geometric approach) could easily understand the relationship between symbolic and graphic representations in problems and thus were able to provide successful solutions.

In both studies presented above, the results of the statistical analysis of C.H.I.C. provided a strong case for the existence of the phenomenon of compartmentalization in students' ways of dealing with different tasks on functions. However, the findings of each of the two studies were substantial and gave different information regarding the acquisition and mastery of the concept of function. Lack of implications and similarity connections among different types of conversion of the same mathematical content in Study 1 indicated that students were not in a position to change systems of representation of the same mathematical content of functions in a coherent way. Lack of implicative and similarity connections between the geometric approach and the algebraic perspective in students' responses in Study 2 provided support for students' deficiency in flexibly employing and selecting the appropriate approach, in this case the geometric one, to sketch a graph or to solve a problem on functions. It can be asserted that registers of representations remained compartmentalized in students' minds and mathematical thinking was fragmentary and limited to the use of particular representations or a particular approach in both types of transformation, that is, treatment and conversion.

Compartmentalization, as indicated by Duval (1993; 2002) and explained empirically in the present paper, is a general phenomenon that appears not only in the learning of functions, but also in the learning of many different concepts, as pointed out at the beginning of this paper. All these findings indicate students' deficits in the coordination of different representations related to various mathematical concepts. Duval (1993; 2002) maintains that the de-compartmentalization of representations is a crucial point for the understanding of mathematical concepts.

Identifying the phenomenon of compartmentalization among the registers of representation in students' thinking on functions indicated that current instructional methods fail to help students develop a deep conceptual understanding of the particular construct. On the basis of the above findings, two current experimental efforts have been designed and carried out for the teaching of functions in order to accomplish de-compartmentalization. The former research effort (Study 3) involved two experimental programs. Experimental Program 1 involved instruction of a classic nature and one widely used at the university level. On the contrary, Experimental Program 2 was based on a continuous interplay between different representations of various functions. The other study (Study 4) involved an experimental program that promoted the exploration and discovery of open ended problems in the environment of a mathematical software program that provided multiple representation capabilities and allowed the students to switch easily between numeric, symbolic and visual representations of information. Students that participated in Experimental Program 1 of Study 3 did not show a significant improvement in the conversion tasks and continued to treat the various representations of function as distinct entities, thus demonstrating a compartmentalized way of working and thinking. As regards Experimental Program 2 of Study 3 and the experimental program of Study 4, despite their distinctive features they were both successful in stimulating a positive change in students' responses and in attaining the de-compartmentalization of representations in their performance. More specifically, the former experimental program succeeded in developing students' abilities in the conversion from one mode of representation to another. The latter program was successful in developing students' flexibility to select the most appropriate approach in solving tasks in functions and to use the geometric approach in function problems efficiently.

6.2. Recommendations for further research

Research on the identification of the phenomenon of compartmentalization in the learning of functions and other concepts should be expanded. The present paper provides support to the systematic use of appropriate statistical tools, such as the implicative statistical analysis of R. Gras (1996), to assess and analyze students' understanding of functions or other mathematical concepts. A continued research focus is needed to find ways to breach the compartmentalized way of thinking in students. The research directed towards finding ways to develop students' flexibility in using different registers of representations of functions and in moving from one to another, described briefly above, continues so as to provide explanations for the success of the two aforementioned experimental programs and to determine those features of the interventions that were particularly effective in accomplishing de-compartmentalization. There is a need for longitudinal studies in the area of registers of representations and problem solving in functions to enhance our understanding of the effectiveness and appropriateness of intervention studies like the aforementioned one. Additional studies of a qualitative nature are also needed to uncover students' difficulties in the particular domain, to expand the knowledge of how students interact with different modes of representations of functions in a conventional setting or a technological environment and how they move from a particular approach, i.e. an algebraic strategy to a more advanced one, i.e. a geometric approach in solving tasks on functions. The results of such attempts may help teachers and researchers at the university and high school levels to place emphasis on certain dimensions of the notion of function and the pedagogical approaches to teaching functions, so that students can be assisted in constructing a solid and deep understanding of the particular concept.

References

Aspinwall, L., Shaw, K. L., & Presmeg, N. C. (1997). Uncontrollable mental imagery: Graphical connections between a function and its derivative. *Educational Studies in Mathematics, 33*, 301-317.

Bodin, A., Coutourier, R., & Gras, R. (2000). *CHIC : Classification Hiérarchique Implicative et Cohésive-Version sous Windows – CHIC 1.2 ,* Rennes : Association pour la Recherche en Didactique des Mathématiques.

Cheng, P.C.H. (2000). Unlocking conceptual learning in mathematics and science with effective representational systems. *Computers and Education, 33*, 109-130.

D'Amore B. (1998). Relational objects and different representative registers: cognitive difficulties and obstacles. *L'educazione matematica*, *1*, 7-28.

Dubinsky, E., & Harel, G. (1992*).* The nature of the process conception of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function. Aspects of Epistemology and Pedagogy* (pp. 85-106). United States: The Mathematical Association of America.

Duval, R. (1993). Registres de Représentation Sémiotique et Fonctionnement Cognitif de la Pensée. *Annales de Didactique et de Sciences Cognitives*, *5*, 37-65.

Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. *Mediterranean Journal for Research in Mathematics Education, 1*(2), 1-16.

Eisenberg, T., & Dreyfus, T. (1991). On the reluctance to visualize in mathematics. In W. Zimmermann & S. Cunningham (Eds.), *Visualization in Teaching and Learning Mathematics* (pp. 9-24). United States: Mathematical Association of America.

Elia, I., Gagatsis, A., & Gras, R. (2005). Can we "trace" the phenomenon of compartmentalization by using the I.S.A.? An application for the concept of function. In R. Gras, F. Spagnolo & J. David (Eds.), *Proceedings of the Third International Conference I.S.A. Implicative Statistic Analysis* (pp. 175-185). Palermo, Italy: Universita degli Studi di Palermo.

Evangelidou, A., Spyrou, P., Elia, I., & Gagatsis, A. (2004). University students' conceptions of function. In M. Johnsen Høines & A. Berit Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education*, (Vol. 2, pp. 351-358). Bergen, Norway: Bergen University College.

Even, R. (1998). Factors involved in linking representations of functions. *The Journal of Mathematical Behavior*, *17*(1), 105-121.

Gagatsis, A., & Elia, I. (2005a).A review of some recent studies on the role of representation in mathematics education in Cyprus and Greece. *Paper presented at CERME 4,* 2005, February, Saint Feliux de Guixols, Spain.

Gagatsis, A., & Elia, I. (2005b). Il concetto di funzione e le sue rappresentazioni nell'educazione secondaria. *Bollettino dei Docenti di Matematica, 50*, 41-54.

Gagatsis, A., & Shiakalli, M. (2004). Ability to translate from one representation of the concept of function to another and mathematical problem solving. *Educational Psychology, 24*(5), 645-657.

Gagatsis, A., Elia, I., & Mougi, A. (2002). The nature of multiple representations in developing mathematical relations. *Scientia Paedagogica Experimentalis, 39*(1), 9-24.

Gagatsis, A., Kyriakides, L., & Panaoura, A. (2004). Assessing the cross-cultural applicability of number lines in conducting arithmetic operations using structural equation modeling : A comparative study between Cypriot, Italian and Greek primary pupils. *World Studies in Education*, *5*(1), 85-101.

Gagatsis, A., Shiakalli, M.,& Panaoura, A.(2003). La droite arithmétique comme modèle géométrique de l' addition et de la soustraction des nombres entiers. *Annales de didactique et de sciences cognitives, 8*, 95-112.

Gras, R. (1996). *L'implication statistique.* Collection associée à ' Recherches en Didactique des Mathématiques'. Grenoble : La Pensée Sauvage.

Hitt, F. (1998). Difficulties in the articulation of different representations linked to the concept of function. *The Journal of Mathematical Behavior, 17*(1), 123-134.

Kaldrimidou, M., & Ikonomou, A. (1998). Factors involved in the learning of mathematics: The case of graphic representations of functions. In H. Stenbring, M.G. Bartolini Bussi & A. Sierpinska (Eds.), *Language and Communication in the Mathematics Classroom* (pp. 271-288). Reston, Va: NCTM.

Kaput, J. J. (1987). Representation systems and mathematics. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 19-26). Hillsdale, NJ: Lawrence Erlbaum.

Knuth, J. E. (2000). Student understanding of the Cartesian Connection: An exploratory study. *Journal of Research in Mathematics Education*, *31*(4), 500-508.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research, 60,* 1–64.

Lerman, I.C. (1981). *Classification et Analyse Ordinale des Données.* Paris: Dunod.

Lesh, R., Post, T. & Behr, M. (1987). Representations and translations among representations in mathematics learning and problem solving. In C. Janvier (Ed.), *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 33-40). Hillsdale, NJ: Lawrence Erlbaum.

Marcou, A., & Gagatsis, A. (2003). Rappresentazioni e apprendimento matematico: applicazioni nel campo delle frazioni. *La matematica e la sua didattica, 2,*124-138.

Markovits, Z., Eylon, B. and Bruckheimer, M. (1986). *Functions today and yesterday.* For the Learning of Mathematics, 6*(2)* 18-28.

Michaelidou, N., Gagatsis, A., & Pitta-Pantazi, D. (2004). The number line as a representation of decimal numbers: A research with sixth grade students. In M. Johnsen Høines & A. Berit Fuglestad (Eds.), *Proceedings of the 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. 3, pp. 305-312). Bergen, Norway: Bergen University College.

Moschkovich, J., Schoenfeld, A. H., & Arcavi, A. (1993). Aspects of understanding: On multiple perspectives and representations of linear relations and connections among them. In T. A. Romberg, E. Fennema, & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 69–100). Hillsdale, NJ: Lawrence Erlbaum Associates.

Mousoulides, N., & Gagatsis, A. (2004). Algebraic and geometric approach in function problem solving. In M. Johnsen Høines & A. Berit Fuglestad (Eds.), *Proceedings of 28th Conference of the International Group for the Psychology of Mathematics Education* (Vol. *3*, pp. 385-392). Bergen, Norway: PME.

Mousoulides, N., Philippou, G., Hoyles, C (2005). Mathematical discovery in the context of number sequences. *Paper presented at the 11th European for Research on Learning and Instruction, 2005, August.*

Mousoulides, N., & Gagatsis, A. (2006). The role of new technologies in improving problem-solving in functions. Pre-print, Department of Education, University of Cyprus.

Romberg, T. A., Fennema, E., & Carpenter, T. P. (Eds.). (1993). *Integrating research on the graphical representation of functions.* Hillsdale, NJ: Erlbaum.

Sfard, A. (1992). Operational origins of mathematical objects and the quandary of reification - The case of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy* (pp. 59-84). United States: The Mathematical Association of America.

Sierpinska, A. (1992). On understanding the notion of function. In E. Dubinsky & G. Harel (Eds.), *The Concept of Function. Aspects of Epistemology and Pedagogy* (pp. 25-28). United States: The Mathematical Association of America.

Vinner, S., & Dreyfus, T. (1989). Images and definitions for the concept of function. *Journal for Research in Mathematics Education, 20*(4), 356-266.

Yerushalmy, M., & Schwartz, J. L. (1993). Seizing the opportunity to make algebra mathematically and pedagogically interesting. In T. A. Romberg, E. Fennema & T. P. Carpenter (Eds.), *Integrating research on the graphical representation of functions* (pp. 41–68). Hillsdale, NJ: Lawrence Erlbaum Associates.

Athanasios Gagatsis
Departament of Education
University of Cyprus
Cyprus

E-mail: gagatsis@ucy.ac.cy

Iliada Elia
Departament of Education
University of Cyprus
Cyprus

E-mail: iliada@ucy.ac.cy

Nicholas Mousoulides
Departament of Education
University of Cyprus
Cyprus

E-mail: n.mousoulides@ucy.ac.cy

Learning Mathematics: Increasing the Value of Initial Mathematical Wealth

Adalira Sáenz-Ludlow [1]

A sign may recall a certain concept or combination of concepts from somebody's memory, and can also prompt somebody to certain actions. In the first case we shall call a sign a symbol, in the second a signal. The (nature of the) effect of the sign depends on context and the actual mental situation of the reader. Van Dormolen, 1986, p.157.

RESUMEN

Usando la teoría de signos de Charles Sanders Peirce, este artículo introduce la noción de riqueza matemática. La primera sección argumenta la relación intrínseca entre las matemáticas, los aprendices de matemáticas, y los signos matemáticos. La segunda, argumenta la relación triangular entre interpretación, objetivación, y generalización. La tercera, argumenta cómo el discurso matemático es un medio potente en la objetivación semiótica. La cuarta sección argumenta cómo el discurso matemático en el salón de clase, media el aumento del valor de la riqueza matemática del alumno, en forma sincrónica y diacrónica, cuando él la invierte en la construcción de nuevos conceptos. La última sección discute cómo maestros, con diferentes perspectivas teóricas, influyen en la dirección del discurso matemático en el salón de clase y, en consecuencia, en el crecimiento de la riqueza matemática de sus estudiantes.

- **PALABRAS CLAVE:** Riqueza matemática, interpretación, relación con signos, la tríada interpretación-objetivación-generalización.

ABSTRACT

Using the Peircean semiotic perspective, the paper introduces the notion of mathematical wealth. The first section argues the intrinsic relationship between mathematics, learners of mathematics, and signs. The second argues that interpretation, objectification, and generalization are concomitant semiotic processes and that they constitute a semiotic triad. The third argues that communicating mathematically is a powerful means of semiotic objectification. The fourth section presents the notion of mathematical wealth, the learners' investment of that wealth, and the synchronic-diachronic growth of its value through classroom discourse. The last section discusses how teachers, with different theoretical perspectives, influence the direction of classroom discourse and the growth of the learner's initial mathematical wealth.

Fecha de recepción: Febrero de 2006/ Fecha de aceptación: Mayo de 2006

1 Department of Mathematics and Statistics. University of North Carolina at Charlotte.

KEY WORDS: Mathematical Wealth, Interpretation, Relationships with Signs, The Triad Intepretation-Objectification-Generalization.

RESUMO

Usando a teoria de signos de Charles Sanders Peirce, este artigo introduz a noção de riqueza matemática. A primeira secção argumenta a relação intrínseca entre a matemática, os aprendizes de matemáticas, e os signos matemáticos. A segunda, argumenta a relação triangular entre interpretação, objetivação e generalização. A terceira, argumenta como o discurso matemático é um potente meio na objetivação semiótica. A quarta seção argumenta como o discurso matemático na sala de aula adequar o aumento do valor da riqueza matemática do aluno, em forma sincrônica e diacrônica, quando ele inverte a construção de novos conceitos. A última seção discute como maestros, com diferentes perspectivas teóricas, influem na direção do discurso matemático na sala de aula e, conseqüentemente, no crescimento da riqueza matemática de sus estudantes.

PALAVRAS CHAVES: Riqueza matemática, interpretação, relação com signos, a tríade interpretação-objetivação-generalização.

RÉSUMÉ

En utilisant la perspective sémiotique peircienne, cet article introduit la notion de richesse mathématique. La première section soutient qu'il y a une relation intrinsèque entre les mathématiques, les apprenants des mathématiques et les signes. La deuxième section soutient que l'interprétation, l'objectivation et la généralisation sont des processus sémiotiques concomitants et qu'ils constituent une triade sémiotique. La troisième section soutient que la communication mathématique est un puissant moyen sémiotique d'objectivation. La quatrième section présente la notion de richesse mathématique, l'investissement de cette richesse par les apprenants et la croissance synchronique et diachronique de sa valeur à travers le discours de la salle de classe. La dernière section discute de la façon dont les enseignantes et enseignants, avec des perspectives théoriques différentes, agissent sur l'orientation de la discussion dans la salle de classe et sur l'enrichissement de la pensée mathématique initiale des apprenants.

MOTS CLÉS: Richesse mathématique, interprétation, relation avec des signes, la triade interprétation-objectivation-généralisation.

Mathematics and its Intrisic Relationship with Signs

Since ancient times, philosophers and mathematicians alike have been concerned with the definition of mathematics as a scientific endeavor and as a way of thinking. These definitions have evolved both according to the state of the field at a particular point in time and according to different philosophical perspectives. Davis and Hersh, assert that "each generation and each thoughtful mathematician within a generation formulates a definition according to his lights" (1981, p. 8). To define *mathematics* is as difficult as to define *signs*. It is not easy to define either one without mentioning the other, as it is not easy to define them in a paragraph and even less in a couple of sentences. Mathematicians make use of and create mathematical signs to represent, "objectify", or encode their creations. On the other hand, learners interpret mathematical signs and their relationships both to decode the conceptual objects of mathematics and to objectify (i.e., encode) their own conceptualizations.

All kinds of signs and sign systems are ubiquitous in our lives but so is mathematics. Given the fascinating and ineludible dance between mathematics and signs, it is not surprising that some mathematicians become semioticians. Peirce, for example, dedicated several volumes to analyze the relationship between mathematical objects and mathematical signs (The New Elements of Mathematics, Vols. I, II, III, IV, 1976) as well as several essays to discuss the essence of mathematics (for example, the one published in Newman's *World of Mathematics*, 1956). Peirce defines mathematics as the science that draws necessary conclusions and its propositions as "fleshless and skeletal" requiring for their interpretation an extraordinary use of abstraction. He also considers that mathematical thought is successful only when it can be generalized. Generalization, he says, is a necessary condition for mathematical thinking.

Rotman (2000), inspired by Peirce's theory, has dedicated a book to define mathematics as a sign. At the beginning of his book, he gives an overarching definition of mathematics to conclude that mathematics is essentially a symbolic practice.

> Mathematics is many things; the science of number and space; the study of pattern; an indispensable tool of technology and commerce; the methodological bedrock of the physical sciences; an endless source of recreational mind games; the ancient pursuit of absolute truth; a paradigm of logical reasoning; the most abstract of intellectual disciplines. In all of these and as a condition for their possibility, mathematics involves the creation of imaginary worlds that are intimately connected to, brought into being by, notated by, and controlled through the agency of specialized signs. One can say, therefore, that mathematics is essentially a *symbolic practice* resting on a vast and never-finished language—a perfectly correct but misleading description, since by common usage and etymology "language" is identified with speech, whereas one doesn't speak mathematics but writes it. (2000, p. ix, emphasis added).

But *where* does this symbolic practice come from? Is mathematics, as an expression of the symbolic behavior of the human species, a part of all cultures? Davis and Hersh (1981) argue that mathematics is in books, in taped lectures, in computer memories, in printed circuits, in mathematical machines, in the arrangement of the stones at Stonehenge, etc., but first and foremost, they say, it must exist first *in people's minds*. They acknowledge that there is hardly a culture, however primitive, which does not exhibit some rudimentary kind of mathematics. There seems to be a common agreement among White (1956), Wilder (1973), Bishop (1988), and Radford (2006a) for whom mathematics is essentially a *cultural symbolic practice* that encapsulates the progressive accumulation of constructions, abstractions, generalizations, and symbolization of the human species. Progress, White contends, would have not been possible if it were not for the human ability *to give ideas an overt expression* through the use of different kinds of signs (or what he calls *the human symbolic behavior*). He asserts that human communication, as the most important and general of all symbolic behaviors, facilitates new combinations and syntheses of ideas that are passed from one individual to another and from one generation to the next. White also stresses that mathematics like language, institutions, tools, the arts, etc. is a cultural expression in the stream of the total culture. In fact, he argues that *mathematics is a synthesizing cultural process* in which concepts react upon concepts and ideas mix and fuse to form new syntheses. For White, culture is the locus of mathematical reality:

> Mathematical truths exist in the cultural tradition in which the individual is born and so they enter his mind from the outside. But apart from cultural tradition, mathematical concepts have neither existence nor meaning, and of course, cultural tradition has no existence apart from the human species. Mathematical realities thus have an existence independent of the *individual mind*, but are wholly dependent upon the *mind of the species*. (1956, pp. 2350-2351, emphasis added)

If mathematics is a symbolic practice, then the understanding of the nature of sign systems (i.e. the networking of signs over signs to create new sign-references according to a particular syntax, grammar, and semantics) is important for the teaching and learning of mathematics. Given that individuals, by nature, possess symbolic behavior and mathematics is a symbolic practice, then why do some students come to dislike mathematics as a subject and very soon fall behind? In general, semiotics theories give us a framework to understand the mathematical and the non-mathematical behavior of our students. Among different theoretical perspectives on semiotics, Peirce's theory of signs helps us to understand how we come to construct symbolic relationships based on associative iconic and indexical ones. A relation is iconic when it makes reference to the similarity between sign and object; it is indexical when it makes reference to some physical or temporal connection between sign and object; and it is symbolic when it makes reference to some formal or merely agreed upon link between sign and object, irrespective of the physical characteristics of either sign or object.

Representation and interpretation are two important aspects of Peirce's theory. He sees *representation* as the most essential mental operation without which the notion of *sign* would make no sense (Peirce, 1903) and considers that the mind comes to associate ideas by means of referential relations between the characteristics of sign-tokens and those of the objects they come to represent. As for *interpretation*, he considers that without the interpretation of signs, communicating with the self and with others becomes an impossible task (Peirce, CP vols. 2 and 4, 1974). That is, without being interpreted, a sign as a sign does not exist. What exists is a thing or event with the potential of being interpreted and with the potential of becoming a sign. Metaphorically speaking, a sign is like a switch; it becomes relevant and its existence becomes apparent only if it is turned on-and-off, otherwise, the switch is just a *thing* with the potential to become a switch. Likewise, a sign-token becomes a sign only when its relationship to an object or event is turned on in the flow of attention of the interpreting mind. That cognitive relationship between the sign-token and the interpreting mind is essential in Peirce's semiotic theory; in fact, it is what distinguishes his theory from other theories of signs. He crystallizes this interpreting relation between the sing-token and the individual as being the *interpretant* of the sign. This *interpretant* has the potential to generate a new sign at a higher level of interpretation and generalization. At this higher level, the new sign could, in turn, generate other iconic, indexical, or symbolic relationships with respect to the object of the sign. However, while the individual generates new *interpretants*, the object represented by the sign undergoes a transformation in the mind of the individual who is interpreting. That is, the object of the sign appears to be filtered by the continuous interpretations of the learner. In summary, Peirce considers the existence of the sign emerging both from the learner's intellectual labor to conceptualize the object of the sign and from the construction of this object in the learner's mind as a result of his intentional acts of interpretation.

> A sign stands *for* something *to* the idea that it *produces or modifies*. Or, it is a *vehicle* conveying into the mind something from without. That for which it stands is called its *object*; that which it conveys, its m*eaning*; and the idea to which it gives rise, its *interpretant*. (CP 1.339; emphasis added)

> By a Sign I mean anything whatever, real or fictile which is capable of a sensible form, is applicable to something other than itself…and that is *capable of being interpreted in another sign* which I call its Interpretant as to communicate *something that may have not been previously known about its* Object. There is thus a *triadic relation* between any Sign, and Object, and an Interpretant. (MS 654. 7) (Quoted in Pamentier, 1985; emphasis added).

Peircean semiotics helps to understand and explain many aspects of the complexity of the teaching and learning of mathematics. For example, teachers' and learners' expressions of their interpretations of mathematical signs by means of writing, reading, speaking, or gesturing; the interrelationship of the multiple representations of a concept without confounding the concept with any of its representations; and the dependency of mathematical notation on interpretation, cultural context, and historical convention. In trying to understand the semiotic nature of the teaching and the learning of mathematics, the above list about the semiotic aspects of the teaching-learning activity is anything but complete.

Brousseau, for example, contends that mathematicians and teachers both perform a "didactical practice" albeit of a different nature. Mathematicians, he says, do not communicate their results in the form in which they create them; they re-organize them, they give them the most general possible form; "they put knowledge into a communicable, decontextualized, depersonalized, detemporalized form" (1997, p. 227). This means, that they encode their creations using mathematical sign systems or they create new signs if necessary. That is, they objectify or symbolize their creations (i.e., knowledge objects) through spacio-temporal signs. On the other hand, the teacher undertakes actions in the opposite direction. She, herself, interprets mathematical meanings embedded in spacio-temporal signs (sign-tokens), decodes conceptual objects, and looks for learning situations that could facilitate the endowment of those sign-tokens with mathematical meanings in the minds of the learners. Thus, mathematicians and teachers of mathematics have a necessary interpretative relationship with the sign systems of mathematics (i.e., semiotic mathematical systems) because they continuously use them to encode, interpret, decode, and communicate the mathematical meanings of conceptual objects.

Teacher's and Learner's Interpretations and Objectifications

The interpretation of signs is important for two reasons. First, signs are not signs if they are not interpreted; being a sign means being a sign *of* something *to* somebody. Second, the meaning of a sign is not only in the sign but also in the mind interpreting that sign. Now the question is: Does a sign objectify? According to Peirce's definition of signs, the answer is yes. A sign does objectify (i.e., It does make tangible) the object (conceptual or material) that it *stands for*. However, the sign not only objectifies but it also communicates (to the interpreting mind) something that has not been previously known about the object. Thus, Peirce's definition of signs implies a continuous process of interpretation and as a consequence, a concomitant process of gradual objectification.

Radford (2006b), on the other hand, considers that to objectify is to make visible and tangible something that could not be perceived before. He defines *objectification* as "an active, creative, imaginative, and interpretative social process of gradually becoming aware of mathematical objects and their properties". This definition is not in contradiction with Peirce's definition of signs. Radford (2003) also defines *means of objectification* as "tools, signs of all sorts, and artifacts that individuals intentionally use in social-meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities" (p. 41). This definition is also in harmony with Peirce's definition of *interpretant.*

Since mathematical objects make their presence manifest only through signs and sign systems, how can teachers help learners to enter into the world of these semiotic systems and break the code, so to speak, to "see" those objects by themselves? Which mathematical objects do learners interpret from signs[2]? Or better, what "objects" do sign-tokens stand for in the minds of learners and teachers? Would

2 Peirce gave several definitions of signs without contradicting previous definitions; instead he extended them. The invariant in his definitions is the triadic nature of the sign. The variation is in the names he gave to the sign-vehicle/ sign-token or material representation of the sign. First, he called *sign* the material representation of the sign, then sign-vehicle, and then representamen. Some mathematics educators have favored the sign triad object-sign-interpretant, others, like myself, have favored the sign triad object-representamen-interpretant because it does not use the word sign to indicate, at the same time, the triad and a term in the triad. In this paper, I use the words representamen, representation, and sign-token interchangeably. However, Peirce used the term representation in the general sense of being a necessary operation of the human mind.

learners and teacher 'interpret' the same mathematical objects (i.e., knowledge objects) from sign relations in mathematical sign systems? Who objectifies what? What are the "products or effects" of teacher's and learners' interpretations and objectifications? What are the teacher's interpretations of the learners' interpretations? It appears that teachers' and learners' interpretations and objectifications go hand in hand in the teaching-learning activity. Because of the triadic nature of the sign, there is a necessary and concomitant relationship between objectification and interpretation; there is no interpretation without objectification and no objectification without interpretation. In addition, these two processes are linked to a third concomitant process, the process of generalization.

Mathematicians objectify their creations inventing new mathematical signs or encoding them, using already established signs and sign systems. Teacher and learners re-create knowledge objects by interpreting mathematical signs in a variety of contexts; by doing so, they undergo their own processes of objectification. There seems to be running, in parallel, three processes of objectification: the objectification of the teacher, the objectification of the learners, and the teacher's objectification of the learners' objectifications. This seems to be a cumbersome play with words, although this is at the heart of the interrelationship between teaching and learning. Obviously, teacher and learners *objectify*, but do they objectify the same thing? Are these objectifications isomorphic or at least do they resemble each other? Is the teacher aware of these processes of objectification? If so, then the teacher has the potential: (a) to question and validate her own interpretations and objectifications; (b) to make hypotheses about the learners' objectifications; (c) to question the learners to validate her hypothesis in order to guide their processes of interpretation and objectification; and (d) to differentiate between her interpretations and objectifications and the learners' interpretations and objectifications.

When teachers and learners engage in the teaching-learning activity, w*ho* interprets and *what* is interpreted is somewhat implied, but it is nevertheless tacit, in the processes of objectification and interpretation. Obviously, in one way or another, teachers appear to play an important role in the learners' processes of interpretation and objectification. Brousseau appears to indicate these levels of interpretation. "The teacher's work … consists of proposing a *learning situation* to the learner in such a way that [the learner] produces her knowing as a *personal answer* to a question and uses it or modifies it in order to satisfy the constraints of the *milieu* [which is managed by necessary contextual and symbolic relationships] and *not* just the *teacher's expectations"* (1997, p. 228, emphasis added). Here, Brousseau points out the difference between learners' interpretations and teachers' interpretations and intentions. The question is whether or not the teacher's intentions and interpretations are realized in the students' interpretations and objectifications. In other words, do the teacher's and the learners' interpretations and objectifications, at least, resemble each other?

The teacher may design learning situations to induce learners' *construction* of mathematical objects and relationships among those objects; or the teacher may design learning situations in which the mathematical object is directly *delivered* as if it were a *cultural artifact* ready to be "seen" and memorized by the learners, while saving them the cost of their own abstractions and generalizations. In the latter case, the learners could be objectifying only the iconic or indexical aspects of the mathematical signs without capturing the symbolic aspects of those signs and their symbolic relations with other signs. In the former case, the learners capture both the symbolic aspects of the signs and their symbolic relations with other signs. This means that the learner is able to unfold those signs to "see" not only the symbolic aspects but also the indexical and iconic aspects embedded in them. Thus, learners and teacher could be interpreting different aspects of the mathematical signs (iconic, indexical, or symbolic) and, in consequence, interpreting the nature of mathematical objects from different levels of generalization and abstraction.

But what is the nature of the mathematical objects? How many types of objects could be interpreted from mathematical signs? Duval (2006) calls our attention to different types of *objects:*

(1) *Objects* as *knowledge objects* when attention is focused on the invariant of a set of phenomena or on the invariant of some multiplicity of possible representations. Mathematical objects like numbers, functions, vectors, etc. are all *knowledge objects.*

(2) *Objects* as *transient phenomenological objects* when the focus of attention is on this or that particular aspect of the representation given (e.g., shape, position, size, succession, etc.).

(3) *Objects* as *concrete objects* when the focus of attention is only on their perceptual organization.

Thus given a sign-token (i.e., a representamen or a representation), one could interpret at face value a *concrete object* if one focuses strictly on the material aspects of this semiotic means of objectification without constructing relationships with other representations. One could also interpret a *phenomenological object* if one goes beyond pure perception and focuses on aspects of those representations in space and time. Or one could also interpret a *knowledge object* if one focuses on the invariant relations in a representation or among representations. For example, Duval (2006) considers that the algebraic equation of a line and its graph could be seen as *phenomenological objects* when one focuses on the material aspects of these representations (i.e., iconic and iconic-indexical aspects of the sign-tokens or representations); they could be *knowledge objects* if one focuses on the invariance of these representations (i.e., symbolic aspects). Once one is able to interpret and to objectify *knowledge objects*, one should be able to unfold the phenomenological (i.e., iconic, iconic-indexical) and material (i.e., iconic) aspects of those objects. However, if one objectifies only *phenomenological* and *concrete objects*, one would not necessarily come up with the symbolic aspects of their corresponding *knowledge objects*.

In a nutshell, Duval's characterization of 'objects' points out the semiotic stumbling blocks of the teaching and learning of mathematics. In this characterization, the manifestation of a *knowledge object* depends not only on its representation but also on the human agency of the interpreter, user, producer, or re-producer of that object. *Objects* could be either the interpretation of pure symbolic relations embedded in the sign-tokens or representations (i.e., knowledge objects or pure signifieds); or they could be pure material tokens with no signifieds (i.e., concrete objects or concrete things); or they could be materially based tokens interpreted in time and space (i.e., phenomenological objects). The best case would be when the knowledge object is objectified in space and time with structured signifieds and with the potential of being used again in private and inter-subjective conceptual spaces; and, vice versa, when mathematical knowledge objects are decoded from the material sign-tokens or representations without escaping their extension in space and their succession in time.

As teachers and learners engage in the teaching-learning activity, *which* objects are the teacher referring to and *which* objects are the learners interpreting, objectifying, and working with? In the best of all scenarios, teacher and learners could interpret, from the same sign-token or representation, the same knowledge object. However, sometimes learners might only be interpreting concrete objects (i.e., concrete marks) or phenomenological objects missing, in the process, the knowledge object; meanwhile the teacher might be interpreting that learners are interpreting knowledge objects. This situation would clearly mark a conceptual rupture between teacher and learners. Therefore, interpreting in the classroom is a process that unfolds at three levels: (1) the level of those who send an intentional message (the teacher or the students); (2) the level of those who receive and interpret the message (the learners or the teacher); and (3) the level of the sender's interpretation of the receiver's interpretation. Thus, in the teaching-learning activity, the interpretation process is not only a continuous process of objectification but it is also a relative process (relative not only to teachers and learners but also relative to their prior knowledge, not to mention their beliefs about knowledge and knowing).

Communicating Mathematically as a Means of Objectification

Communication in the mathematics classroom was viewed as depending exclusively on language (syntax and grammar), the active and passive lexicon of the participants, and the nature of the content of the message (Austin and Howson, 1979). Now, we have become aware that communication depends not only on natural language but also on the specific sublanguages of different fields of study, on linguistic and non-linguistic semiotic systems, and on a variety of social and cultural contexts in which the content of the message is embedded (Halliday, 1978; Habermas, 1984; Bruner, 1986; Vygotsky, 1987; Steinbring et al. 1998). Communication is also influenced by the behavioral dispositions and expectations of the participants as well as by their intersubjective relations of power (Bourdieu, 1991). Thus, perspectives on communication, in general, appear to have gained in complexity rather than in simplicity. Hence, perspectives on communication in the mathematics classroom have changed. This communication depends on natural language, mathematical sublanguage, and mathematical sign systems that mediate teacher's and learners' interpretations of mathematical objects.

Rotman (2000) points out a special feature of mathematical communication. He contends that in order to communicate mathematically, we essentially write. He contends that writing plays not only a *descriptive* but also a *creative* role in mathematical practices. He asserts that those *things* that are described (thoughts, signifieds, and notions) and the means by which they are described (*scribbles*) make up each other in a reciprocal manner. Mathematicians, as producers of mathematics, Rotman says, think their scribbles and scribble their thinking. Therefore, one is induced to think that learners of mathematics should do the same in order to produce and increase their personal 'mathematical wealth' as a product of their own mathematical labor. Such wealth does not accumulate all at once, but rather, it accumulates gradually in a synchronic as well as in a diachronic manner. We will enter the discussion of mathematical wealth and its synchronic-diachronic formation in the next section.

It appears that communicating mathematically is first and foremost an *act of writing* in the form of equations, diagrams, and graphs, supported all along by the specialized sublanguage of mathematics (mathematical dictionaries are a living proof that a mathematical sublanguage exists). We also need to consider that writing is not an isolated act. *Acts of writing* are concomitant with *acts of reading*, *listening*, *interpreting*, *thinking*, and *speaking*. All these acts intervene in semiotic processes of objectification resulting from personal processes of interpretation by means of contextualization and de-contextualization, concretization and generalization. That is, communicating mathematically depends on the synergy of the processes of interpretation, objectification, and generalization.

Gay (1980), Rossi-Landi (1980), and Deacon (1997) argue that any semiological system only has a finite lexicon but its semantics accounts for an unlimited series of acceptable combinations and that some of these combinations propose original ways of describing linguistic and extralinguistic reality. By the same token, the semiotic system of mathematics has a finite number of tokens and a finite set of axioms, theorems, and definitions (Ernest, 2006). When these elements are combined, they account for a large number of acceptable combinations that describe or justify, create or interpret, prove or verify, produce or decode already culturally structured mathematical objects. In discovering, constructing, apprehending, reproducing, or creating mathematical objects, reading and writing, listening and speaking become essential means for producing and interpreting combinations of referential relations (whether iconic, indexical, or symbolic) in a space that is both visible and intersubjective.

Vygotsky (1987) contends that in any natural language the writing and speaking acts are of different nature. Writing, he says, is a monological activity in which context is mental rather than physical and therefore it does not benefit from the immediate stimulation of others. This makes writing a demanding mental activity that requires not only the syntax and grammar of the language in use, but also the conceptual objects (i.e., knowledge objects) to be encoded or decoded using particular signs or combination of signs. In contrast, Vygostsky argues that oral dialogue is characterized by the dynamics of turn-taking determining the direction of speech: in oral dialogue, questions lead to answers and puzzlements lead to explanations. Written speech, instead, is not triggered by immediate responses as in oral dialogue. In writing, the unfolding of an argument is based much more on the personal and private labor of the individual. What Vygotsky argues about written and oral speech in the context of language can be transferred to the context of mathematical communication inside and outside of the classroom. It is one thing to clarify one's mathematical ideas when debating them and another to produce them as the result of one's own isolated mental labor and personal reflection. Both types of communication are commonly used among mathematicians (Rotman, 2000). In the last decades, oral and written modes of interacting in the classroom have been accepted as appropriate ways of communicating mathematically in the classroom (National Council of Teachers of Mathematics, Standards, 2000).

Rotman (2000) also considers that writing and thinking are interconnected and co-terminous, co-creative, and co-significant. There is no doubt that for professional mathematicians who are in the business of producing mathematical knowledge this should be the case. But are writing and thinking always interconnected, co-creative, and co-significant activities for the learners? Or are the learners using writing to take into account only the perceptual level of mathematical signs (i.e., sign-tokens or concrete objects) to automatically perform algorithmic computations in order to survive academically? Do multiple-choice exams interfere with the development of the learners' thinking-writing capacity? Do teachers make learners aware that reading, writing, listening, and speaking are effective means of objectifying mathematical knowledge objects? Do teachers make learners aware that communicating mathematically is also constituted by justifying in terms of explanation, verifications, making valid arguments, and constructing proofs?

To communicate mathematically in the classroom, the teacher has: (a) to flexibly *move* within and between different semiotic systems (e.g., ordinary language, mathematical sub-language, mathematical notations, diagrams, graphs, gestures, etc.) (Duval 2006); (b) to *refer* to mathematical objects that are other than visible and concrete (e.g., patterns, variance, and invariance across concepts) (see for example, Radford, 2003); (c) to *address* the learners in ways that are supposed to be meaningful to them (see for example, Ongstad, 2006); and (d) to *express* (verbally and nonverbally) the encoding and decoding of mathematical objects (Ongstad, 2006). Thus communicating mathematically between teacher and learners also requires the triad referring-addressing-expressing within and between several semiotic systems.

Interpreting mathematical signs is, in essence, a dynamic process of objectification in which the individual gradually becomes aware of knowledge objects represented in verbal, algebraic, visual, and sometimes imaginary representations (Davis and Hersh, 1981) and these representations have their own inherent properties. Becoming aware of knowledge objects through a variety of representations is in itself a demanding intellectual labor because of the characteristics of different representations. Skemp (1987), for example, points out differences between visual and verbal/algebraic representations: (1) Visual representations, such as diagrams, manifest a more *individual and analog* type of thinking; in contrast, verbal/algebraic representations manifest a more *socialized* type of thinking. (2) Visual representations tend to be *integrative or synthetic*; in contrast, verbal/algebraic representations are *analytical* and show detail. (3) Visual representations tend to be *simultaneous*; in contrast, verbal/algebraic representations tend to be *sequential*. (4) Visual representations tend to be *intuitive*; in contrast, verbal/algebraic representations tend to be *logical*. All these tacit differentiations are part and parcel of the tacit knowledge underpinning the classroom mathematical discourse and they may create difficulties for some learners (Presmeg, 1997). Yet another source of tacit knowledge in the classroom discourse is the variety of speech genres in mathematical discourse, for example, debating, arguing, justifying, and proving (Seeger, 1998). For Rotman, persuading, convincing, showing, and demonstrating are discursive activities with the purpose of achieving intersubjective agreement, generalization, and semiotic objectification.

This kind of *tacit knowledge* is not even remotely considered to be a part of the institutionalized school curriculum and many teachers are not even aware of it. The lack of explicitness of the tacit knowledge (expected to be understood by the learners) contributes to their abrupt and foggy entrance into the territory of the mathematical world, where those who will successfully accumulate 'mathematical wealth' are the ones who have the capacity of making *explicit* for themselves the *tacit* underpinnings of mathematical discourse and the triadic nature of the process of conceptualization (interpretation, objectification, and generalization).

To summarize, the emergence of mathematical objects and their meanings are in no way independent from intentional acts of interpretation and objectification mediated by reading and writing, speaking and listening. These acts are essential in the gradual mathematical growth of the *mathematical wealth* of the learners. Communicating mathematically in terms of reading, writing, listening, and debating should be considered *means of interpretation and objectification*. Hence, knowledge of semiotics appears to be a necessary conceptual tool in the classroom, not only for theoretical and explanatory purposes but also for pragmatic ones.

Communicating Mathematically and Mathematical 'Wealth'

We would like to consider *mathematical wealth* as a metaphor to refer to the learner's continuous accumulation of mathematical knowledge as the product of his *intellectual labor* in an *intra-subjective* or *inter-subjective space*. This mathematical wealth is personal, although socially and culturally constituted, in addition to continuously being in the making.

As learners initiate and continue their journey in a mathematical world (which is planned by the institutionalized curriculum and/or by the learners' own interests), they continuously invest their existing mathematical wealth in order to increase its value. This investment is a continuous process of evolution, development, and transformation of the learner's referential relations using signs of iconic, indexical, and symbolic nature. Sign-tokens are not inherently icons, indices, or symbols; they are so only if interpreted in that way. The learner's interpretation of the referential relations of signs is manifested in his verbal and written responses. Say for example, that a learner is capable of keeping in memory the expression "*positive times positive is positive and negative times negative is positive*"(*). What is the meaning of this expression for a learner at different phases of his mathematical journey? Does it change? Does it remain the same?

It could be that he has memorized this expression as we memorize prayers when we are little; they just stick in our minds and we regurgitate them, even if we do not know what they mean. It could be that the learner interprets that expression as follows: "I remember that with a '-'and a '-' I can make a '+''; and with a '+' and a '+' I can only make a '+'". In these cases, the learner has only an iconic relationship *with* the expression (*). The learner is trying to make sense by focusing on the physical resemblances of the sign-tokens. Would he be able to ascend from the level of having an iconic relation *with* the expression (*) to the level of having an indexical relation *with* it? If the learner says, for example, "I know that 2 times 3 is 6 and -2 times -3 is 6", then the learner has an iconic-indexical relation *with* the expression (*) because he has a particular case that, in a way, indicates the *possibility* of the generality of this statement. However, when the learner comes to transform the above expression into an expression like "xy>0 only in cases when x>0 and y>0 or when x<0 and y<0" or to recognize that "-x" could be positive or negative depending on the value of x; then the learner has a symbolic relation *with* the expression (*). In the latter, the learner has come to enrich the meaning of the expression (*) as he works with variable quantities in the context of algebra.

In fact, as the learner comes to develop a symbolic relationship *with* this expression, or the expression (*) *becomes* symbolic *for* the learner, he will also come to have an iconic and iconic-indexical relationship *with* it. This is to say that once a learner has a symbolic relation *with* a sign, he would be able to unfold it into iconic and iconic-indexical relations whenever necessary. But the other way around is not necessarily true. A learner, who has an iconic or an iconic-indexical relationship *with* a sign-token (in this case the expression (*)) may not necessarily have a symbolic relationship *with* it (i.e., the sign-token does not yet stand for a knowledge object in the mind of the learner). What does this mean in terms of objects? A learner who has constructed either a concrete or a phenomenological object may very well have not yet constructed a knowledge object. However, if the learner has constructed a knowledge object, one can safely infer that he also has constructed the corresponding concrete and phenomenological objects (i.e., the learner could be able to deconstruct the knowledge object into phenomenological and concrete objects).

When a learner repeats the expression "positive times positive is positive and negative times negative is positive", it means that he could have an iconic, an iconic-indexical, or a symbolic relationship *with* the expression. What is the relationship that the learner has constructed? This is not evident until the learner has the opportunity to use it in different contextual situations. How does the teacher, who is in charge of guiding the learner, interpret the kind of relationship that the learner has *with* the expression? The teacher could have a symbolic relationship *with* the expression (*) and think that the learner also has a symbolic relationship *with* it. In addition, if the teacher considers that any sign-token or representation is inherently symbolic, independently of the learner's interpretation, she would firmly believe that the learner could have only a symbolic relationship *with* it. Henceforth, the teacher will not change her interpretation of the learner's interpretation, and this might rupture the semantic link in the communication between the teacher and the learner. The teacher's expectations would run at a level higher than the current level of the learner's possibilities. This could prompt the teacher to misjudge the capabilities of the learner and to give up on the learner instead of creating new learning situations to induce the construction of the learner's symbolic relationship *with* sign-tokens (in this case the expression (*)). The worst case would be when the learner stops increasing the value of his initial mathematical wealth and soon falls behind others and with feelings of not having any intellectual capacity for mathematics.

The teacher needs to understand that the expression (*) or any other sign could have iconic, iconic-indexical, or iconic-indexical-symbolic meanings *for* the learner at different points of his mathematical journey. That is, the teacher should be aware that what one routinely calls "symbols" are nothing else than sign-tokens that can be interpreted at different levels of generalization. The teacher who comes to understand *what* is symbolic and for *whom, what* is iconic-indexical and for *whom, what* is iconic and for *whom,* should also come to see her teaching deeply rooted in her own learning of mathematics and in her learning of her students' learning.

A teacher unaware of hers and the learners' possible iconic, indexical, and symbolic relationships *with* signs has no grounds for making hypotheses about the learners' interpretations. Then, the teacher will only interpret her own interpretations but not those of the learners. That is, the teacher comes to collapse the three levels of interpretation (her own interpretation, the learners' interpretation, and her interpretation of the learner's interpretation) making it only one muddled level that barely reflects the cognitive reality of those involved in the teaching-learning activity. In doing so, the teacher loses cognitive contact with the learner and thus the opportunity to support his personal processes of re-organization and transformation of his prior knowledge. It is not surprising, then, that Bauersfeld (1998) noticed that learners are alone in making their own interpretations and that there is a difference between "the matter taught" and "the matter learned". In our framework, this would translate as the existence of a difference between the matter interpreted by the teacher, the matter taught by the teacher, and the matter interpreted by the learners.

At any given moment, learners start with a particular set of mathematical conceptualizations to be transformed and re-organized. This initial set of conceptual elements, with whatever mathematical value (iconic, indexical, or symbolic) , is what we would like to call the *initial mathematical wealth*. This wealth, if invested *in* well designed learning situations using a variety of contexts, will allow the learner to embed iconic relationships into iconic-indexical relationships and to embed iconic-indexical relationships into symbolic ones. By doing so, the learner will come to construct mathematical patterns (at different levels of generalization), and regulated combinations of mathematical signs according to the structure of the mathematical sign systems he is working with at that moment. For example, learners' generalization, in the natural numbers, that multiplication makes bigger and division makes smaller, has to be re-conceptualized or re-organized when they start working with decimals. Later on, multiplication needs to be generalized as an operation with particular properties. And even later, division needs to be re-cognized and re-organized as a particular case of multiplication. That is, the learner's relationship *with* multiplication and its results needs to be transcended and attention needs to be focused on the nature of the operation itself, leaving implicit the indexicality of particular results as well as the iconicity of the sign-tokens "times" or "x" (like in 4 times 2 or $4x2$) used for multiplication in grade school. That is, multiplication, in the long run, should become a *symbolic operation* in the mind of the learner and not only the mere memorization of multiplication facts and the multiplication algorithm.

Hence, the nature of the investment of the learner's mathematical wealth resides in his capacity to produce new levels of interpretations and concomitantly new objects (concrete, phenomenological, and knowledge objects) at different levels of generality (iconic, indexical, or symbolic). This kind of investment increases the learner's mathematical wealth and goes beyond the manipulation of "sign-tokens [3]". That is, the value of the investment increases as the learner's interpretation of signs ascends from iconic, to iconic-indexical, to iconic-indexical-symbolic along his recursive and continuous personal processes of interpretation, transformation, and re-organization. Moreover, what becomes symbolic at a particular point in time in the learner's conceptual evolution could become the iconic or iconic-indexical root of a new symbolic sign at a higher level of interpretation. For example, our middle school knowledge of the real numbers with the operations of addition and multiplication becomes the root for interpreting, later on, the field structure of real numbers (i.e., the set of real numbers with the operations of addition and multiplication constitutes an additive group and a multiplicative group respectively and also the operation of multiplication distributives over the operation of addition).

In summary, learners who become mathematically wealthy are those who, along the way, are able to *interpret knowledge objects* from concrete *sign-tokens* and, in the process, are able to transcend their phenomenological aspects (i.e., iconic-indexical) and ascend to symbolic relationships *with* them through continuous acts of interpretation, objectification, and generalization. No matter through what lens one sees teaching and learning (i.e., learners discover, construct, or apprehend mathematical concepts), this triadic intellectual process (interpretation-objectification-generalization) is in reality a continuous recursive synchronic-diachronic process in their intellectual lives. This process is not only synchronic. It would be impossible for the learner to appreciate, all at once, current and potential meanings embedded in contextual interpretations of mathematical signs. Only when the learners have traveled the mathematical landscape for some time, they are able to "see" deeper meanings in mathematical signs as they interpret them in new contexts and in new relationships with other signs. Hence, the process is also diachronic. In the diachronicity of the process, the learner comes to understand the meaning potential of different signs.

Continuity and recurrence (i.e., going back in thought to consider something again under a new light) is the essence of this synchronic-diachronic process. Continuity and recursion allow learners (1) to carry on with their personal histories of conceptual development and evolution and (2) to transcend conceptual experiences in particular contexts through the observation of invariance and regularities as they see those experiences from new perspectives. That is, the sequential nature of the synchronic-diachronic process upholds all personal acts of interpretation, objectification, and generalization as well as of self-persuasion. Essentially, this is a mediated and a dialectical process between learner's knowing and knowledge in the permanent presence of the continuous flow of time, not only synchronically (in the short lived present) but also diachronically (across past, present, and future). As learners travel through the world of school mathematics, they construct and interpret for themselves a network of mathematical conceptualizations that is continuously re-organized through mathematical discourse

2 It is worthwhile to notice that the expression *manipulation of symbols* becomes an oxymoron in Peirce's theory of signs and it could be replaced by the expression *manipulation of sign-tokens*.

(reading, writing, speaking, and listening) and de-contextualized through abstraction and generalization. As the learners' networks of mathematical conceptualizations become increasingly re-organized and transformed over time, the earlier value of their *mathematical wealth* also increases.

Where do Learners Build up and Consolidate their Mathematical Wealth?

As learners travel through a particular territory of the mathematical world (e.g., the institutionalized school curriculum) they become *mathematically wealthier* because they become better acquainted with the ins and outs of the territory (i.e., they are able to produce symbolic interpretations of signs, or they relate to signs iconically and indexically but in a systematic manner). Others have a bird's eye view of the territory (i.e., they are able to produce only isolated iconic, or indexical interpretations of signs or they relate to signs iconically or indexically but in an unsystematic manner) and soon forget they have seen the landscape because they have made no generalizations. Still others are able to finish their journey traveling on automatic mode (i.e., using calculators and memorized manipulations) to establish their own peculiar relationships *with* the mathematical code or mathematical semiotic systems. Henceforth, they are able to produce, at best, only iconic interpretations from signs that soon will be forgotten.

The learners' mathematical wealth is built in a socio-cognitive classroom environment grounded on collective mathematical discourse as opposed to the unidirectional discourse from the teacher to the students. The quality of this discourse and the teacher's focus of attention on the learners' mathematical arguments influence the ways in which learners invest their mathematical wealth and how they become mathematically wealthier. It is well known that teachers, who are in charge of directing the classroom discourse, guide their practices according to conscious or unconscious theoretical perspectives on mathematics and the teaching of mathematics and they focus their attention on different aspects of classroom discourse. Sierpinska (1998) delineates the theoretical perspectives of teachers within three ample frameworks: *constructivist*, *socio-cultural*, and *interactionist* theories. Constructivist perspectives focus primarily on the learners' actions and speech while the actions and speech of the teacher are seen as secondary; that is, the constructivist teacher focuses essentially on the learners and their mathematics. Socio-cultural (i,e.,Vygotskian) perspectives focus on the social and historical character of human experience, the importance of intellectual labor, the mediating role of signs as mental tools, and the role of writing in the individual's intellectual development; that is, the socio-cultural teacher focuses essentially on culture and mediated socio-cognitive relations. Interactionist perspectives focus on language as a social practice; that is, the interactionist teacher focuses essentially on discourse and intersubjectivity. The *behaviorist* perspective could be added to those emphasized by Sierpinska. The behaviorist teacher focuses essentially on the learners' performance and pays little attention, if any, to the learners' ways of thinking. Finally, eclectic teachers seem to intertwine one or more theoretical frameworks according to the needs of the learners and their personal goals as teachers.

In any classroom, one needs to be cautions about what could be considered successful classroom communication. Successful classroom discourse may not be an indication of successful mathematical communication. Steinbring et al. (1998) contend that learners may be successful in learning only the rituals of interaction with their teachers or the routine and stereotyped frames of communication (like the well-known initiation-response-evaluation and funneling patterns). This kind of communication, they argue, leaves the learners speechless mathematically although keeping the appearance of an exchange of mathematical ideas. Brousseau (1997), and Steinbring et al. (1988), among others, present us with classical examples in which teachers, consciously or unconsciously, hurry up or misguide learners' processes of interpretation. Thus, communicating mathematically is more than simple ritualistic modes of speaking or the manipulation of sign-tokens; it is based on a progressive folding of meaningful interpretations passing from iconic, to iconic-indexical, to iconic-indexical-symbolic, and vice versa the unfolding of these relations in the opposite direction. Or as Deacon (1997) puts it: "symbolic relationships are composed of indexical relationships between sets of indices, and indexical relationships are composed of iconic relationships between sets of icons" (p. 75). That is, more complex forms of objectification emerge from simpler forms (i.e., simpler forms are transcended but remain embedded in more complex ones).

This is to say that the learner's process of mathematical interpretation is mediated by mathematical sign systems (icons, indexes, or symbols and their logical and operational relations) to constitute networks of conceptualizations and strategies for meaning-making. Communicating mathematically is, in fact, a *continuous* semiotic process of *interpretation, objectification,* and *generalization.* The construction of generalizations takes the learner from simple iconic relations, to indexical relations, and then to symbolic relation (i.e., folding of iconic relations into indexical ones, and then embedding indexical relations into symbolic ones) in order to make new interpretations and new objectifications that produce new generalizations. Moreover, deconstructing generalizations takes the learner in the opposite direction (i.e., unfolding of symbolic relations into iconic-indexical ones, and unfolding iconic-indexical relations into iconic ones) in order to exemplify, in particular cases, the skeletal invariance arrived at in generalization. Both directions are necessary because, together, they manifest not only the recursive and progressive constructive power of individual minds but also they manifest the human and socio-cultural roots of mathematical thinking.

Concluding Remarks

Using a Peircean perspective on semiotics, this paper argues the notion of *mathematical wealth.* The initial cognitive mathematical wealth of any learner begins early in life. In his years of schooling and with the guidance of teachers, this initial wealth is progressively invested and its value gradually increased. The process of investment is, in essence, a mediated-dialectical process of decoding a variety of semiotic systems and, conversely, the encoding of thoughts and actions in those semiotic systems that intervene. Such systems could be of socio-cultural, pedagogical, or mathematical nature.

For mathematical wealth to increase in value in the process of investment, the learner has to decode not only the mathematical code but also the tacit code of socio-cognitive rules of engagement in the classroom. A priori and implicitly, he is expected to understand, that reading and writing, constructing and interpreting mathematical arguments, listening and speaking, and justifying in the form of explanation, verification, and proof are necessary activities for the learning of mathematics. He also has to understand that these activities can effectively mediate the appropriation and construction of mathematical meanings *from* mathematical signs and the encoding of his own interpretations and meaning-making processes back *into* mathematical signs.

The paper also argues three levels of interpretation in the classroom: (a) the learner's level of mathematical interpretation; (b) the teacher's own level of mathematical interpretation; and (c) the teacher's level of interpretation of the learners' mathematical interpretations. It is also argued that mathematical meanings are not only inherent in mathematical signs but also inherent in the learner's cognitive relationship *with* those signs. Such relationships could be of an iconic, indexical, or symbolic nature. These relationships are not necessarily disconnected since an iconic relationship could ascend and become an indexical relationship, and the latter could ascend and become a symbolic relationship. Vice versa, a symbolic relationship could be unfolded into an indexical relationship, and the indexical relationship could be unfolded into an iconic relationship. In fact, when learners manipulate sign-tokens, it is sometimes necessary, for efficiency, to keep symbolic relations implicit in one's mind. Keeping the ascending and descending directions of relationships *with* signs and sign systems allow learners to move from the particular to the general and from the general to the particular. The learners' relationships *with* mathematical signs and sign systems are the result of mediated-dialectical processes between the learner's knowing and knowledge in the synchronic and diachronic triadic process of *interpretation, objectification, and generalization*. The reader is referred to Radford (2003) and Sáenz-Ludlow (2003, 2004, and 2006) for other instances of learners' processes of interpretation, objectification, and generalization.

References

Austin, J. L. & Howson A. G. (1979). Language and mathematical education.*Educational Studies in Mathematics*, *10*, 161-197.

Bauersfeld, H. (1998). About the notion of culture in mathematics education. In F. Seeger, J. Voigt, & U. Waschescio (Eds.), *The culture of the mathematics classroom*, (pp. 375-389). Cambridge, UK: Cambridge University Press.

Bishop, A. J. (1988). *Mathematical enculturation: A cultural perspective on mathematics education.* Dordrecht: Kluwer Academic Publishers.

Bourdieu, P. (1991). *Language and symbolic power.* Cambridge, Massachusetts: Harvard University Press.

Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Edited by Nicolas Balacheff, Martin Cooper, Rosamund Sutherland, and Virginia Warfield. Dordrecht, The Netherlands: Kluwer Academic Press.

Bruner, J. S. (1986). *Actual minds, possible worlds*. Cambridge, Massachusetts: Harvard University Press.

Davis, P. J. & Hersh, R. (1981). *The mathematical experience*. Boston: Houghton Mifflin Company.

Deacon, T. (1997). *The symbolic species: The coevolution of language and the brain*. New York: W. W. Norton & Company.

Duval, R. (2006). The cognitive analysis of problems of comprehension in the learning of mathematics. In A. Sáenz-Ludlow, and N. Presmeg (Eds.), *Semiotic perspectives on epistemology and teaching and learning of mathematics, Special Issue, Educational Studies in Mathematics, 61*, 103-131.

Ernest, P. (2006). A semiotic perspective of mathematical activity: The case of number. In A. Sáenz-Ludlow, & N. Presmeg (Eds.), *Semiotic perspectives on epistemology and teaching and learning of mathematics, Special Issue, Educational Studies in Mathematics, 61*, 67-101.

Gay, W. (1980). Analogy and metaphor. *Philosophy and Social Criticism*, *7*(3-4), 299-317.

Habermas, J. (1984). *The theory of communicative action*. *2*. Boston: Beacon Press.

Halliday, M. A. K. (1978). *Language as social semiotics*. London: Arnold.

National Council of Teachers of Mathematics (2000). *Principles and Standards of School Mathematics*. Reston, Virginia: National Council of Teachers of Mathematics.

Ongstad, S. (2006). Mathematics and mathematics education as triadic communication? In A. Sáenz-Ludlow, and N. Presmeg (Eds.), *Semiotic perspectives on epistemology and teaching and learning of mathematics, Special Issue, Educational Studies in Mathematics, 61*, 247-277.

Parmentier, R. J. (1985). Signs' place in medias res: Peirce's concept of semiotic mediation. In E. Mertz & R.J. Parmentier (Eds.), *Semiotic mediation* (pp. 23-48). Orlando, Florida: Academic Press.

Peirce, C. S. (1903). The three normative sciences. In *The Essential Peirce*, *2*, (pp. 1893-1913) edited by The Peirce Edition Project, (pp. 196-207). Bloomington, Indiana: Indiana University Press.

Peirce, C. S. (1956). The essence of mathematics. In James R. Newman (Ed.), *The World of Mathematics*, *3*, (pp. 1773-1783), New York: Simon and Schuster.

Peirce, C. S. (1974). *Collected Paper (CP)*. C. Hartshorne, and P. Weiss (Eds.), *1-4*. Cambridge, Massachusetts: Harvard University Press. (Reference is made to volumes and paragraphs).

Peirce, C. S. (1976). *The New Elements of Mathematics (NEP)*. Carolyn Eisele (Ed), *1-4*. The Hague: Mouton Publishers.

Presmeg, N. C. (1997). Generalization using imagery in mathematics. In L. D. English (Ed.), *Mathematical reasoning: Analogies, metaphors, and images* (pp. 299-312). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Radford, L. (2003). Gestures, speech, and the sprouting of signs: A semiotic-cultural approach to learners' types of generalization. *Mathematical Thinking and Learning*, *5*(1), 37-70.

Radford, L. (2006a). The anthropology of meaning. In A. Sáenz-Ludlow, & N. Presmeg (Eds.), *Semiotic perspectives on epistemology and teaching and learning of mathematics, Special Issue, Educational Studies in Mathematics, 61*, 39-65.

Radford, L. (2006b). Glossary of internal document of the symposium of the Symbolic Cognition Group. Vermont, January 3-9, 2006.

Radford, L. (in press). Semiótica cultural y cognición. In R. Cantoral & O. Covián (Eds.), *Investigación en Matemática Educativa en Latinoamérica*. México.

Rossi-Landi, F. (1980). On linguistic money. *Philosophy and Social Criticism*, *7*(3-4), 346-372.

Rotman, B. (2000). *Mathematics as sign*. Stanford, California: Stanford University Press.

Sáenz-Ludlow, A. (2003). A collective chain of signification in conceptualizing fractions. *Journal of Mathematical Behavior, 22*, 181-211.

Sáenz-Ludlow, A. (2004). Metaphor and numerical diagrams in the arithmetical activity of a fourth-grade class. *Journal for Research in Mathematics Education, 1*(35), 34-56.

Sáenz-Ludlow, A. (2006). Classroom interpreting games with an illustration. In A. Sáenz-Ludlow, & N. Presmeg (Eds.), *Semiotic perspectives on epistemology and teaching and learning of mathematics, Special Issue, Educational Studies in Mathematics, 61*, 183-218.

Seeger, F. (1998). Discourse and beyond: On the ethnography of classroom discourse. In H. Steinbring, M. G. Bartolini Bussi, & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp.85-101). Reston, Virginia: National

Council of Teachers of Mathematics.

Sierpinska, A. (1998). Three epistemologies, three views of classroom communication: Constructivism, sociocultural approaches, interactionism. In H. Steinbring, M. G. Bartolini Bussi, & A. Sierpinska (Eds.), *Language and communication in the mathematics classroom* (pp. 30-62). Reston, Virginia: National Council of Teachers of Mathematics.

Skemp, R. (1987). *The psychology of learning mathematics*. Lawrence Erlbaum Associates.

Steinbring, H., Bartolini Bussi, M. G., & Sierpinska, A. (Eds.) (1998). *Language and communication in the mathematics classroom*. Reston, Virginia: National Council of Teachers of Mathematics.

Van Dormolen, J. (1986). Textual analysis. In B. Chirstiansen, A. G. howson, & M. Otte (Eds.), *Perspectives on mathematics education*, (pp. 141-171). Boston: B. Reidel Publishing Company.

Vygotsky, L. S. (1987). *Thinking and speech*. New York: Plenum Press.

White, L. A. (1956). The locus of mathematical reality: An anthropological footnote. In James R. Newman (Ed.), *The* World *of Mathematics*, *4*, (pp. 2348-2364). New York: Simon and Schuster.

Wilder, R. (1973/1968). *Evolution of mathematical concepts*. Milton Keyness, England: The Open University Press.

Adalira Sáenz-Ludlow
Department of Mathematics and Statistics
University of North Carolina at Charlotte
USA

E-mail: sae@email.uncc.edu

Everyday and Mathematical Language 100 Years After the Publication of "On Denoting" by Bertrand Russell

Giorgio T. Bagni [1]

RESUMEN

El artículo "On denoting" (*en torno a la denotación*) de B. Russell, publicado en 1905, es un hito de la reflexión filosófica sobre el lenguaje. En este artículo, examinamos la reacción de los alumnos, de una frase inspirada de un ejemplo célebre introducido por Russell, y de un aserto expresado en lenguaje matemático. Apartándonos del análisis de los datos experimentales que encierra la interpretación de los conceptos clásicos de realidad y de racionalidad, proponemos algunas reflexiones que pasan por alto "la objetividad epistémica estándar de la certeza privada hacia la práctica de la justificación en el interior de una comunidad comunicativa" (J. Habermas). Concluimos que el lenguaje constituye un momento muy importante en el cual el sentido de una expresión está fijo; sin embargo, mantenemos presente en nuestra mente que "el lenguaje, así como cualquier otro sistema semiótico, funciona en el interior de una red de significados culturales" (L. Radford).

- **PALABRAS CLAVE:** Lenguaje, justificación, sentido, racionalidad, verdad, validez.

ABSTRACT

The article "On denoting" by B. Russell, published in 1905, is a milestone in philosophical reflection on language. In the present paper, we examine pupils' reactions both to a sentence inspired by a celebrated example introduced by Russell and to a statement expressed in mathematical language. We move away from an interpretation of experimental data confined to the classical concepts of truth and rationality and propose instead some reflections that shift "the standard of epistemic objectivity from the private certainty of an experiencing subject to the public practice of justification within a communicative community" (J. Habermas). We conclude that language is a very important moment in which the meaning of an expression is fixed, but we keep in mind that "language, like any other semiotic system, functions inside a cultural network of significations" (L. Radford).

- **KEY WORDS:** Language, justification, meaning, rationality, truth, validity.

Fecha de recepción: Marzo de 2006/ Fecha de aceptación: Abril de 2006

1 Department on Mathematics and Computer Science. University of Udine (Italy)

RESUMO

O artigo “On denoting” (*em torno da denotação*) de B. Russell, publicado en 1905, é um sinal da reflexão filosófica sobre a linguagem. Neste artigo, examinamos a reação dos alunos, de uma frase inspirada em um exemplo célebre introduzido por Russell, e de uma afirmação expressada na linguagem matemática. Nos afastando da análise dos dados experimentais que contém a interpretação dos conceitos clássicos de realidade e de racionalidade, propomos algumas reflexiones que passam por alto “a objetividade epistemica padrão da certeza privada em direção à prática da justificação no interior de uma comunidade comunicativa” (J. Habermas). Concluímos que a linguagem constitui um momento muito importante no qual o sentido de uma expressão está fixo; entretanto, mantemos presente em nossa mente que “a linguagem, assim como qualquer outro sistema semiótico, funciona no interior de uma rede de significados culturais” (L. Radford).

- **PALAVRAS CHAVE:** Linguagem, justificação, significado, racionalidade, verdade, validade.

RÉSUMÉ

L’article “On denoting” (*De la dénotation*) de B. Russell, publié en 1905, est un jalon de la réflexion philosophique sur le langage. Dans cet article, nous examinons la réaction des élèves à une phrase inspirée d’un célèbre exemple introduit par Russell et à une assertion exprimée en langage mathématique. En nous écartant de l’analyse des données expérimentales qui limite l’interprétation aux concepts classiques de vérité et de rationalité, nous proposons quelques réflexions qui amènent « l’objectivité épistémique standard de la certitude privée vers la pratique publique de la justification à l’intérieur d’une communauté communicative » (J. Habermas). Nous concluons que le langage constitue un moment très important par lequel le sens d’une expression est fixé, mais nous gardons présent à l’esprit le fait que « le langage, ainsi que n’importe quel autre système sémiotique, fonctionne à l’intérieur d’un réseau de significations culturelles » (L. Radford).

- **MOTS CLÉS :** Langage, justification, sens, rationalité, vérité, validité.

1. Introduction

Many recent works show that culture and mathematical thinking are strictly linked (see for instance Wartofsky, 1979; Crombie, 1995; Radford, 1997; Furinghetti & Radford, 2002). And language is an important element in this link. A quotation by Radford (making reference to Ilyenkov, 1977, p. 79) will help us to frame more precisely the focus of our work and its educational relevance: Radford states that “language is *one* of the means of objectification (albeit a very important one), but … there are also several others”; moreover, “as a means of objectification, language does not objectify indiscriminately. Language, like any other semiotic system, functions inside a cultural network of significations, from whence grammar and syntax draw their meaning” (Radford, 2003a, p. 141; 2003b)[2]. The question with which we are going to deal in this paper is the following: firstly, can we consider language as a sort of favourite or absolute moment in which the meaning of an expression is fixed? (Let us notice, for instance, that paradigmatic analysis seeks to identify the different pre-existing sets of signifiers which can be related to the content of texts: Sonesson, 1998). Secondly, let us remember that, according to R. Rorty, the discipline presently called philosophy of language has two different sources: one of them is the cluster of problems “about how to systematize our notions of meaning and reference in such a way as to take advantage of quantificational logic”; the latter, explicitly epistemological, “is the attempt to retain Kant’s picture of philosophy as providing a permanent ahistorical framework for inquiry in the form of a theory of knowledge” (Rorty, 1979, p. 518). In this paper we are going to discuss, on the basis of some experimental data, whether or not we can always make reference to a definite set of meanings for linguistic expressions and, in particular, to a clear notion of truth.

From the historical viewpoint, G. Vattimo points out that “almost all the most important and subtle problems of contemporary language philosophy were elaborated and faced, for the first time, in the Middle Ages” (Vattimo, 1993, p. 640; in this paper the translations are ours). The medieval doctrine of *suppositio* is deemed significant (Bocenski, 1956, pp. 219-230; Kneale & Kneale, 1962). According William of Shyreswood, “meaning is the representation [*praesentatio*] of an idea in the mind. The *suppositio* is the co-ordination [*ordinatio*] of the concept under another concept” (Bocenski, 1956, p. 217); Petrus Hispanus, too, in his *Summulae logicales*, pointed out the difference between *significatio* and *suppositio* (Geymonat, 1970, I, p. 549; Bagni, 1997); and in his *Summa Logicae* (I, 63, 2) William of Ockham (1281-1349) stated that the *suppositio* “is a property belonging to a term, just because [it is included] in a proposition” (Bocenski, 1956, p. 219).

Nevertheless we cannot completely develop this interesting issue through reference to the Logic of the Middle Ages. We shall introduce the subject of our study through a theoretical framework based upon some elements of 20th-century philosophical research: in section 2 we shall make reference to the paper *On denoting* by Bertrand Russell (1872-1970), published a century ago, together with its historical connection to Meinong and Frege (2.1); some positions of Wittgenstein’s (2.2), Quine’s and Brandom’s (2.3) will allow us to introduce Apel’s and Habermas’ approaches (2.4), which are to be considered crucial for our work. Through these we shall discuss (section 5) experimental data (sections 3 and 4).

2. Theoretical framework

2.1. Frege and Russell

Let us consider first some reflections on “definite descriptions” (Penco, 2004, p. 54): we shall compare some ideas put forward by Gottlob Frege (1848-1925) and by

[2] Aristotle distinguished men from animals because of the presence of the logos (logos, often translated by “reason”; but H.G. Gadamer suggests a proper translation of this term by “language”: Gadamer, 2005, p. 155).

Russell. In order to introduce the problem, it is recalled that since Aristotle we have known that "through language we can correctly refer to things that do not exist [...] or to elements whose existence is possible but that can hardly be proved" (Lo Piparo, 2003, p. 165). It is moreover worth mentioning the theoretical approach of Alexius Meinong (1853-1920), who stated that "objects of knowledge do not necessarily exist" (Meinong, 1904, p. 27; Orilia, 2002).

The Fregean approach is based upon the Compositionality Principle (Frege, 1923, p. 36), according to which a statement containing a term without denotation has no truth value: for instance, a statement referring to a non-existing person is neither true nor false (Frege, 1892). On the contrary, according to Russell, statements containing definite descriptions (e.g. *the current President of the Italian Republic*) imply the existence of an individual (*Mr. Carlo Azeglio Ciampi*) to whom the considered property is referred (and this individual is unique), at least at the time when the sentence is stated (March 2006). The problem is that some definite descriptions (and names are definite descriptions too) do not refer to existing individuals: when we talk about *Ares* or *the father of Phobos and Deimos* we are not making reference to an existing individual.

In order to avoid ambiguity, in his article entitled *On denoting*, published in *Mind* a century ago, Russell suggested making the logical form of a definite description explicit. So, a proposition like *The father of Phobos and Deimos is the Greek god of war* would be *There is one and only one individual of whom it can be said: he is the father of Phobos and Deimos, and he is the Greek god of the war*. Frege's and Russell's approaches are very different. Let us consider, for instance, the sentence *The King of France is bald*: according to Frege it is neither true nor false because the term *the king of France* has no reference; according to Russell it is false because we can write it in the form: *There is one and only one king of France and he is bald* (Wittgenstein will make reference to a similar position: Wittgenstein, 1969a, p. 173).

Many years after the publication of *On denoting*, P.F. Strawson (1950) underlined an important distinction between a *sentence* and an *utterance* and this led us to distinguish between *denotation* and *reference*. Denotation links an expression and what it denotes (taking into account conventions and linguistic rules); reference links an expression and the object to which the speaker wants to make reference (Bonomi, 1973; Penco, 2004, p. 84). With *The King of France is bald*, Russell deals only with denotation, while Frege considers the speaker's idea to make reference to a non-existing object, so he concludes that the sentence has no truth value, such a reference being impossible. Of course if we consider a different context, e.g. a legend or a fiction where the king of France is actually bald, we would have to revise our position (it should be remembered that according to Frege, words must be considered only within a proposition: for instance, *Phobos and Deimos* could indicate either the sons of Ares and Aphrodite or the satellites of Mars; see: Frege, 1923).

2.2. Wittgenstein: from "Tractatus" to "Philosophical Investigations"

The position of Russell's most important pupil, Ludwig Wittgenstein (1889-1951), is rather complex because it must be divided into two very different periods. In his *Tractatus logico-philosophicus* (published in 1921 with a preface by Russell himself)

Wittgenstein reprises (sometimes critically) and develops some ideas of Frege’s and of Russell’s: while Frege considers natural language as unavoidably imperfect, Russell wants to point out its logical form (Russell, 1910) and Wittgenstein states that “in fact, all the propositions of our everyday language, just as they stand, are in perfect logical order” (Wittgenstein, 1922, § 5.5563; but Wittgenstein’s position expressed in his *Tractatus*, reveals some tension; see: Marconi, 2000a, p. 54); so if our language “looks ambiguous, we must recognise that its essence or its true logical form are hidden” (Penco, 2004, p. 60).

The so-called second Wittgenstein proposed a very different approach (his *Philosophical Investigations* were published in 1953, two years after the philosopher’s death): the meanings of words must be identified in their uses within a context. The concept of ‘language-game» is fundamental: it is a context of actions and words in which an expression assumes its meaning; so a language game is both a tool for the study of the language and the “starting point” where “we can reflect on the language by describing the differences and similarities of language games, instead of looking for its essence, as done in the *Tractatus*” (Penco, 2004, p. 105; concerning the continuity between the first and the second Wittgenstein, see: Marconi, 2000b, pp. 95-101). In addition, Hilary Putnam developed this approach and concluded that the meaning of a word is to be found in (and in some ways distributed among) the community of speakers (Putnam, 1992).

Let us now examine a remark by Habermas (that we shall reprise later): through his descriptive approach to the use of language, Wittgenstein levels its cognitive dimension; as soon as the truth conditions that we must know in order to employ propositions correctly are derived just from linguistic praxis *to which we are used*, the difference between validity and social value vanishes (Habermas, 1999, p. 80): this suggests a revision of the concepts of ‘validity’ and ‘truth’. Of course Habermas’ position must be considered critical: he underlines that the justification cannot be based upon life, but rather must be related to fundability (Habermas, 1983, p. 80). We shall reprise this point later.

2.3. Some ideas by Quine and Brandom

Willard Van Orman Quine (1908-2000) makes reference to the modality *de dicto* and *de re* (Quine, 1960; Kneale, 1962): “a *de re* belief is a belief expressed by the speaker about some properties of a certain object in the world; a *de dicto* belief is a belief expressed by the speaker about a proposition” (Penco, 2004, p. 161; interesting historical references can be found in: von Wright, 1951, pp. 25-28 and Prior, 1955, pp. 209-215). For instance, the proposition *John believes that Ares is the Greek god of war*, referring to a *de dicto* belief, cannot be replaced by *John believes that the father of Phobos and Deimos is the Greek god of war*: as a matter of fact we cannot be sure that John knows that *Ares* is *the father of Phobos and Deimos*. On the contrary, the proposition *about Ares John believes he is the Greek god of war*, referring to a *de re* belief, can be replaced by *concerning the father of Phobos and Deimos, John believes he is the Greek god of war*, where the speaker characterised *Ares* through a personal description, even if John does not know it. Some similar situations have been studied by Frege (see for instance: Frege, 1892; Origgi, 2000, pp. 110-123) and we shall reprise them in order to discuss our experimental data.

Brandom tries to revise some of Wittgenstein’s ideas and proposes replacing his language games with his 'game of giving and asking for reasons' (Brandom, 1994 and 2000). Although Brandom's conception of language has been considered restrictive (it does not consider aspects like calling, ordering etc.), his approach will be relevant to our research (see moreover: Habermas, 1999, pp. 102 and 140).

2.4. Apel and Habermas

According to Karl-Otto Apel (1987), every speaker implicitly makes reference to norms for *meaningful* and intelligible discourse, *truth* (romantic correspondence between sentence and reality), *veracity* (correct expression of the speaker's state) and *normative correctness* (respect of community rules). As a consequence we are able to acquire the conditions for 'ideal' communication, which assumes the role of normative principle: the discussion's impartiality and the possibility to reach some agreement among the bargaining parties depend on those conditions (see moreover the "rational" discussion as introduced in Lakoff & Johnson, 1980, p. 111, and the "conversation", p. 102).

According to Habermas, the rationality of judgements does not imply their truth, but only their justified acceptability in a particular context (Habermas, 1999, p. 102). Jürgen Habermas distinguishes between the truth of a statement and its rational affirmability (Habermas, 1999, p. 11) and reprises Apel's ideas (criticised in Davidson, 1990) in order to highlight the fundamental possibility of an 'ideal' communication: he underlines the importance of the inclusion in a universal world of well-ordered interpersonal relations, and the crucial element in order to do that is the rigorous condition of communication (Habermas, 1999, p. 279).

Intersubjective validity does not derive only from a convergence that can be observed with reference to the ideas of different individuals: Habermas refers epistemic authority to a community of practice and not only to individual experience (Habermas, 1999, pp. 136 and 238). The structure of the discourse creates a connection between the structures of rationality itself. As a matter of fact, it has three different roots, closely related one to another: the predicative structure of knowledge at an institutional level (Cassirer, 1958, III, p. 329), the teleological structure of the action and the communicative structure of the discourse (Habermas, 1999, p. 99). These Habermasian considerations will be very important in interpreting our experimental data.

3. Methodology

In this work, we are going to analyse the discussion of a group of students aged 15-16 years (5th class of a *Ginnasio-Liceo Classico*, in Treviso, Italy) regarding a question about the truth of two sentences in some ways similar to *The King of France is bald* (Russell, 1905)[3].

During a lesson in the classroom, pupils were divided into groups of three pupils each. The division was at random. The researcher (who was not the mathematics teacher of the pupils but who was however present in the

3 The *Ginnasio-Liceo Classico* is a school with high educational standards, in which pupils are asked to study many classical subjects, in disciplines such as Italian Literature, Latin and Greek Literature, History and so on; the mathematical curriculum is based upon elementary Algebra and Euclidean Geometry, and some basic elements of Logic are included (in particular, pupils knew the notion of proposition as a 'statement that can assume one and only one truth value, true or false': for instance, sentences including predicates related to subjective evaluations cannot be considered propositions).

classroom with the teacher and the pupils) proposed two sentences to the pupils and invited all the groups to decide if the given sentences were true or false. In particular, we are going to focus on the discussion that occurred in one of the groups.

The question was proposed while taking into account the importance of avoiding the suggestion of a strict dilemma (‘true or false?’), forcing the students to give a specific answer. As we shall see, the first sentence (*The King of the inhabitants of the Moon is bald*) makes reference to Russell’s aforementioned example; after some minutes, the second sentence ($1/0+1/0+1$ *is odd*) was added, in order to see the effect of asking such kinds of questions in sentences expressed in algebraic language[4].

4. Experimental data

The researcher writes the first sentence on the blackboard. The second sentence will be added after ten minutes:

For each sentence say: Is it a true sentence? Is it a false sentence?

(1) The King of the inhabitants of the Moon is bald (2) …

Discuss your answer in the group and write it on a sheet of paper.

Here is the (translated) transcription of the conversation that took place in the group formed by A., B., C.

4.1. Transcription

- [01] A.: (*smiles*) What is it?
- [02] B.: (*in a low voice*) *The King of the inhabitants of the Moon is bald*.
- [03] A.: The King of the inhabitants of the Moon, what does that mean?
- [04] C.: Well, I say, the Moon is something with no hair, if we consider the sun and its beams…
- [05] B.: (*ironically*) But what are you talking about?
- [06] C.: No, no, I am joking, there are no inhabitants on the Moon. If they existed, I would be able to state something.
- [07] B.: (*looks around*) But what does it mean, true or false?
- [08] A.: I do not know who the inhabitants of the Moon are, and then, come on, there are no inhabitants on the Moon and so there is not a king.
- [09] B.: Then it is false.
- [10] C.: It’s not as easy as it seems, in my opinion there is something unclear. They are playing with words and so we don’t understand. Let’s read carefully. There is not a king, and the inhabitants, what does it mean? On the Moon there is nobody, hence the king of the Moon is the Moon itself.
- [11] A.: Perhaps there are some micro-organisms, something that we cannot see, entities different from us.
- [12] C.: (*gesticulating*) Or think, perhaps someone saw an astronaut with his helmet, so that he looks bald and when he talks about it, so perhaps it is true.
- [13] A.: (*sure*) No, it is not relevant, it says the inhabitants of the Moon, it doesn’t say the Moon or the king of the Moon is false, I mean bald. We must see the inhabitants and then the king.
- [14] B.: Well, in this case it’s false, there are no inhabitants, no king, hence of course he’s not bald.

4 Of course, a full evaluation of this important aspect ought to be based upon particular and detailed research.

- [15] A.: Watch out, perhaps there's a trap, as he says (*C.*), perhaps the exercise cannot be done.
- [16] B.: I'll divide this sentence up: when I say that the king of the inhabitants does not exist, full stop, it is false, and what follows is also false. If I say, later, that he is bald or not, this is not important, do you understand?
- [17] A.: (*doubtful*) So let's say that the sentence… would be false.
- [18] C.: Yes, the simplest thing to do is to answer that it's false. But if the question deals with a film or a tale with a king of the Moon that is bald, in that tale it's true.
- [19] A.: Just a moment, it's better to emphasize the king of the Moon, in our answer. The king is false. If we want to say that the whole sentence is false we must be able to see the king, with his hair and …
- [20] B.: (*interrupting*) No, it's impossible to see him, he doesn't exist. (*To C.*) It's no tale, otherwise they would have told us. So it's false.
- [21] A.: (*after a while*) In short, one thing is to say that a sentence is false, I say that something is not true and so there is something wrong in the sentence. Another thing is to talk about someone and then say he is, for instance, bald or not; when I talk about a person, I suppose he exists.
- [22] B.: No, wait, but in your opinion is it enough to say something about someone who doesn't exist in order to make him real? If he doesn't exist, he's false.
- [23] A.: He is not false, the king; the problem is whether it's false that he is bald. Let's think carefully, before answering. It seems false, but perhaps it's not so.
- [24] B.: Listen, think about the question as a whole, they say the king is bald, it can be false because the king is not bald or because there is no king at all. If we want it to be true we must have the king and he must be bald.
- [25] C.: (*looking at A., a bit impatient*) Come on, it's clearly false! You make us wrong, if you say that it's not false, then it is true, and what do you mean? Do you mean that the inhabitants of the Moon are bald?
- [26] A.: Eh, it's not true, it's obvious. However it is not easy to understand. (*Looking at B.*) No, you are right, let's write false, I agree.

Now the Researcher completes the task on the blackboard:

For each sentence say: Is it a true sentence? Is it a false sentence?

(1) The King of the inhabitants of the Moon is bald (2) $1/0+1/0+1$ *is odd*

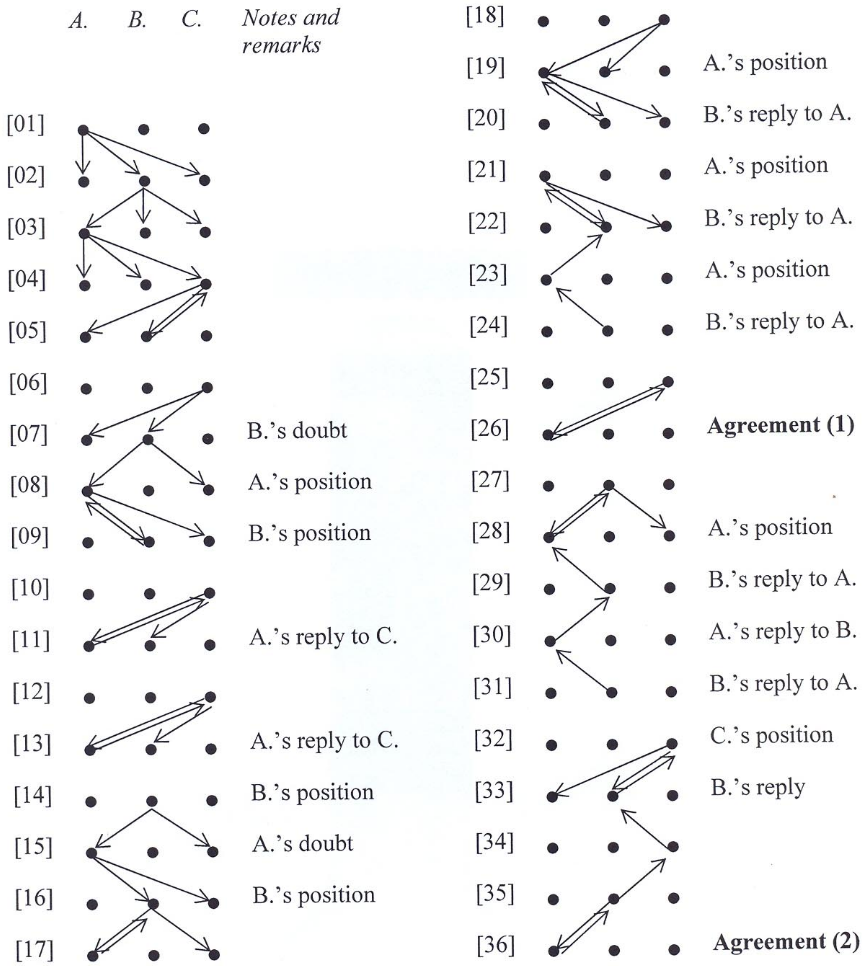
Discuss your answer in the group and write it on a sheet of paper.

- [27] B.: Yes, it's like before. False.
- [28] A.: (*doubtful*) Just a moment… if we say false, it's even. Maybe this exercise is impossible.
- [29] B.: No, why do you think even? It's different. Here it's odd, we must look at this sentence.
- [30] A.: Watch out, it's not like the first sentence. And what about if they had said even?
- [31] B.: False. It would be false, 1/0 is not a number.
- [32] C.: $1/0$ means infinity.
- [33] B.: No, the teacher told us it isn't true, $1/0$ is impossible.

- [34] C.: It's not infinity but it's a very very big number. How can I say if it's odd or even?
- [35] B.: No, no, it's not a number, it would be very big but actually it doesn't exist.
- [36] A.: Come on, there is a trick: they make you think it's odd because it's like $2+2+1$ that would be 5, but the starting number doesn't exist. It's false, once again.

4.2. Interaction flow chart

In the following flow chart (Sfard & Kieran, 2001; Ryve, 2004) different arrow directions are used to distinguish *proactive* and *reactive* utterances. In the case considered, the essential connection with everyday language prompted us to avoid the distinction between *object-level* and *non-object-level* utterances.



In the next section we are going to analyse our experimental data (transcriptions and flow chart) on the basis of our framework.

5. Discussion

5.1. First sentence

In [03] A. proposes the problem of reference and in [04] C. seems to suggest the possibility of an unusual interpretation of ‘bald’ (“the Moon is something with no hair, if we consider the sun and its beams…”). However the student himself, turns back in [06] to a more usual meaning (“No, no, I am joking, there are no inhabitants on the Moon”). A.’s next utterance, [08], can be connected to the Compositionality Principle: “Come on, there are no inhabitants on the Moon and so there is not a king”.

C.’s utterance [10] is interesting: “they are playing with words and so we don’t understand. Let’s read carefully. There is not a king, and the inhabitants; what does it mean? On the Moon there is nobody, hence the king of the Moon is the Moon itself”. He does not recognise the “perfect logical order” of common language (Wittgenstein, 1922, p. 5.5563): as well as ‘referential opacity’ (Quine, 1960), he considers the semantic aspect and proposes an unusual *suppositio* (if “on the Moon there is nobody”, we could say that “the king of the Moon is the Moon itself”).

C.’s next utterance [12] is also interesting (“Or think, perhaps someone saw an astronaut with his helmet, so that he looks bald and when he talks about it, so perhaps it is true”): the communication function of the language is explicitly considered (Dummett, 1993, p. 166; see moreover: Habermas, 1999, p. 105) and this is the one point in which falsehood, although in *de dicto* modality, does not refer only to the problem of existence. A’s utterance [13] (“no, it is not relevant, it says the inhabitants of the Moon, it doesn’t say the Moon or the king of the Moon is false, I mean bald. We must see the inhabitants and then the king”) is not completely clear, but brings the discussion back to the main question.

Now we can consider the direct comparison of B.’s ideas with A.’s. In [14] B. says: “well, in this case it’s false, there are no inhabitants, no king, hence of course he’s not bald”. A.’s utterance [15] expresses some doubts (“perhaps the exercise cannot be done”): he seems to choose a ‘Fregean’ approach, and a conclusion avoiding the assignment of a truth value, but in [16] B. expresses his viewpoint further: “I’ll divide this sentence up: when I say that the king of the inhabitants does not exist, full stop: it is false, and also what follows is false. If I say, later, that he is bald or not, this is not important, do you understand?” The Compositionality Principle is once again followed, but B. seems to consider a ‘Russellean’ denotation. A.’s utterance [17] (“so let’s say that the sentence… would be false”) does not show conviction.

C.’s utterance [18] refers to the importance of the context (see moreover the *suppositio*): now the connection between an expression’s meaning and its use in a context is clear: “but if the question deals with a film or a tale with a king of the Moon that is bald, in that tale it’s true”.

In [19] A. declares his willingness to accept the falsehood of the sentence considered, but underlines that it mainly refers to the existence of the king of the Moon: “just a moment, it’s better to emphasize the king of the Moon, in our answer. The king is false”. This point is interesting: like in [17], A. shows a positive frame of mind with reference to B.’s position, but according to him “if we want to say that the whole sentence is false we must be able to see the king, with his hair and …”.

After B.’s reply [20], taking into account C.’s objections too (“It’s no tale, otherwise they would have told us”) and after a while, in [21] A. says: “One thing is to say that a sentence is false. I say that something is not true and so there is something wrong in the sentence. Another thing is to talk about someone and then say he is, for instance, bald or not; when I talk about a person I suppose he exists.” So A. seems to propose a distinction between a *de dicto* modality and a *de re* modality: the pupil would distinguish a statement like *I say that the king of the Moon is bald* and a statement like *I say about the king of the Moon that he is bald* (Penco, 2004, p. 191). The second expression, in A.’s opinion, would be divided up in the following way: *I am talking about the king of the Moon and (later) I say he is bald*: so the expressions examined would bind the speaker.

As we can see from the flow-chart, a direct comparison between A. and B. now resumes ([21]-[24]): B.’s reply [22] is interesting (“but in your opinion is it enough to say something about someone who doesn’t exist in order to make him real?” This brings to mind Meinong’s position according to which “objects of knowledge do not necessarily exist”: Meinong, 1904, p. 27). Nevertheless, A. is not completely persuaded and certainly, in this ‘ game of giving and asking for reasons’: he acknowledges in [23] the plausibility of B.’s conclusions (“it seems false, but perhaps it’s not so”) but at the same time confirms his ‘Fregean’ approach (“he is not false, the king; the problem is whether it’s false that he is bald”). However, the first part of the discussion is about to finish: as a matter of fact, in [24] B. states once again his ‘Russellean’ viewpoint: “listen, think about the question as a whole, they say the king is bald, it can be false because the king is not bald, or because there is no king at all. If we want it to be true we must have the king and he must be bald”.

While [14], [16] and [22] did not completely persuade A., this utterance is crucial and conclusive (C.’s utterance [25], “come on, it’s clearly false” can be compared with a well-known note of Wittgenstein’s: “all I should further say as a final argument against someone who did not want to go that way, would be: ‘Why, don’t you see…!’ – and that is no *argument*”: Wittgenstein, 1956, I,§ 34). In [26], after pointing out the lack of clarity in the expression examined (“Eh, it’s not true, it’s obvious, however it is not easy to understand”: and A. makes reference to a ‘non-truth’, perhaps in order to underline its difference from a ‘falsehood’) A. accepts B.’s conclusions.

With reference to Apel’s perspective, A.’s doubts do not seem to be related to comprehension of the meaning of the discourse: its ‘truth’ (correspondence between sentence and reality) is connected with or perhaps set against its *normative correctness* (respect of community rules), mainly if we consider the features of a critical analysis of the sentence itself, of the “definite descriptions” (Penco, 2004, p. 54) that we find in it and of the coordination of its parts ([24]: “it can be false because the king is not bald, or because there is no king at all”). If we keep in mind the distinction between the truth of a statement and its rational affirmability (Habermas, 1999, p. 11) and if we interpret ‘correctness’ as acceptability according to rigorous conditions of communication (Habermas, 1999, p. 279), we can say that A. is induced to accept the correctness of the shared final choice thanks to the argument developed by the group of students (in particular by B.). We shall reprise these considerations in the final section of our work.

5.2. Second sentence

B.’s role is now sure and, as shown by the flow-chart, the discussion about the second sentence can be divided into two moments: a first debate between A. and B. ([27]-[31]) and a second debate between C. and B. ([32]-[35]). In both these moments, B. expresses his positions properly, taking into account the results of the previous discussions about the first sentence (see for instance the utterance [27]).

A.'s doubt [28] is interesting (the utterance is similar to [15], but now it is based upon a different argument). According to A., to say that '$1/0+1/0+1$ *is odd' is false* would correspond to saying that '$1/0+1/0+1$ *is even' is true*: let us note that a similar argument (to say that *'The king of the inhabitants of the Moon is bald' is false* would correspond to saying that *'The king of the inhabitants of the Moon is hairy' is true*) was not considered by A. in the previous part of the discussion (only C.'s utterance [25] can be connected to this argument). Such a difference seems to be related to the different contexts: the mathematical one, with its particular language and symbols, can suggest the use of *tertium non datur*.

B.'s strong utterance [31] ("$1/0$ isn't a number") is very important: the student interprets the sentence $1/0+1/0+1$ *is odd* as $1/0+1/0+1$ *is an odd number* and, more precisely, $1/0+1/0+1$ *is a number and this number is odd*. The first part of this sentence is false (the analogy with B.'s utterance [16] is clear: we have once again a 'Russellean' denotation) so all the sentence must be considered false.

The discussion between C. and B. deals with the 'nature' of $1/0$: in [32] C. states "1/0 means infinity" and, because of B.'s objection ([33]: "no, the teacher told us it isn't true, $1/0$ is impossible"), in [34] C. changes his mind and states that "it's a very very big number", so "how can I say if it's odd or even?" However in [35] B. points out: "no, no, it's not a number, it would be very big but actually it doesn't exist" and the discussion leads A. to accept B.'s justified position explicitly ([36]: "the starting number doesn't exist. It's false, once again").

It should be noted that the syntactic structure $n+n+1$ to which the second sentence makes reference can lead the students to consider an odd number. This element is very relevant, and in our opinion this is the crucial point with reference to the role of algebraic language: in the first sentence, the existence of the king of the inhabitants of the Moon would have no consequences about his hair, but now if n is an integer, $n+n+1$ would really be an odd number (in [36] A. says that "they make you think it's odd because it's like $2+2+1$ that would be 5, but the starting number doesn't exist"). But this factor did not influence the students.

6. Concluding remarks

Let us now turn back to the questions proposed in the Introduction. Clearly experimental data can lead us to state once again that language is a very important moment in which the meaning of an expression is fixed; but clearly we must also keep in mind that "language, like any other semiotic system, functions inside a cultural network of significations" (Radford, 2003a, p. 141). It is impossible to make reference to a completely sure set of meanings and to a single, absolute notion of truth (moreover, relevant issues concern the connection between the acquisition of a representation, namely a linguistic one, with the full conceptual acquisition of an object: D'Amore, 2001b; see moreover: Duval, 1998, D'Amore, 2001a, 2003a and 2003b).

The experience described brings to mind a position held by Putnam (1992) according to which the meaning (and we are thinking about a whole sentence, more than about a single word) is to be found in the community of the speakers and refers to different ways of considering the sentence (and, as we shall see, to the three “different roots of rationality”: Habermas, 1999, p. 99). Rorty notices that a merely ‘subjective’ argument must be disregarded by the reasonable partners of a conversation (Rorty, 1979, p. 368): we realized that a meaning has been built by collective negotiation, a real ‘game of giving and asking for reasons’ (Brandom, 2000); but in our opinion it is trivial to conclude that both arguments by B. and by A. are plausible (Strawson, 1950). As a matter of fact, this plausibility of both positions and their evolution lead us to posit: is it correct to propose a similar ‘truth evaluation’?

Of course both sentences were ambiguous, while the choice true-false can be considered only if the assigned sentence is a real ‘proposition’: but how can our pupils recognise real ‘propositions’? The traditional answer ‘a proposition is a statement that assumes one and only one truth value’, in this case, can be circular. Moreover, it is important to realize that the ambiguity considered is not connected to the structure of the assigned sentences (for instance, $3/6+3/6+1$ *is odd* is clearly a… perfect proposition!).

The task considered is neither connected only to an isolated epistemic rationality, nor refers only to coherence (Rorty, 1979, p. 199; Williams, 1996, p. 267; certain and coherent proofs can coexist with “conceptual confusion”: Wittgenstein, 1953, pp.II-XIV) or analogy: the comparison [27]-[31] demonstrates that the difference in the contexts (the first sentence is expressed in common language, the second refers to a mathematical context) does not authorize us to transfer the truth value from the first to the second sentence uncritically. Moreover, the term ‘false’ can have different values in different contexts (Lakoff & Johnson, 1980 p. 153).

So, should we doubt everything? This question is misleading (“if you tried to doubt everything you would not get as far as doubting anything. The game of doubting itself presupposes certainty”: Wittgenstein, 1969b, p. 115; from the logical viewpoint we agree with Lolli, 2005, p. 13-17). Furthermore, a charge of a conventionalistic reduction of the concept of truth would be groundless (Andronico, 2000, p. 252); Wittgenstein himself would reply: “‘So you are saying that human agreement decides what is true and what is false?’ – It is what human beings *say* that is true and false; and they agree in the *language* they use. That is not agreement in opinions but in form of life” (Wittgenstein, 1953, p. 241).

As noted in 2.4, this position has been elaborated by some authors. It is important to consider our traditional notions of ‘truth’ and ‘validity’: knowledge’s objectivity criterion is founded on public praxis instead of private certainty, so ‘truth’ becomes a ‘three members’ concept of validity (Habermas, 1999, p. 239), a validity justified with reference to a public (Schnädelbach, 1992).

The discussion of our experimental data does not allow us to conclude only that working together (in groups) is useful: such a conclusion would be induced by our opting to propose the exercise to some groups of pupils. The final common decision of the students was achieved after an active discussion, and had some consequences (Habermas, 1999, p. 137; in our case, for instance, the group must declare its decision to the Researcher, to the Teacher and to other students); so we must surpass the sphere of propositions (and texts) and take into account the sphere of actions, e.g. in using a predicate (as noticed by Kambartel, 1996, p. 249). With regard to the students' behavior, the discussion (in the perspective of a decision to be taken) seems to interpret the mentioned position and to develop the different roots of rationality (Habermas, 1999, p. 99). Of course the debate, under the explicit influence of the text of the assigned exercise, is still far from the 'ideal' communication described by Habermas and by Apel (C.'s role, for instance, is often minor, although his utterances related to the *suppositio* are really interesting); in other groups of students, the discussion developed without a final agreement (Lakoff & Johnson, 1980); nevertheless our experimental data (in particular utterances [19], [21]-[24], [28]-[31] and [32]-[35], too) enables us to state that the discussion did not lead the pupils only to a convergence of different ideas, but to a real change of viewpoint (see Habermas, 1999, p. 238 e 254). This fundamental moment can be highlighted in the utterances [24] and [35].

We would like to make a final reflection: we provided out students with a stimulating question about the truth (and the falsehood) of some sentences in different contexts, and this is quite a traditional exercise; but how can we speak about 'truth' with any certainty? Rorty asks himself if the truth of a sentence can really be considered as independent from the context of the justification (Rorty, 1994) and our experience seems to bear out his doubt: the behavior of some students did change after the passage from a non-mathematical context to a mathematical one; for instance, in [28]-[30] and in [36] the influence of algebraic syntax is clear (A.: "they make you think it's odd because it's like $2+2+1$ that would be 5, but the starting number doesn't exist"; let us remember that the mathematical curriculum of the Italian *Ginnasio-Liceo Classico* includes several chapters devoted to algebraic syntax; nevertheless, as previously noted, algebraic language's general role in pupils' behavior should be investigated more deeply).

Reflection on these issues is important (Lakoff & Johnson, 1980, p. 197-222): a distinction between 'validation' (*Geltung*) and 'validity' (*Gültigkeit*) is fundamental and can lead us to weaken the traditional distinction between the 'validation' of a statement that is approved and the 'validity' of a statement that *deserves* intersubjective acknowledgment because it is true (Habermas, 1999, p. 277). If we accept that a truth predicate can be considered (also) in the language game of the argumentation, we can point out its importance (also) with reference to its functions in this language game and hence in the pragmatic dimension of a particular use of the predicate (Habermas, 1999, p. 246) and we must take into account some important consequences. Truth itself must be related to a particular culture (to a particular language system): probably students belonging to different cultures would express their arguments in a different way (as previously noted, in Italy, the *Ginnasio-Liceo Classico* is considered a school with high educational standards). Truth is relative to comprehension, so there are no points of view allowing us to obtain 'absolutely objective truth' (Lakoff & Johnson, 1980, p. 236 and 283).

Thus, the intercultural aspect must be considered and this point is expressed in Wittgenstein too: "if anyone believes that certain concepts are absolutely the correct ones, and that having different ones would mean not realizing something that we realize – then let him imagine certain very general facts of nature to be different from what we are used to, and the formation of concepts different from the usual ones will become

intelligible to him” (Wittgenstein, 1953,§ II-XII). This point of view has been examined by M. Messeri, who concludes: “so there is something intrinsically misleading in ethnocentric behavior according to which different cultures are incomplete, rough and unsatisfactory” (Messeri, 2000, p. 190).
Moreover, some influences of didactical contract can be considered: probably students’ arguments would be different if used outside the school, in a different context. So, does the predicate of truth have different uses? Is ‘school rationality’ different from ‘everyday rationality’? What are the consequences in the educational sphere? Further research can be devoted to examining these important points more deeply.

References

Andronico, M. (2000). Giochi linguistici e forme di vita. In Marconi, D. (Ed.), *Guida a Wittgenstein*. Laterza, Roma-Bari, 241-288.

Apel, K.-O. (1987). Fallibilismus, Konsenstheorie der Wahrheit und Letzbegründung. Forum f. Philosophie. *Philosophie und begründung*. Suhrkamp, Frankfurt a.M., 116-211.

Bagni, G.T. (1997). *Elementi di storia della logica formale*. Pitagora, Bologna.

Bocenski, J.M. (1956). *Formale Logik*. Verlag Karl Alber, Freiburg-München (page numbers refer to the Italian translation: *La logica formale.* Einaudi, Torino 1972).

Bonomi, A. (1973). *La struttura logica del linguaggio*. Bompiani, Milano.

Brandom, R. (1994). *Making it Explicit*. Harvard University Press, Cambridge MA.

Brandom, R. (2000). *Articulating Reasons. An Introduction to Inferentialism*. Harvard University Press, Cambridge MA.

Cassirer, E. (1958). *Philosophie der symbolischen Formen*. WBG, Darmstadt.

Crombie, A.C. (1995). Commitments and styles of European scientific thinking. *History of Sciences 33*, 225-238.

D’Amore, B. (2001a). Une contribution au débat sur les concepts et les objets mathématiques. *Scientia Paedagogica Experimentalis* XXXVIII, *1*, 17-46.

D’Amore, B. (2001b). Conceptualisation, registres de représentations sémiotiques et noétique: interactions constructivistes dans l’apprentissage des concepts mathématiques et hypothèse sur quelques facteurs inhibant la dévolution. *Scientia Paedagogica Experimentalis* XXXVIII, *2*, 143-168.

D’Amore, B. (2003a). La complexité de la noétique en mathématiques ou les raisons de la dévolution manqué. *For the learning of mathematics, 23*(1), 47-51.

D'Amore, B. (2003b). The noetic in mathematics. *Scientia Pedagogica Experimentalis* XXXIX, *1*, 75-82.

Davidson, D. (1990). The Structure and Content of Truth. *The Journal of Philosophy, 87*, 279-328.

Dummett, M. (1993). Language and Communication. In *The Seas of Language*. Oxford University Press, Oxford, 166-187.

Duval, R. (1998). Signe et objet (I). Trois grandes étapes dans la problématique des rapports entre représentation et objet. *Annales de Didactique et de Sciences Cognitives, 6*, 139-163.

Frege, G. (1892). Über Sinn und Bedeutung. *Zeitschrift für Philosophie und philosophische Kritik, 100*, 25-50 (Senso e riferimento. *Senso, funzione e concetto*, Laterza, Roma-Bari, 2001).

Frege, G. (1923). Logische Untersuchungen, Dritter Teil: Gedankegefuge. *Beitrage zur Philopsophie des Deutschen Idealismus, 3*, 36-51 (page numbers refer to Italian translation: *Ricerche logiche*. Guerini, Milano 1992).

Furinghetti, F. & Radford, L. (2002). Historical conceptual developments and the teaching of mathematics: from philogenesis and ontogenesis theory to classroom practice. In English L. (Ed.), *Handbook of International Research in mathematics education*. Erlbaum, Hillsdale, 631-654.

Gadamer, H.G. (2005). *Linguaggio*. Laterza, Roma-Bari.

Geymonat, L. (1970). *Storia del pensiero filosofico e scientifico*. Garzanti, Milano.

Habermas, J. (1983). *Moralbewusstsein und kommunikatives Handeln*. Suhrkamp, Frankfurt a.M. (page numbers refer to Italian translation: *Etica del discorso*. Laterza, Roma-Bari 1985).

Habermas, J. (1999). *Wahrheit und Rechtfertigung. Philosophische Aufsätze*. Suhrkamp, Frankfurt a.M. (page numbers refer to Italian translation: *Verità e giustificazione*. Laterza, Roma-Bari 2001).

Ilyenkov, E. (1977). The concept of the ideal. In *Philosophy in the USSR. Problems of Dialectical Materialism*. Progress Publishers, Moskwa, 71-79.

Kambartel, F. (1996). Universalität, richtig verstanden. *Deutsche Zeitschrift für Philosophie, 44*, 249.

Kneale, W.C. (1962). *Modality "De Dicto" and "De Re"*. In Nagel, E., Suppes, P. & Tarsky, A. (Eds.), *Logic, Methodology and Philosophy of Science*, I. The University Press, Stanford, 622-633.

Kneale, W.C. & Kneale, M. (1962). *The Development of Logic*. Clarendon Press, Oxford.

Lakoff, G. & Johnson, M. (1980). *Metaphors we live by*. University of Chicago Press, Chicago (page numbers refer to Italian translation: *Metafora e vita quotidiana*. Bompiani, Milano 1998).

Lolli, G. (2005). *QED Fenomenologia della dimostrazione*. Bollati Boringhieri, Torino.

Lo Piparo, F. (2003). *Aristotele e il linguaggio*. Laterza, Roma-Bari.

Marconi, D. (2000a). Il Tractatus. In *Guida a Wittgenstein*. Laterza, Roma-Bari, 15-58.

Marconi, D. (2000b). Transizione. In *Guida a Wittgenstein*. Laterza, Roma-Bari, 59-102.

Meinong, A. (1904). Über Gegenstandstheorie. In Meinong, A., Ameseder, R. & Mally, E., *Untersuchungen zur Gegenstandstheorie und Psychologie*. Barth, Leipzig, 1-50 (page numbers refer to Italian translation: *Teoria dell’oggetto*. Quodlibet, Macerata 2003).

Messeri, M. (2000). Seguire la regola. In Marconi, D. (Ed.), *Guida a Wittgenstein*. Laterza, Roma-Bari, 151-192.

Origgi, G. (2000). *Introduzione a Quine*. Laterza, Roma-Bari.

Orilia, F. (2002). *Ulisse, il quadrato rotondo e l’attuale re di Francia*. ETS, Pisa.

Penco, C. (2004). *Introduzione alla filosofia del linguaggio*. Laterza, Roma-Bari.

Prior, A.N. (1955). *Formal Logic*. Oxford University Press, London.

Putnam, H. (1992). Significato, riferimento e stereotipi. In Bottani A. & Penco, C. (Eds.), *Significato e teorie del linguaggio*. Franco Angeli, Milano.

Quine, W.V.O. (1960). *Word and Object*. MIT Press, Cambridge MA.

Radford, L. (1997). On psychology, historical epistemology and the teaching of mathematics: towards a socio-cultural history of mathematics. *For the Learning of mathematics*, *17*(1), 26-33.

Radford, L. (2003a). On the epistemological limits of language. Mathematical knowledge and social practice in the Renaissance. *Educational Studies in Mathematics*, *52*(2), 123-150.

Radford, L. (2003b). On Culture and Mind. A post-Vygotskian Semiotic Perspective, with an Example from Greek Mathematical Thought, In Anderson, M. & Al. (Eds.), *Educational Perspectives on Mathematics as Semiosis*. Legas, Ottawa, 49-79.

Rorty, R. (1979). *Philosophy and the Mirror of Nature*. Princeton University Press, Princeton NJ (page numbers refer to the German translation: *Der Spiegel der Natur*. Suhrkamp, Frankfurt a.M. 1981).

Rorty, R. (1994). Sind Aussagen universelle Geltungsansprüche? *Deutsche Zeitschrift für Philosophie*, *42*(6), 975-988.

Ryve, A. (2004). Can collaborative concept mapping create mathematical productive discourses? *Educational Studies in Mathematics*, *26*, 157-177.

Russell, B. (1905). On Denoting. *Mind*, *14*, 479-493.

Russell, B. (1910). Knowledge by Acquaintance and Knowledge by Description. *Proceedings of the Aristotelian Society*, *11*, 108-128.

Schnädelbach, H. (1992). Thesen über Geltung und Wahrheit. *Zur Rehabilitierung des animal rationale*. Suhrkamp, Frankfurt a.M., 104-115.

Sfard, A. & Kieran, C. (2001). Cognition as communication. Rethinking learning-by-talking through multi-faceted analysis of students' mathematical interactions. *Mind, Culture, Activity*, *8*(1), 42-76.

Sonesson, G. (1998). The Concept of Text in Cultural Semiotics. *Trudy po znakyvym sistemam - Sign System Studies 26*, Tartu University Press, Taru, 88-114.

Strawson, P.F. (1950). On Referring. *Mind*, 59, 320-344 (Flew, A., Ed.: *Essays in Conceptual Analysis*. Macmillan, London 1960, 21-52).

Vattimo, G. (Ed.) (1993). *Enciclopedia Garzanti di Filosofia*. Garzanti, Milano.

Von Wright, G.E. (1951). *An Essay in Modal Logic*. North-Holland, Amsterdam.

Wartofsky, M. (1979). Perception, representation and the forms of action: towards an historical epistemology. In *Models. Representation and the scientific understanding*. Reidel, Dordrecht, 188-209.

Williams, M. (1996).*Unnatural doubts*. Princeton University Press, Princeton NJ.

Wittgenstein, L. (1922). *Tractatus logico-philosophicus*. Routledge and Kegan Paul, London.

Wittgenstein, L. (1953). *Philosophische Untersuchungen*. Blackwell, Oxford.

Wittgenstein, L. (1956). *Bemerkungen über die Grundlagen der Mathematik*. Blackwell, Oxford.

Wittgenstein, L. (1969a). *Philosophische Grammatik*. Blackwell, Oxford (page numbers refer to the Italian translation: *Grammatica filosofica*. La Nuova Italia, Firenze 1990).

Wittgenstein, L. (1969b). *Über Gewissheit*. Blackwell, Oxford.

Giorgio T. Bagni
Dipartimento di Matematica e Informatica
Università di Udine
Italia

E-mail: bagni@dimi.uniud.it

Semiosis as a Multimodal Process

Ferdinando Arzarello [1]

RESUMEN

Las aproximaciones semióticas clásicas resultan ser muy estrechas para investigar los fenómenos didácticos del salón de clase de matemáticas. Además de los recursos semióticos estándares utilizados por los alumnos y los maestros (como los símbolos escritos y el lenguaje hablado), otros recursos semióticos importantes son los gestos, las miradas, los dibujos y los modos extra-lingüísticos de expresión. Sin embargo, estos últimos caben difícilmente en las definiciones clásicas de los sistemas semióticos. Para superar esta dificultad, en este artículo adopto una perspectiva vygotskiana y presento una noción extendida de sistema semiótico, el haz semiótico*,* que se revela particularmente útil para incluir todas los recursos semióticos que encontramos en los procesos de aprendizaje de las matemáticas. En este artículo subrayo algunos puntos críticos en la descripción usual de los sistemas semióticos; discuto acerca del paradigma multimodal y encarnado que ha venido emergiendo en los últimos años en investigaciones realizadas en psicolinguística y neurociencia y analizo los gestos desde un punto de vista semiótico. Luego, introduzco la noción de paquete semiótico y lo ejemplifico a través de un estudio de casos.

- **PALABRAS CLAVES:** Recursos semióticos, encarnamiento, multimodalidad, gestos, inscripciones.

ABSTRACT

Classical semiotic approaches are too narrow to investigate the didactical phenomena in the mathematics classroom. In addition to the standard semiotic resources used by students and teachers (e.g. written symbols and speech), other important semiotic ressources include also gestures, glances, drawings and extra-linguistic modes of expressions. However, these semiotic ressurces fit with difficulties within the constraints of the classical definitions of semiotic systems. To overcome such difficulties I adopt a vygotskian approach and present an enlarged notion of semiotic system, the *semiotic bundle*, which reveals particularly useful for framing all the semiotic resources we find in the learning processes in mathematics. The paper stresses some critical points in the usual description of the semiotic systems; it discusses the multimodal and embodied paradigm, which is emerging in these last years from researches in psycholinguistics and neuroscience and analyses gestures from a semiotic point of view. Then it introduces the notion of semiotic bundle and exemplifies it through a case study.

Fecha de recepción: Junio de 2006/ Fecha de aceptación: Julio de 2006

1 Dipartimento di Matematica, Università di Torino, Italy.

KEY WORDS: Semiotic resources, embodiment, multimodality, gestures, inscriptions.

RESUMO

As aproximações clássicas semióticas resultam ser muito limitadas para investigar os fenômenos didáticos de sala de aula de matemática. Além aos recursos padrão dos semióticos usados pelos estudantes e pelos professores (como os símbolos escritos e a língua falada), outros recursos importantes dos semióticos são os gestos, os olhares, os desenhos e as maneiras extra-lingüísticas da expressão. Não obstante, estes últimos não se adaptam bem nas definições clássicas dos sistemas dos semióticos. A fim superar esta dificuldade, neste artigo eu adoto um perspectiva vygotskiana e apresento uma noção estendida do sistema do semiótico ao pacote semiótico que é particularmente útil incluir todos os recursos dos semióticos que nós encontramos nos processos da aprendizagem da matemática. Neste artigo eu enfatizo alguns pontos críticos na descrição usual dos sistemas semióticos.Discuto sobre o paradigma multimodal e personificado que tem emergido nos últimos anos das investigações feitas na psicolinguística e na neurociência e analiso os gestos sob um ponto da vista do semiótico. Logo, eu introduzo a noção do pacote do semiótico e a exemplifico com um estudo dos casos.

PALAVRAS CHAVE: Recursos semióticos, significação, multimodalidade, gestos, inscrições.

RÉSUMÉ

Les approches sémiotiques classiques sont trop étroites pour étudier les phénomènes didactique de la salle de classe de mathématiques. En plus des ressources sémiotiques traditionnelles (comme les symboles écrits et la langue) utilisées par les élèves et les enseignants, d'autres ressources sémiotiques importantes comprennent les gestes, les regards, les dessins et les modes extra-langagiers d'expression. Ces dernières rentrent difficilement dans les définitions classiques des systèmes sémiotiques. Afin de surmonter cette difficulté, dans cet article j'adopte une perspective vygotskienne et je présente une notion élargie de système sémiotique, le faisceau sémiotique, qui s'avère particulièrement utile afin d'inclure toutes les ressources sémiotiques que nous rencontrons dans les processus d'apprentissage des mathématiques. Dans cet article je souligne quelques points critiques concernant la description usuelle des systèmes sémiotiques; j'offre une discussion du paradigme multimodal et incarné lequel a émergé ces dernières années dans le cadre des recherches menées en psycholinguistique et neuroscience. Suite à cela j'analyse les gestes d'un point de vue sémiotique. Après j'introduis la notion de paquet sémiotique et l'exemplifie à travers une étude de cas.

MOTS CLÉS: Ressources sémiotiques, incarnement, multimodalité, gestes, inscriptions.

Introduction.

Semiotics is a powerful tool for interpreting didactical phenomena. As Paul Ernest points out,

> *"Beyond the traditional psychological concentration on mental structures and functions 'inside' an individual it considers the personal appropriation of signs by persons within their social contexts of learning and signing. Beyond behavioural performance this viewpoint also concerns patterns of sign use and production, including individual creativity in sign use, and the underlying social rules, meanings and contexts of sign use as internalized and deployed by individuals. Thus a semiotic approach draws together the individual and social dimensions of mathematical activity as well as the private and public dimensions. These dichotomous pairs of ideas are understood as mutually dependent and constitutive aspects of the teaching and learning of mathematics, rather than as standing in relations of mutual exclusion and opposition."* (Ernest, 2006, p.68)

However, the classical semiotic approach places strong limitations upon the structure of the semiotic systems it considers. They generally result in being too narrow for interpreting the complexity of didactical phenomena in the classroom. As we shall discuss below, this happens for two reasons:

(i) As observed by L. Radford (2002), there are a variety of semiotic resources used by students and teachers, like gestures, glances, drawings and extra-linguistic modes of expression, which do not satisfy the requirements of the classical definitions for semiotic systems as discussed in literature (e.g. see Duval, 2001).

(ii) The way in which such different registers are activated is multimodal. It is necessary to carefully study the relationships within and between registers, which are active at the same moment and their dynamics developing in time. This study can only partially be done using the classic tools of semiotic analysis.

To overcome these two difficulties, I adopt a Vygotskian approach for analyzing semiotic resources and present an enlarged notion of semiotic system, which I have called *semiotic bundle*. It encompasses all the classical semiotic registers as particular cases. Hence, it does not contradict the semiotic analysis developed using such tools but allows us to get new results and to frame the old ones within a unitary picture.

This paper is divided into three main chapters. *Chapter 1* summarizes some salient aspects of (classical) Semiotics: it shows its importance for describing learning processes in mathematics (§ 1.1), points out two opposite tendencies in the story of Semiotics, which reveal the inadequacy of the classical approach when it is used in the classroom (§1.2), and discusses the semiotic role of artefacts, integrating different perspectives from Vygotsky to Rabardel (§1.3).

Chapter 2 develops the new concept of semiotic bundle (§2.1), discusses the multimodal and embodied paradigm, which has emerged in recent years from research in psycholinguistics and neuroscience

(§2.2), and analyses gestures from a semiotic point of view (§2.3).

Chapter 3 introduces a case study, which concretely illustrates the use of semiotic bundles in interpreting the didactical phenomena.

A *Conclusion*, with some comments and open problems, ends the paper.

1. The semiotic systems: a critical approach

1.1 Semiotics and mathematics

Charles S. Peirce points out a peculiar feature of mathematics which distinguishes it from other scientific disciplines:

> *"It has long been a puzzle how it could be that, on the one hand, mathematics is purely deductive in its nature, and draws its conclusions apodictically, while on the other hand, it presents as rich and apparently unending a series of surprising discoveries as any observational science. Various have been the attempts to solve the paradox by breaking down one or other of these assertions, but without success. The truth, however, appears to be that all deductive reasoning, even simple syllogism, involves an element of observation; namely, deduction consists in constructing an icon or diagram, the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning, of experimenting upon this image in the imagination, and of observing the result so as to discover unnoticed and hidden relations among the parts. ... As for algebra, the very idea of the art is that it presents formulae, which can be manipulated and that by observing the effects of such manipulation we find properties not to be otherwise discerned. In such manipulation, we are guided by previous discoveries, which are embodied in general formulae. These are patterns, which we have the right to imitate in our procedure, and are the icons par excellence of algebra".* (Hartshorne & Weiss, 1933, 3.363; quoted in Dörfler, n.d.).

In fact, mathematical activities can develop only through a plurality of palpable registers that refer to its ideal objects:

> *"…the **oral** register, the **trace** register (which includes all graphic stuff and writing products), the **gesture** register, and lastly the register of what we can call the **generic materiality**, for lack of a better word, namely the register where those ostensive objects that do not belong to any of the registers above reside"* ([2]).
>
> (Bosch & Chevallard, 1999, p. 96, emphasis in the original)

These observations are the root of all semiotic approaches to mathematical thinking, some of which I shall briefly review below.

---

2 "...[le] registre de *l'oralité*, registre de *la trace* (qui inclut graphismes et écritures), registre de *la gestualité*, enfin registre de ce que nous nommerons, faute de mieux, *la matérialité quelconque*, où prendront place ces objets ostensifs qui ne relèvent d'aucun des registres précédemment énumérés. "

Peirce's observations point out different aspects of the semiotic approach:

(i) the introduction of signs, namely perceivable (spatio-temporal) entities, like *"icons or diagrams, the relations of whose parts shall present a complete analogy with those of the parts of the object of reasoning"*;

(ii) the manipulation of signs, namely *"experimenting upon this image in the imagination"* and/or *"manipulating it"* concretely and *"observing the effects of such manipulation"*;

(iii) the emergence of rules and of strategies of manipulation: *"in such activities we are guided by previous discoveries, which are embodied in the signs themselves*", e.g. in the general formulae of algebra, and *"that become patterns to imitate in our procedure*". Typical examples are the signs of Algebra and of Calculus, Cartesian graphs, arrow diagrams in Graph Theory or Category Theory, but also 2D figures or 3D models in Geometry. Generally speaking, such signs are *"kind[s] of inscriptions of some permanence in any kind of medium (paper, sand, screen, etc)"* (Dörfler, n.d.) that allow/ support what has been sometimes called (e.g. Dörfler, *ibid.*) *diagrammatic reasoning*. The paper of Dörfler provides some examples, concerning Arithmetic, Algebra, Calculus and Geometry. Other examples, albeit with different terminology, are in Duval (2002, 2006).

However, as the quotation from Peirce shows, the semiotic activities are not necessarily limited to the treatment of inscriptions since they also deal with images that are acted upon in imagination (whatever it may mean): *"A sign is in a conjoint relation to the thing denoted and to the mind. If this relation is not of a degenerate species, the sign is related to its object only in consequence of a mental association, and depends upon a habit."* (Hartshorne & Weiss, 1933, 3.360).

I shall discuss this point below after having considered the more standard approaches to semiotic systems, which study inscriptions (signs in a more or less wide sense) and operations upon them. E.g., according to Ernest (2006, pp. 69-70), a *semiotic system* consists of three *components*:

1. A set of signs, the tokens of which might possibly be uttered, spoken, written, drawn or encoded electronically.

2. A set of rules of sign production and transformation, including the potential capacity for creativity in producing both atomic (single) and molecular (compound) signs.

3. A set of relationships between the signs and their meanings embodied in an underlying meaning structure.

An essential feature of a semiotic system has been pointed out by Duval (2002), who introduced the concept of *semiotic representations.* The signs, relationships and rules of production and transformation are semiotic representations insofar as they bear an intentional character (this is also evident in the quotation of Peirce). This intentional character is not intrinsic to the sign, but concerns people who are producing or using it. For example, a footprint in the sand generally is not a semiotic representation in this sense: a person who is walking on the beach has no interest in producing or not producing it; however, the footprint that Robinson Crusoe saw one day was the sign of an unsuspected inhabitant of the deserted island, hence he gave it a semiotic function and for him the footprint became a semiotic representation.

Other important aspects of semiotic systems are their *semiotic functions,* which can be distinguished as *transformational* or *symbolic* (see: Duval, 2002 and 2006; Arzarello et al., 1994).

The *transformational* function consists in the possibility of transforming signs within a fixed system or from one system to another, according to precise rules (algorithms). For example, one can transform the sign $x(x+1)$ into $(x^2 + x)$ within the algebraic system (register) or into the graph of a parabola from the Algebraic to the Cartesian system. Duval (2002, 2006) calls *treatment* the first type of transformation and the second one *conversion*. According to Duval (2002), conversions are crucial in mathematical activities:

> *"The characteristic feature of mathematical activity is the simultaneous mobilization of at least two registers of representation, or the possibility of changing at any moment from one register to another."*

The *symbolic* function refers to the possibility of interpreting a sign within a register, possibly in different ways, but without any material treatment or conversion on it. E.g. if one asks if the number $n(n+1)$ is odd or even one must interpret n and $(n+1)$ with respect to their oddity and see that one of the two is always even. This is achieved without any transformation on the written signs, but rather by interpreting differently the signs n, $(n+1)$ and their mutual relationships: the first time as odd-even numbers and then as even-odd numbers. The symbolic function of signs has been described by different authors using different words and from different perspectives: C.S. Peirce, C.K.Ogden & I. A. Richards (semiotics); G. Frege (logic); L. Vygotsky (psychology) and others: see Steinbring (2005, chapter 1) for an interesting summary focusing on the problem from the point of view of mathematics education. The symbolic function possibly corresponds to the activity of *"experimenting upon an image in the imagination"*, mentioned by Peirce. All of the aforementioned authors point out the triadic nature of this function, namely that it consists in a complex (semiotic) relationship among three different components (the so called semiotic triangle), e.g. using Frege´s terminology, among the Sense (Sinn), the Sign (Zeichen) and the Meaning (Bedeutung). Peirce spoke of *"a triple relation between the sign, its object and the mind"*; Frege (1969) was more cautious and avoided putting forward in his analysis what he called the *third world*, namely the psychological side.

Semiotic systems provide an environment for facing mathematics not only in its structure as a scientific discipline but also from the point of view of its learning, since they allow us to seek the cognitive functioning underlying the diversity of mathematical processes. In fact, approaching mathematical activities and products as semiotic systems also allows us to consider the cognitive and social issues which concern didactical phenomena, as illustrated by the quotation of Ernest in the Introduction.

Transformational and symbolic functions of signs are the core of mathematics and they are very often intertwined. I shall sketch here a couple of examples. An interesting historical example, where both transformational and symbolic functions of semiotic registers are present is the method of completing the square in solving second order equations. This can be done within the algebraic as well as the geometric register. Another important example of the creative power of the symbolic function is given by the novelty of the Lebesgue integral (of a real function f in an interval $[a,b]$) with respect to the Riemann one. In the latter, one collects data forming the approximating integral sums subdividing the interval $[a,b]$ in intervals Δ_i, each of length δ_i less than some δ : the basic signs are the products $l_i\delta_i$, where l_i is some value of the function f in Δ_i (or its sup or inf in it) and the final sum $\sum l_i\,\delta_i$ is made considering the values i corresponding to all the intervals Δ_i of the subdivision. In the former, the subdivision is made considering, for each value l of f, the set Δ_l of x's such that $f(x) = l$: the basic signs are the products $l\,|\Delta_l|$, and the final sum $\sum l\,|\Delta_l|$, is made considering all the values l that the function assumes while x varies in $[a,b]$.

1.2 Two opposite tendencies

Within the main components of a semiotic system (signs and operations on them), there is a tension between two opposite modalities, which is particularly evident when a semiotic lens is used to analyse didactical processes and not only mathematical products. This tension is in fact a by-product of the two contrasting features of mathematics pointed out by Peirce, that is, its apodictic and observational aspects.

The first one consists in the strong tendency to formalize in mathematics:

> *"The more important for the mathematical practice is the availability of a calculus which operates on diagrams (function terms) and permits to evaluate derivatives, anti-derivatives and integrals according to established diagrammatic operation rules. … Here again we find the striving for manipulable diagrams which can be taken to accurately reflect the related non-diagrammatic structures and processes."* (Dörfler, n.d.)

Different crucial examples of this tendency are: the algebraic language, which (Harper, 1987) introduced suitable formalism to treat classes of arithmetic problems (equations included); Cartesian geometry, which allowed for the translation of the geometric figural register into the algebraic one; and arrow-diagrams in Category Theory. All such new inscriptional entries also allowed for new forms of reasoning and solving problems and hence had a strong epistemological and cognitive impact. A culminating case in this tendency toward formalization consists in the idea of formal system, elaborated by Hilbert (see Detlefsen, 1986).

The construction of a (formal) axiomatization in the sense of Hilbert's formalist program can be considered another method of translating into diagrams. Let us take, for instance, an axiom system for the structure of real numbers: it consists of formulas in a precise formal language together with the rules inference, e.g. first order predicate logic. These can be viewed as diagrams in the sense intended by Peirce. Proofs and theorems are then obtained by manipulating such diagrams and observing the outcomes of the manipulations (the logical deductions). One could therefore interpret (formal) axiomatization as a kind of diagrammatization (see Dörfler, n.d.).

Moreover, if one looks carefully at some logical ideas in Mathematical Logic developed at the turn of the twentieth century, the tendency toward formalism shows a further mathematical aspect of semiotic conversions, namely the idea of the interpretation of one theory into another. As an example, I call to mind the second part of the book *Foundation of Geometry* (Hilbert, 1962), where Hilbert typically interprets geometrical objects and statements into real numbers or into some subfield of reals to build models where some specific axiom of geometry does not hold. The concept of interpretation is the logical and mathematical counterpart of the idea of conversion from one register to another. Its roots are in the conversion/interpretation of one model into another one: typically, the interpretation of a model for hyperbolic geometry within the Euclidean model, namely the Klein disk and the Poincaré disk or half-plain. The rationale behind such logical approaches is that the relationships among objects represented in different ways within different registers can be shown better in one register than in another, exactly because of the specificity of the register, possibly because of the symbolic function it promotes. For instance, we can note the validity or less of an axiom of geometry in the usual Euclidean model (first register) or in a model built using only a subfield of real numbers (second register). A very recent area of research that has developed in line with this approach is the project of *Reverse Mathematics* (Sympson, 1999), where typically an important theorem T is proved carefully within a formal system ***S*** using some logical hypothesis H. For example, the Heine-Borel theorem in Analysis using as logical hypothesis a (weak) form of König lemma. Reverse Mathematics then tries to answer to the following 'reverse' question: does it exist within ***S*** a proof of H using T as hypothesis? Namely, one tries to prove the equivalence between T and H within a suitable system ***S***, namely the equivalence between sentences whose meaning is within two different registers (e.g. the analysis and the logical one).

The concept of interpretation has carefully refined the transformational and symbolic functions of mathematical signs during the years, from the pioneering semantic interpretations of geometrical models to the elaborate formal theories studied in Reverse Mathematics.

On the one hand, this approach has enlarged the horizon of semiotic systems from within mathematics (*inner enlargement*): think of the different models of reasoning induced by the Calculus inscriptions with respect to those pertaining to the algebraic ones, or to those induced by the «reasoning by arrows» in Category Theory. But on the other hand, it has also narrowed the horizon within which mathematical semiotic activities are considered, limiting them to their strictly formal aspects.

Unfortunately, this is not enough when cognitive processes must be considered, e.g. in the teaching-learning of mathematics. In such a context, it is the same notion of signs and of operations upon them that needs to be considered with a greater flexibility and within a wider perspective. In the classroom, one observes phenomena which can be considered as signs that enter the semiotic activities of students[3] but which are not signs as defined above and are not processed through specific algorithms. For example, observing students who solve problems working in group, their gestures, gazes and their body language in general are also revealed as crucial semiotic

[3] *Semiotic Activity* is classically defined as any "communicative activity utilizing signs. This involves both sign 'reception' and comprehension via listening and reading, and sign production via speaking and writing or sketching." The main purpose of the paper is to widen this definition.

resources. Namely, non-written signs and non-algorithmic procedures must also be taken into consideration within a semiotic approach. Roughly speaking, it is the same notion of sign and of operations upon them that needs to be broadened. In fact, over the years, many scholars have tried to widen the classical formal horizon of semiotic systems, also taking into consideration less formal or non formal components.

While formalism represents the first tendency of the aforementioned tension in Semiotics, these broadening instances from outside mathematics constitute the other tendency (*outer enlargement*). This tendency can already be found in the complex evolution of the sign definition in Peirce and is also contained in some pioneering observations by Vygotsky concerning the relationships between gestures and written signs, such as the following:

> *"The gesture is the initial visual sign that contains the child's future writing as an acorn contains a future oak. Gestures, it has been correctly said, are writing in air, and written signs frequently are simply gestures that have been fixed."* (Vygotsky, 1978, p. 107; see also: Vygotsky, L. S. 1997, p. 133.).

This was also anticipated by Ludwig Wittgenstein, who changed his mind about the centrality of propositions in discourse and the role of gestures, passing from the *Tractatus* to the *Philosophische Untersuchungen,* as the following well known episode illustrates:

> *"Wittgenstein was insisting that a proposition and that which it describes must have the same 'logical form', the same 'logical multiplicity', Sraffa made a gesture, familiar to Neapolitans as meaning something like disgust or contempt, of brushing the underneath of his chin with an outward sweep of the finger-tips of one hand. And he asked: 'What is the logical form of that?' Sraffa's example produced in Wittgenstein the feeling that there was an absurdity in the insistence that a proposition and what it describes must have the same 'form'. This broke the hold on him of the conception that a proposition must literally be a 'picture' of the reality it describes."* (Malcom & Wright, 2001, p. 59)

But it is specifically in some recent research in the field of Mathematical Education that semiotic systems are being studied explicitly within a wider (outer) approach (e.g. see: Duval, 2002, 2006; Bosch & Chevallard, 1999; Steinbring, 2005, 2006; Radford, 2003a; Arzarello & Edwards, 2005). Such research deepens the original approaches by people like Peirce, Frege, Saussurre, Vygotsky and others.

I will sketch some examples: the semiotic means of objectification, the notion of semiotic systems (both due to Luis Radford), the concept of Representational Infrastructure (due to J. Kaput and to R. Noss) and the so-called extra-linguistic modes of expressions (elaborated by psycholinguists). Radford introduces the notion of *semiotic means of objectification* in Radford (2003a). With this seminal paper, Radford makes explicit the necessity of entertaining a wider notion of semiotic system. He underlines that:

> *"Within this perspective and from a psychological viewpoint, the objectification of mathematical objects appears linked to the*

> *individuals' mediated and reflexive efforts aimed at the attainment of the goal of their activity. To arrive at it, usually the individuals have recourse to a broad set of means. They may manipulate objects (such as plastic blocks or chronometers), make drawings, employ gestures, write marks, use linguistic classificatory categories, or make use of analogies, metaphors, metonymies, and so on. In other words, to arrive at the goal the individuals rely on the use and the linking together of several tools, signs, and linguistic devices through which they organize their actions across space and time."*

Hence he defines this enlarged system as semiotic means of objectification, that is:

> *"These objects, tools, linguistic devices, and signs that individuals intentionally use in social meaning-making processes to achieve a stable form of awareness, to make apparent their intentions, and to carry out their actions to attain the goal of their activities."*

The semiotic means of objectification constitute many different types of signs (e.g. gestures, inscriptions, words and so on). They produce what Radford calls *contextual generalization*, namely a generalization which still refers heavily to the subject's actions in time and space and in a precise context, even if he/she is using signs that have a generalizing meaning. In contextual generalization, signs have a two-fold semiotic nature: they are going to become symbols but are still indexes. We use these terms in the sense of Peirce (see: Hartshorne, C. & Weiss, 1933): an index gives an indication or a hint on the object, like an image of the Golden Gate makes you think of the town of San Francisco *("it signifies its object solely by virtue of being really connected with it"*, Hartshorne & Weiss, 1933, 3.361). A symbol is a sign that contains a rule in an abstract way (e.g. an algebraic formula).

The semiotic means of objectification also embody important cultural features. In this sense, Radford speaks of *semiotic systems of cultural meanings* (Radford, this volume; previously called *Cultural Semiotic Systems*, Radford, 2003a), that is, those systems which make available varied sources for meaning-making through specific social signifying practices; such practices are not to be considered strictly within the school environment but within the larger environment of society as a whole, embedded in the stream of its history. Furthermore, cultural semiotic systems are an example of outer enlargement of the notion of semiotic system.

A similar example of enlargement of the notion of semiotic system is the concept of *representational infrastructure*, introduced by J. Kaput et al. (2002), which exploits some cultural and social features of signs. Discussing the appearance of new computational forms and literacies that are pervading the social and economic lives of individuals and nations alike, they write:

> *"…The real changes are not technical, they are cultural. Understanding them… is a question of the social relations among people, not among things. The notational systems we use to present and re-present our thoughts to ourselves and to others, to create and communicate records across space and time, and to support reasoning and computation constitute a central part of any civilization's infrastructure. As with*

infrastructure in general, it functions best when it is taken for granted, invisible, when it simply 'works'." (Kaput et al., 2002, p. 51).

An example both of cultural semiotic system and of representational infrastructure, discussed in Radford (2003a) and in Kaput et al. (2002), consists in the developing of algebraic symbolism, which *"in more than one millennium gradually freed itself from written natural language and developed within a representational infrastructure".*

As a last example of a broader notion of semiotic system, I refer to the distinction made by psycho-linguists between linguistic and extra-linguistic modes of expression. They describe the former as the communicative use of a sign *system*, the latter as the communicative use of a *set* of signs (Bara & Tirassa, 1999):

> *"Linguistic communication is the communicative use of a symbol system. Language is compositional, that is, it is made up of constituents rather than parts... Extra-linguistic communication is the communicative use of an open set of symbols. That is, it is not compositional: it is made up of parts, not of constituents. This makes for crucial differences from language..."*

1.3 The semiotic mediation of artefacts

In keeping with this perspective, artefacts as representational infrastructures also enter into semiotic systems. Realizing the semiotic similarity between signs and artefacts constitutes a crucial step in the story of outer semiotic enlargements. This similarity has two aspects. One is ergonomic and is properly focused if one considers the dialectic between artefact and instrument developed by Verillon & Rabardel (1995) who introduced the notion of *instrumental genesis*. The other is psychological and has been pointed out by Vygotsky, who described the dialectic relationships between signs and instruments by what he called *process of internalization*. I shall describe both in some detail since they allow us to understand more deeply the relevance of the outer enlargements sketched above and are at the basis of my definition of *semiotic bundle*, which I shall introduce below.

Let me start with the ergonomic theory of Verillon and Rabardel[4]: an artefact has its schemes of use (for example, the rules according to which one must manage a compass or a software) and as such it becomes an instrument in the hands of the people who are using it. This idea develops in a fresh way the notion of transformation on a semiotic system. In the ergonomic approach, the technical devices are considered with two interpretations. On the one side, an object has been constructed according to a specific knowledge that assures the accomplishment of specific goals; on the other side, a user interacts with this object, using it (possibly in different ways). The object in itself is called an artefact, that is, a particular object with its features realized for specific goals and it becomes an instrument, that is, an artefact with the various modalities of use, as elaborated by the individual who is using it. The instrument is conceived as the artefact together with the actions made by the subject, organized in collections of operations, classes of invariants and

4 This part of the paper is taken from Arzarello & Robutti (2004).

utilizations schemes. The artefact, together with the actions, constitutes a particular instrument: thus, the same subject can use the same artefact as different instruments.

The pair instrument-artefact can be seen as a semiotic system in the wider sense of the term. The instrument is produced from an artefact introducing its rules of use and, as such, it is a semiotic representation with rules of use that bear an intentional character: it is similar to a semiotic representation. As semiotic representations, instruments can play a fundamental role in the objectification and in the production of knowledge. For example: the compass is an artefact which can be used by a student to trace a circle as the locus of points in a plane at the same distance from a fixed point. A cardboard disk can be used for the same purpose as the compass, but the concept of circle induced by this use may be different.

The transformation of the artefact into an instrument is made through suitable treatment rules, e. g. for the compass, the action of pointing it at a point and tracing a curve with a fixed ray; for the cardboard disk, the action of carefully drawing a line along its border. In a similar way, students learn to manage algebraic symbols: the signs of Algebra or of Analysis, e.g., a^2-b^2 or $\mathrm{D}x^2$, are transformed according to suitable treatment rules, e.g. those producing $(a+b)(a-b)$ or $2x$. Just like an artefact becomes an instrument when endowed with its using rule, the signs of Algebra or of Analysis become symbols, namely signs with a rule (recall the Peirce notion quoted above), because of their treatment rules (see also the discussion about techniques and technologies in Chevallard, 1999).

In both cases, we get semiotic systems with their own rules of treatment. As the coordinated treatment schemes are elaborated by the subject with her/his actions on/with the artefacts/signs, the relationship between the artefact/signs and the subject can evolve. In the case of concrete artifacts, it causes the so-called process of *instrumental genesis*, revealed by the schemes of use (the set of organized actions to perform a task) activated by the subject. In the example above, the knowledge relative to the circle is developed through the schemes of use of the compass or of the cardboard. In the case of algebraic signs, the analogous of the instrumental genesis produced by syntactic manipulations may produce different types of knowledge relative to the numerical structures (see the notion of theory as emerging from the techniques and the technologies, discussed in Chevallard, 1999). Hence, the ergonomic analysis points to an important functional analogy between artefacts and signs [5].

Within a different perspective, Vygotsky had also pointed out a similar analogy between tools [6], which can support human labour, and signs, which can uphold the psychological activities of subjects:

> *"...the invention and use of signs as auxiliary means of solving a given psychological problem (to remember, compare something, report, choose and so on) is analogous to the invention of tools in one psychological*

5 A similar analogy is achieved within a different framework by Chevallard (1999).

6 In the Cambridge Dictionary, a tool is defined as *"something that helps you to do a particular activity"*, an instrument is *"a tool that is used for doing something",* while an artefact is an *"object".* Following this definition, I consider the instrument as a specific tool.

> *respect. The signs act as instrument of psychological activity in a manner analogous to the role of a tool in labour."* (Vygotsky, 1978, p. 52)

As I anticipated above, this common approach to signs and tools is based on the notion of *semiotic mediation* [7], which is at the core of the Vygotskian frame: for a survey see Bartolini & Mariotti (to appear) a paper from which I take some of the following comments.

Vygotsky pointed out both a functional analogy and a psychological difference between signs and instruments. The analogy is illustrated by the following quotation, which stresses their semiotic functions:

> *"...the basic analogy between sign and tools rests on the mediating function that characterizes each of them"* (*ibid.,* p. 54).

The difference between signs and tools is so described:

> *"the tool's function is to serve as the conductor of human influence on the object of activity; it is externally oriented...The sign, on the other hand, changes nothing in the object of a psychological operation. It is a means of internal activity aimed at mastering oneself: the sign is internally oriented."* (*ibid.*, p. 55)

This distinction is central in the Vygotskyan approach, which points out the transformation from externally oriented tools to internally oriented tools (often called *psychological tools*) through the process of *internalization*. According to Vygotsky, in the process of internalization, interpersonal processes are transformed into intrapersonal ones. The process of internalization (through which the 'plane of consciousness' is formed, see Wertsch & Addison Stone, 1985, p.162) occurs through semiotic processes, in particular by the use of semiotic systems, especially of language, in social interaction:

> *"...the Vygotskian formulation involves two unique premises...First, for Vygotsky, internalisation is primarly concerned with social processes. Second, Vygotsky's account is based largely on the analysis of the semiotic mechanisms, especially language, that mediate social and individual functioning....Vygotsky's account of semiotic mechanisms provides the bridge that connects the external with the internal and the social with the individual...Vygotsky's semiotic mechanisms served to bind his ideas concerning genetic analysis and the social origins of behaviour into an integrated approach...it is by mastering semiotic mediated processes and categories in social interaction that human consciousness is formed in the individual"* (Wertsch & Addison Stone, 1985, pp.163-166)

As Bartolini Bussi & Mariotti (Bartolini & Mariotti, to appear) point out, Vygotsky stresses the role and the dynamics of semiotic mediation: first, externally oriented, a sign or a tool is used in action to accomplish a specific task; then, the actions with the sign or the tool (semiotic activity, possibly under the guidance of an expert), generate new signs (words included), which foster the internalization process and produce a new

---

7 It is described in Vygotsky (1978, especially p. 40 and ff).

psychological tool, internally oriented, completely transformed but still maintaining some aspects of its origin.

Vygotsky describes such dynamics without any reference to mathematics; hence, his observations are general; many recent studies have adapted his framework to fit the specificity of mathematics (e.g. see Radford, 2003a; Bartolini & Mariotti, to appear).

2. A new theoretical frame: the semiotic bundle

2.1 Definition and examples

My framework is also specific for mathematics; it allows for better combining the two issues described above, the one from semiotics, in the spirit of the quoted Ernest definition of semiotic systems, and the other from psychology, according to the Vygotskian approach. Both pictures are essential for analyzing the learning processes in mathematics; they are here integrated within a wider model.

On the one hand, it is necessary to broaden the notion of semiotic system in order to encompass all the variety of phenomena of semiotic mediation in the classroom, as already suggested by Radford, who introduced a new notion of semiotic system:

> *The idea of semiotic system that I am conveying includes classical system of representations – e.g. natural language, algebraic formulas, two or three-dimensional systems of representation, in other terms, what Duval (2001) calls discursive and non-discursive registers – but also includes more general systems, such as gestures (which have an intuitive meaning and to a certain extent a fuzzy syntax) and artifacts, like calculators and rulers, which are not signs but have a functional meaning.* (Radford, 2002, p. 21, footnote 7).

On the other hand, the psychological processes of internalization, so important in describing the semiotic mediation of signs and tools, must fill a natural place within the new model.

A major step towards the common frame consists in reconsidering the notion of semiotic system along the lines suggested by Radford. Once we have a more suitable notion of semiotic system, we shall come back to the Vygotskian approach and show that this fresh notion encompasses it properly, allowing for a deeper understanding of its dynamics.

This fresh frame takes into account the enormous enlargement of the semiotic systems horizon, both from the inner and from the outer side that has been described above. Once the semiotic systems have been widened to contain gestures, instruments, institutional and personal practices and, in general, extra-linguistic means of expression, the same idea of operation within or between different registers changes its meaning. It is no longer a treatment or conversion (using the terminology of Duval) within or between semiotic representations according to algorithmic rules (e.g. the conversion from the geometric to the Cartesian register). On the contrary, the operations (within or between) must be widened to also encompass phenomena that may not be strictly algorithmic: for example, practices with instruments, gestures and so on.

At this point of the discussion, the above definition by Ernest can be widened to encompass all the examples we have given. We thus arrive at the notion that I have called *semiotic bundle* (or bundle of semiotic sets). To define it, I need first the notion of *semiotic set*, which is a widening of the notion of semiotic system.

A semiotic set is:

a) A set of signs which may possibly be produced with different actions that have an intentional character, such as uttering, speaking, writing, drawing, gesticulating, handling an artefact.

b) A set of modes for producing signs and possibly transforming them; such modes can possibly be rules or algorithms but can also be more flexible action or production modes used by the subject.

c) A set of relationships among these signs and their meanings embodied in an underlying meaning structure.

The three components above (signs, modes of production/transformation and relationships) may constitute a variety of systems, which span from the compositional systems, usually studied in traditional semiotics (e.g. formal languages) to the open sets of signs (e.g. sketches, drawings, gestures). The former are made of elementary constituents and their rules of production involve both atomic (single) and molecular (compound) signs. The latter have holistic features, cannot be split into atomic components, and the modes of production and transformation are often idiosyncratic to the subject who produces them (even if they embody deeply shared cultural aspects, according to the notion of *semiotic systems of cultural meanings* elaborated by Radford, quoted above). The word set must be interpreted in a very wide sense, e.g. as a variable collection.

A semiotic bundle is:

(i) A collection of semiotic sets.
(ii) A set of relationships between the sets of the bundle.

Some of the relationships may have conversion modes between them.

A semiotic bundle is a *dynamic structure* which can change in time because of the semiotic activities of the subject: for example, the collection of semiotic sets that constitute it may change; as well, the relationships between its components may vary in time; sometimes the conversion rules have a genetic nature, namely, one semiotic set is generated by another one, enlarging the bundle itself (we speak of *genetic conversions*).

Semiotic bundles are semiotic representations, provided one considers the intentionality as a relative feature (see the above comment on the sand footprint).

An example of semiotic bundle is represented by the unity speech-gesture. It has been a recent discovery that gestures are so closely linked with speech that "*we should regard the gesture and the spoken utterance as different sides of a single underlying mental process*" (McNeill, 1992, p.1), namely "*gesture and language are one system*" (*ibid.*, p.2). In our terminology, gesture and language are a semiotic bundle, made of two deeply intertwined semiotic sets (only one, speech, is also a semiotic system). Research on gestures has uncovered some important relationships between the two (e.g. match and mismatch, see Goldin-Meadow, 2003). A semiotic bundle must not be considered as a juxtaposition of semiotic sets; on the contrary, it is a unitary system and it is only for the sake of analysis that we distinguish its components as semiotic sets. It must be observed that if one limits oneself to examining only the semiotic systems and their bundles, many interesting aspects of human discourse are lost: only by considering bundles of semiotic sets can new phenomena be discovered.

This wider approach is particularly fruitful when the processes and activities of people learning mathematics are scrutinized. In the research carried out by the Turin team[8] we investigate semiotic bundles made of several semiotic sets: e.g. gesture, speech and written inscriptions (e.g. mathematical symbols, drawings). The results consist in describing some of the relationships and conversion rules within such a complex bundle.

Semiotic bundles allow us to frame the Vygotskian notion of semiotic mediation sketched above in a more comfortable setting. The dynamics in the process of internalization, according to Vygotsky, is based on semiotic activities with tools and signs, externally oriented, which produce new psychological tools, internally oriented, completely transformed but still maintaining some aspects of their origin. According to Vygotsky, a major component in this internalization process is language, which allows for the transformations. Moreover, such transformations ‘curtail' the linguistic register of speech into a new register: Vygotsky calls it *inner speech* and it has a completely different structure. This has been analyzed by Vygotsky in the last (7th) chapter of *Thought and Language* (Vygotsky, 1992), whose title is *Thought and Word*. Vygotsky distinguishes two types of properties that allow us to distinguish the inner from the outer language: he calls them *structural* and *semantic properties*.

The structural properties of the inner language are its *syntactic reduction* and its *phasic reduction*: the former consists in the fact that inner language reduces to pure juxtaposition of predicates minimizing its syntactic articulation; the latter consists in minimizing its phonetic aspects[9], namely curtailing the same words.

According to Vygotsky's frame, the semantic properties of the inner language are based on the distinction made by the French psychologist Frederic Pauhlan between the *sense* and the *meaning* of a word and by *“the preponderance of the sense [smysl] of a word over its meaning [znachenie]”* (Vygotsky, 1978, p. 244):

> *“the sense is...the sum of all the psychological events aroused in our consciousness by the word. It is a dynamic, fluid, complex whole, which has several zones of unequal stability. Meaning is only one of the zones of sense, the most stable and precise zone. A word acquires its sense from the context in which it appears; in different contexts, it changes its sense. ”* (*ibid.,* p. 244-245).

8 This is being done by our colleagues Luciana Bazzini and Ornella Robutti, by some doctoral and post-doc students, like Francesca Ferrara and Cristina Sabena, and by many teachers (from the elementary to the higher school level) that participate actively to our research, like Riccardo Barbero, Emilia Bulgarelli, Cristiano Dané, Silvia Ghirardi, Marina Gilardi, Patrizia Laiolo, Donatella Merlo, Domingo Paola, Ketty Savioli, Bruna Villa and others.

9 To make an analogy with the outer language, Vygotsky recalls an example, taken from Le Maitre (1905), p. 41: a child thought to the French sentence “Les montagnes de la Suisse sont belles” as “L m d l S s b” considering only the initial letters of of the sentence. Curtailing is a typical feature of inner language.

In inner language, the sense is always overwhelming the meaning. This prevailing aspect of the sense has two structural effects on inner language: the *agglutination* and the *influence*. The former consists in gluing different meanings (concepts) into one expression[10]; the latter happens when the different senses 'flow' together[11] into one unity.

To explain the properties of inner speech, Vygotsky uses analogies that refer to the outer speech and these give only some idea of what he means: in fact, he uses a semiotic system (written or spoken language) to describe something which is not a semiotic system. The grounding metaphors through which Vygotsky describes inner speech show its similarity to semiotic sets: properties like *agglutination* and *influence* make inner speech akin to some semiotic sets, like drawings, gestures and so on. Also, the syntactic phenomena of syntactic and phasic reduction mean that the so-called linear and compositional properties of semiotic systems are violated. Vygotsky's description through the lens of semiotic systems makes this aspect only partially evident.

The notion of semiotic bundle properly frames the most important point in Vygotsky's analysis, namely, the semiotic transformations that support the transformation from outer to inner speech (internalization). The core of Vygotsky's analysis, namely, the internalization process, consists exactly in pointing out a genetic conversion within a semiotic bundle: it generates a fresh semiotic component, the inner speech, from another existing one, the outer speech. The description is given using the structure of the former, which is clearly a semiotic system, to build grounding metaphors in order to give an idea of the latter, which is possibly a semiotic set. The whole process can be described as the enlarging of a bundle through a genetic conversion process.

The main point of this paper consists in using the notion of semiotic bundle to frame the mathematical activities that take place in the classroom. I will argue that learning processes happen in a multimodal way, namely in a dynamically developing bundle, which enlarges through genetic conversions and where more semiotic sets are active at the same moment. The enlargement consists both in the growing of (the number of) active semiotic sets within the bundle and in the increase of the number of relationships (and transformations) between the different semiotic sets.

Their mutual relationships will be analyzed through two types of lenses, which I have called synchronic and diachronic since they analyze the relationship among processes that happen simultaneously or successively in time. The two approaches, which will be discussed below, allow us to frame many results in a unitary way: some are already known but some are new. In particular, I shall investigate the role of gestures in the mathematical discourses of students[12]. I will argue that they acquire a specificity in the

---

10 Vygotsky makes the analogy with the outer language alluding to so-called agglutinating languages which put together many different words to constitute a unique word.

11 To give an idea of influence, Vygotsky makes reference to *The Dead Souls* by N.V. Gogol whose title, by the end of the book, should mean to us *"not so much the defunct serfs as all the characters in the story who are alive physically but dead spiritually"* (*ibid.*, p. 247)

12 Another research project that our group is pursuing concerns the role of teachers' gestures with respect to the learning processes of students: how they are shared by students and how they influence their conceptualization processes.

construction of meaning in mathematical activities because of the rich interplay among three different types of semiotic sets: speech, gestures and written representations (from sketches and diagrams to mathematical symbols). They constitute a semiotic bundle, which dynamically evolves in time.

To properly describe this interplay and the complex dynamics among the different semiotic sets involved in the bundle, I need some results from psychologists, who study gesture. In the next two sections (2.2 and 2.3) I will sketch out some of these.

2.2 Semiotic bundles and multimodality

In mathematics, semiotic representations are deeply intertwined with mental ones (see the discussion in Duval, 2006, pp. 106-107). On the one side, there is a genetic relationship between them: *«the mental representations which are useful or pertinent in mathematics are always interiorized semiotic representations»* (Duval, 2002, p.14). See also the discussion on the internalisation processes in Vygotsky.

On the other side, very recent discoveries in Neuropsychology underline the *embodied* and *multimodal* aspects of cognition. A major result of neuroscience is that *"conceptual knowledge is embodied, that is, it is mapped within the sensory-motor system"* (Gallese & Lakoff, 2005, p.456). *"The sensory-motor system not only provides structure to conceptual content, but also characterizes the semantic content of concepts in terms of the way in which we function with our bodies in the world"* (*ibid.*). The sensory-motor system of the brain is multimodal rather than modular; this means that

> *"an action like grasping…(1) is neurally enacted using neural substrates used for both action and perception, and (2) that the modalities of action and perception are integrated at the level of the sensory-motor system itself and not via higher association areas." (ibid., p. 459).*
>
> *"Accordingly, language is inherently multimodal in this sense, that is, it uses many modalities linked together—sight, hearing, touch, motor actions, and so on. Language exploits the pre-existing multimodal character of the sensory-motor system."* (ibid., p. 456).

The paradigm of multimodality implies that *"the understanding of a mathematical concept rather than having a definitional essence, spans diverse perceptuomotor activities, which become more or less active depending of the context."* (Nemirovsky, 2003; p. 108).

Semiotic bundles are the real core of this picture: they fit completely with the embodied and the multimodal approach. At least one consequence of this approach is that the usual *transformations* and *conversions* (in the sense of Duval) from one register to the other must be considered as the basic producers of mathematical knowledge. Furthermore, its essence consists in the *multimodal interactions* among the different registers within a unique integrate system composed of different modalities: gestures, oral and written language, symbols, and so on (Arzarello & Edwards, 2005; Robutti, 2005). Also, the symbolic function of signs is absorbed within such a picture.

Once the multimodal nature of processes is on the table, manipulations of external signs and of mental images show a common psychological basis: transformational and symbolic functions are revealed as processes that have a deep common nature.

I will argue that if we mobilize a rich semiotic bundle with a variety of semiotic sets (and not only semiotic systems) with their complex mutual relationships (of transformation, conversion, symbolic functions as multimodal interactions among them) students are helped to construct integrated models for the mathematical knowledge they are supposed to learn and understand. In fact, mathematical activity is featured by the richness of the semiotic bundle that it activates. However, things may not be so in the school, where two negative phenomena can push the process in the opposite direction. I call them the Piaget and the Wittgenstein effect, respectively:

a) (*Piaget effect*). Piaget made the *search for isomorphisms* one of the key principles for analyzing knowledge development in children. This emphasis risks underestimating the relevance of the different registers of representation:

> « *Dismissing the importance of the plurality of registers of representation comes down to acting as if all representations of the same mathematical object had the same content or as if the content of one could be seen from another as if by transparency!"* (Duval, 2002, p.14).

b) (*Wittgenstein effect*). Recall the story about Sraffa and Wittgenstein. The author of *Tractatus* in the first phase of his research revealed a sort of *blindness to semiotic sets* (in that case, the gesture register). This is also the case for many mathematicians and teachers: they are possibly interested in semiotic systems as formal systems, while the wider semiotic sets are conceived as something that is not relevant for mathematical activities, especially at the secondary school level.

A consequence of these effects in the classroom is that only some semiotic systems are considered, while semiotic bundles (generally not even restricting oneself to the bundles of semiotic systems) are not taken into account. And even when different semiotic systems are considered, they are always conceived as signifiers of the same object. On the contrary, the representations within a semiotic bundle have their own specificity in promoting an integrated mental model according to the multimodal paradigm, as we shall show in the next chapter.

2.3 Gestures within semiotic bundles

Among the components of semiotic bundles, the semiotic set of gestures has an important role, especially when its relationship with speech and written signs are considered within a multimodal picture. Psychologists have mainly studied gestures in day to day conversation: I shall go over some of their findings in the remaining part of this chapter and I will describe the relationship of gestures (and speech) to written signs in Chapter 3. To do this, I will elaborate upon some of the papers in Arzarello & Edwards (2005), especially the Introduction, and I will also quote some results of Bucciarelli (in print).

Two main points from psychology are important to discuss the way gestures enter into the multimodal semiotic analysis within which we frame the understanding of mathematical concepts in students.

The first point concerns the so-called *Information Packaging Hypothesis*. It expands the idea that "*gestures, together with language*, <u>*help constitute thought*</u>" (McNeill, 1992, p. 245). According to McNeill (p. 594-5), gesture plays a role in cognition—not just in communication—since it is involved in the *conceptual planning* of the messages and plays a role in speech production because it plays a role in the process of conceptualization. Gesture *"helps speakers organize rich spatio-motoric information into packages suitable for speaking [...] by providing an alternative informational organization that is not readily accessible to analytic thinking, the default way of organizing information in speaking"* (Kita, 2000).

Spatio-motoric thinking (constitutive of what Kita calls representational gestures) provides an alternative informational organization that is not readily accessible to analytic thinking (constitutive of speaking organization). Analytic thinking is normally employed when people have to organize information for speech production, since speech is linear and segmented (composed of smaller units); namely, it is a semiotic system. On the other hand, spatio-motoric thinking is instantaneous, global and synthetic, not analyzable into smaller meaningful units, namely, it is a semiotic set. This kind of thinking and the gestures that arise from it are normally employed when people interact with the physical environment, using the body (interactions with an object, locomotion, imitating somebody else's action, etc.). It is also found when people refer to virtual objects and locations (for instance, pointing to the left when speaking of an absent friend mentioned earlier in the conversation) and in visual imagery. Within this framework, gesture is not simply an epiphenomenon of speech or thought; gesture can contribute to creating ideas:

> *"According to McNeill, thought begins as an image that is idiosyncratic. When we speak, this image is transformed into a linguistic and gestural form. ... The speaker realizes his or her meaning only at the final moment of synthesis, when the linear-segmented and analyzed representations characteristic of speech are joined with the global-synthetic and holistic representations characteristic of gesture. The synthesis does not exist as a single mental representation for the speaker until the two types of representations are joined. The communicative act is consequently itself an act of thought. ... It is in this sense that gesture shapes thought."* (Goldin-Meadow, 2003, p. 178).

A second point, claimed by Bucciarelli (in press), concerns the relationships between Mental Models (see Johnson Laird, 1983, 2001) and gestures. Many studies in psychology claim that the learning of declarative knowledge involves the construction of mental models. Bucciarelli argues that gestures accompanying discourse can favour the construction of such models (and therefore of learning). In Cutica & Bucciarelli (2003) it is shown that when gestures accompany discourse the listener retains more information with respect to a situation in which no gestures are performed: *"The experimental evidence is in favour of the fact that gesture do not provide redundancy, rather they provide information not conveyed by words"* (Bucciarelli, in press).

Hence, gestures lead *"to the construction of rich models of a discourse, where all the information is posited in relation with the others"* (*ibid.*).

In short, the main contribution of psychology to the theory of semiotic bundles consists in this: the multimodal approach can favour the understanding of concepts because it can support the activation of different ways of coding and manipulating the information (e.g. not only in an analytic fashion) within the semiotic bundle. This can foster the construction of a plurality of mental models, whose integration can produce deep learning.

Of course these observations are general and concern general features of learning. In the next chapter, I shall discuss how this general frame can be adapted to the learning of mathematics.

This attention to semiotic bundles underlines the fact that mathematics is inseparable from symbolic tools but also that it is *"impossible to put cognition apart from social, cultural, and historical factors"* (Sfard & McClain, 2002, p. 156), so that cognition becomes a *"culturally shaped phenomenon"* (*ibid.*). In fact, the embodied approach to mathematical knowing, the multivariate registers according to which it is built up and the intertwining of symbolic tools and cognition within a cultural perspective are the basis of a unitary frame for analyzing gestures, signs and artefacts. The existing research on these specific components finds a natural integration in such a frame (Arzarello & Edwards, 2005).

In the next chapter, I will focus the attention on the ways in which semiotic bundles are involved in the processes of building mathematical knowledge in the classroom.

3. Semiotic bundles in mathematics learning.

3.1 Synchronic and diachronic analysis

In this chapter, I will illustrate how the notion of semiotic bundle can suitably frame the mathematising activities of young students who interact with each other while solving a mathematical problem. What we will see is a consequence of these social interactions, which can happen and develop because of the didactical situations to which the students are exposed. As I shall sketch below, they are accustomed to developing mathematics discussions during their mathematics hours. The richness of the semiotic bundle that they use depends heavily on such a methodology; in a more traditional classroom setting, such richness may not exist and this may be the cause of many difficulties in mathematical learning: see the comments in Duval (2002, 2006), already quoted, about this point.

The example under consideration concerns elementary school and has been chosen for two reasons: (1) it is emblematic of many phenomena that we have also found at different ages; (2) the simplicity of the mathematical content makes it accessible for everyone.

In the example, I shall show that students in a situation of social interaction use a variety of semiotic sets within a growing semiotic bundle and I shall describe the main mutual relationships among them. To do that, I will use two types of analysis, each focusing on a major aspect of such relationships. The first one is *synchronic analysis*, which studies the relationships among different semiotic sets activated simultaneously by the subject. The second is *diachronic analysis*, which studies the relationships among semiotic sets activated by the subject in successive moments. This idea has been introduced by the authors in Arzarello & Edwards (2005) under the names of *parallel* and *serial analysis*. I prefer the terminology "à la Saussurre" ([13]) because it underlines the

[13] Saussure distinguishes between *synchronic* (static) linguistics and *diachronic* (evolutionary) linguistics. Synchronic linguistics is the study of language at a particular point in time. Diachronic linguistics is the study of the history or evolution of language.

time component that is present in the analysis. However, our time grain is at a different scale, that is, while Saussurre considers long periods of time concerning the historical evolution of at most two semiotic systems (spoken and written language), I consider the interactions among many different semiotic sets over very short periods of time.

Synchronic analysis, even if under a different name, is present in the study of gestures: e.g. the distinction made by Goldin-Meadow between matching and mismatching considers gesture and speech produced at the same moment and conveying equal or different information. Another example of synchronic analysis can be made in mathematics when considering the production of drawings (or formulas) and of speech by students who are solving a problem (see e.g. Arzarello, 2005; but the literature is full of examples). A further example is the semiotic node, discussed by Radford et al. (2003b).

Also, diachronic analysis is not completely new in the literature on signs: e.g. see the notion of mathematical objectification in Radford, or that of conversion in Duval, both discussed above. The power of diachronic analysis changes significantly when one considers the semiotic bundles. In fact, the relationship between sets and systems of signs cannot be fully analyzed in terms of translation or of conversion because of the more general nature of the semiotic sets with respect to the semiotic systems. The modes of conversion between a semiotic set and a semiotic system make evident a genetic aspect of such processes, since a genuine transformation (conversion) is a priori impossible. In fact, a transformation presupposes an action between two already existing systems like in the translation from one language to another. In our case, on the contrary, there is a genesis of signs from a set or a system to a system or a set. The fresh signs with the new set (system) are often built preserving some features of the previous signs (e.g. like the icon of a house preserves some of the features of a house according to certain cultural stereotypes). The preservation generally concerns some of the extralinguistic (e.g. iconic) features of the previous signs, which are generating new signs within the fresh semiotic set (or system); possibly, the genesis continues with successive conversions from the new sets (systems) into already codified systems. Hence, the process of conversion described by Duval concerns mainly the last part of the phenomenon, which involves the transformation between already existing systems. Our analysis shows that such process starts before and has a genetic aspect, which is at the root of the genesis of mathematical ideas.

The main point is that only considering semiotic sets allows us to grasp such a phenomenon, possibly through a diachronic analysis. In fact, nothing appears if one considers only semiotic systems or considers synchronic events.

One could think that such a genesis is far from the sophisticated elaborations of more advanced mathematics. But things are not so; I have examples of this genesis concerning the learning of Calculus (see: Arzarello & Robutti, to appear).

The two analyses, synchronic and diachronic, allows us to focus on the roles that the different types of semiotic sets involved (gestures, speech, different inscriptions, from drawings to arithmetic signs) play in the conceptualization processes of pupils. The general frame is that of multimodality, sketched above.

3.2 The example [14]

The activity involves pupils attending the last year of primary school (5th grade, 11 y.o.); the teacher gives them a mathematical story that contains a problem to solve, taken from the legend of Penelope's cloth in Homer's Odyssey. The original text was modified to get a problem-solving situation that necessitated that the students face some conceptual nodes of mathematics learning (decimal numbers; space-time variables). The text of the story, transformed, is the following:

> *… On the island of Ithaca, Penelope had been waiting twenty years for the return of her husband Ulysses from the war. However, on Ithaca a lot of men wanted to take the place of Ulysses and marry Penelope. One day the goddess Athena told Penelope that Ulysses was returning and his ship would take 50 days to arrive in Ithaca. Penelope immediately summoned the suitors and told them: "I have decided: I will choose my bridegroom among you and the wedding will be celebrated when I have finished weaving a new piece of cloth for the nuptial bed. I will begin today and I promise to weave every two days; when I have finished, the cloth will be my dowry." The suitors accepted. The cloth had to be 15 spans in length. Penelope immediately began to work, but one day she would weave a span of cloth, while the following day, in secret, she would undo half a span… Will Penelope choose another husband? Why?*

When the Penelope's story was submitted to the students (Dec. 2004- Feb. 2005) they were attending the last year of primary school (5th grade). Later, in April-May 2005, in the same school six more teachers submitted the story to their classrooms, as part of an ongoing research project for the Comenius Project DIAL-Connect (Barbero et al., in press). Students were familiar with problem solving activities, as well as with interactions in group. They worked in groups in accordance with the didactical contract that foresaw such a kind of learning. The methodology of the mathematical discussion was aimed at favouring the social interaction and the construction of shared knowledge. As part of the didactical contract, each group was also asked to write a description of the process followed to reach the problem solution, including doubts, discoveries, heuristics, etc. The students' work and discussions were videotaped and their written notes were collected. The activity consisted of different steps that we can summarize as follows. First, the teacher reads the story and checks the students' understanding of the text; the story is then delivered to the groups. Different materials are at the students' disposal, among which paper, pens, colours, cloth, scissors, glue. In a second phase, the groups produce a written solution. The teacher invites the groups to compare the solutions in a collective discussion; she analyses strategies, difficulties, misconceptions, thinking patterns and knowledge content to be strengthened. Then, a poster with the different groups' solutions is produced. In the final phase, the students are required to produce a number table and a graph representing the story; they work individually using Excel to construct the table and the graph of the problem solution. Again, they discuss about different solutions and share conclusions.

---

[14] This part of the paper is partially taken from Arzarello et al. (2006), with the permission of the other authors.

The part of the activity analyzed below is a small piece of the initial phase (30'); it refers to a single group composed of five children: D, E, M, O, S, all of them medium achievers except M, who is weak in mathematical reasoning.

3.3 Analysis: a story of signs under the lenses of diachronic and synchronic analysis.

The main difficulty of the Penelope problem is that it requires two registers to be understood and solved: one for recording the time, and one for recording the successive steps of the cloth length. These registers must be linked in some way, through some relationship (mathematicians would speak of a function linking the variables time and cloth length). At the beginning, these variables are not so clear for the students. So, they use different semiotic sets to disentangle the issue: gestures, speech, written signs. They act with and upon them; they interact with each other; they repeatedly use the text of the story to check their conjectures; they use some arithmetic patterns.

We see an increasing integration of these components within a semiotic bundle: in the end, they can grasp the situation and objectify a piece of knowledge as a result of a complex semiotic and multimodal process. We shall sketch some of the main episodes and will comment a few key points in the final conclusion (numbers in brackets indicate time).

Episode 1. The basic gestures (synchronic analysis).

After reading the text, the children start rephrasing, discussing and interpreting it. To give sense to the story, they focus on the action of weaving and unraveling a span of cloth which is represented by different gestures: a hand sweeping across the desk (Fig. 1), the thumb and the index extended (Fig. 2), two hands displaced parallel on the desk (Figs. 3 and 4). Some gestures introduced by one student are easily repeated by the others and become a reference for the whole group.

This is the case of the two parallel hands shown in Figs. 3 and 4. Attention is focused on the action, and the gestures occur matching either the verbal clauses or the "span", as we can see from the following excerpt:

(6'58'') **S:** She makes a half (*hand gesture in Fig. 2*), then she takes some away (*she turns her hand*), then she makes… (*again, her hand is in the position of Fig. 2*) […]

Figure 1

Figure 2

Figure 3

Figure 4

E: "It is as if you had to make a piece like this, it is as if you had to make a piece of cloth like this, she makes it (*gesture in Fig. 3*). Then you take away a piece like this (*gesture in Fig. 5*), then you make again a piece like this (*gesture in Fig. 3*) and you take away a piece like this (*gesture in Fig. 5*)"

O: "No, look… because… she made a span (*Fig. 4*) and then, the day after, she undid a half (*O carries her left hand to the right*), and a half was left… right? … then the day after…"

D: (*D stops O*) "A half was always left"

Figure 5

The dynamic features of gestures that come along with speech condense the two essential elements of the problem: time passing and Penelope's work with the cloth. Their existence as two entities is not at all explicit at this moment, but, through gesturing, children make the problem more tangible. The function of gestures is not only to enter into the problem, but also to create situations of discourse whose content is accessible to everyone in the group. The rephrasing of similar words and gestures by the students (see the dispositions of the hands in Fig. 4) starts a dynamics for sharing various semiotic sets, with which the group starts to solve the problem. At the moment, the semiotic bundle is made up of their gestures, gazes and speech .

Episode 2. A new semiotic set: from gestures to written signs (diachronic analysis).

After having established a common understanding of what happens in Penelope's story, the children look for a way to compute the days. S draws a (iconic) representation of the work Penelope does in a few days, actually using her hand to measure a span on paper. The previous gesture performed by different students (Figs. 3-5) now becomes a written sign (Fig. 6). As had happened before with words and gestures, the drawing is also imitated and re-echoed by the others (Fig. 7): even these signs, generated by the previous gestures, contribute to the growth of the semiotic bundle. The use of drawings makes palpable to the students the need of representing the story using two registers. See the two types of signs in Figs. 7-8: the vertical parallel strokes (indicating spans of cloth) and the bow sign below them (indicating time).

Episode 3. The mutimodality of semiotic sets I: towards a local rule (diachronic + synchronic analysis).

In the following excerpts, the children further integrate what they have produced up to now (speech, gestures and written representations) and also use some arithmetic; their aim is to grasp the rule in the story of the cloth and to reason about it. They can now use the written signs as *"gestures that have been fixed"* (Vygotsky, 1978; p. 107) and represent the story in a condensed way (see Fig. 8); moreover, they check their conjectures reading again the text of the problem:

(10’30’’) **S:** From here to here it is two spans (*she traces a line, mid of Fig. 8*). If I take half, this part disappears (*she traces the horizontal traits in Fig. 8*) and a span is left; therefore in two days she makes a span

O: No, in four days, in four, because…

S: In four days she makes two spans, because (*she traces the curve under the traits in Fig. 8*)…plus this

O: In four days she makes one, because (*she reads the text*), one day she wove a span and the day after she undid a half…

As one can see in Fig. 7, S tries to represent on paper Penelope’s work of weaving and also of unraveling, which causes troubles, because of the necessity of marking time and length in different ways. These two aspects naturally co-existed in gestures of Figg. 1-3. O finds the correct solution (4 days for a span), but the group does not easily accept it and O gets confused. The drawing introduced by S (Fig. 8) represents the cloth, but with holes; due to the inherent rigidity of the drawing, students easily see the span, but not half a span. A lively discussion on the number of days needed to have a span begins. Numbers and words are added to the drawings (Figs. 9-10) and fingers are used to compute (Fig. 11). New semiotic resources enter the scene within different semiotic sets which are integrating each other more and more, not by juxtaposition or translation but by integration of their elements: they all continue to be active within the semiotic bundle, even later, as we shall see below.

Episode 4. The multimodality of semiotic sets II: towards a global rule (diachronic analysis).

Once the local question of “how many days for a span” is solved, the next step is to

Figure 6

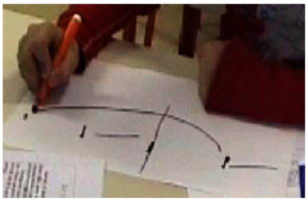


Figure 7

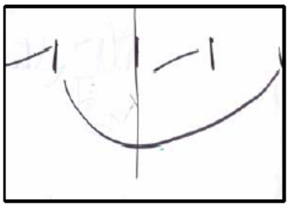


Figure 8

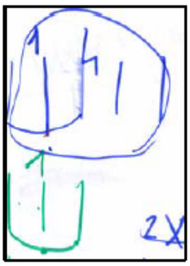


Figure 9

Figure 10

Figure 11

solve the problem globally. To do that, the rule of "4 days for a span" becomes the basis (Fig.12) of an iterative process:

(13'30'') **O, E:**… it takes four days to make a whole span (*E traces a circle with the pen all around*: Fig. 12)

D: and another four to make a span (*D shows his fingers*) and it adds to 8 (*D counts with fingers*)

S: so, we have to count by four and arrive at 50 days (*forward strategy*: Fig. 13) [...]
(14'25'') **O**: no, wait, for 15 spans, no, 4 times 15

S: no, take 15, and always minus 4, minus 4, minus 4 (or: 4 times 5), minus 2, no, minus 1 [*backward strategy*: Fig. 14]

Two solving strategies are emerging here: a *forward strategy* (counting 4 times 15 to see how many days are needed to weave the cloth) and a *backward strategy* (counting "4 days less" 15 times to see if the 50 days are enough to weave the cloth). The two strategies are not so clear to the children and conflict with each other.

In order to choose one of them, the children use actual pieces of paper, count groups of four days according to the forward strategy and so they acquire direct control over the computation. Only afterwards do they compute using a table and find that 60 days are needed for 15 spans of cloth. In this way, they can finally answer the question of the problem and write the final report: Penelope will not choose another bridegroom.

Conclusions

The story of signs described in the example illustrates the nature of semiotic bundles. The first signs (gestures, gazes and speech) constitute a first basic semiotic bundle, through which the children start their semiotic activities. Through them, the bundle is enriched with new semiotic sets (drawings and numbers) and with a variety of fresh relationships among them. The enlargement occurs through genetic conversions, namely through a genetic process, where the previous semiotic sets (with their mutual relationships) generate new semiotic components and change because of this genesis, becoming enriched with fresh mutual relationships. By so doing, not only do the students produce new semiotic sets, but the sense—in the Vygotskian meaning of the word—of the older ones is transformed, still maintaining some aspects of their origin. All these processes develop within a gradually growing and multimodal cognitive environment that we have analyzed through the lens of the semiotic bundle.

The story of the bundle starts with the gesture of the two hands displaced parallel on the desk (episode 1). This gesture later generates a written iconic representation

Figure 12

Figure 13

Figure 14

Figure 15

(episode 2), successively enriched by numerical instances (episode 3) and by arithmetic rules (episode 4), expressed through speech and (new and old) gestures. Gesture, speech, written signs and arithmetic representations grow together in an integrated way supporting the semiotic activities within the semiotic bundle which enlarges more and more. Students develop their semiotic activities and share them: it is exactly through such activities that they can grasp the problem, explore it and elaborate solutions.

All the components are active in a multimodal way up to the end. This is even evident when the students discuss how to write the solution in the final report (Fig. 15: 27' 32"). Gestures and speech intervene first as cognitive means for understanding the story of the cloth; later as means of control for checking the conjectures on the rule. Information is condensed in gestures, entailing a global understanding of the story. The two variables (time and cloth development), first condensed in gesture (an agglutination example in the sense of Vygotsky), generate two different signs in the fresh semiotic set (drawings) that they themselves have generated within the semiotic bundle: it is exactly this disentanglement that allows children to grasp the story separating its structural elements. On its own, speech objectifies the structure of the story, first condensing the local rule in a sentence (episode 3), then exploiting the general rule as an iterative process (episode 4).

The semiotic objectification in this story happens because of the semiotic activities within the semiotic bundle. It is evident that it constitutes an integrated semiotic unity; the activity within it does not consist of a sequence of transcriptions from one register to another, as posited in other studies (e.g. Duval, 1993). On the contrary, it develops in a growing, holistic and multimodal way, which, in the end, produces the objectification of knowledge.

The lenses of semiotic bundles allow us to frame the semiotic phenomena that occur in the classroom within a unitary perspective. Moreover, a semiotic bundle also incorporates dynamic features, which can make sense of the complex genetic relationships among its components, e.g. the genetic conversions and the Vygotskian internalization processes.

This study leaves many problems open: I list only some of those I am interested in studying in the near future:

1. Elsewhere (Arzarello, in press), I introduced the notion of *Space of Action, Production and Communication* (APC-space) as an environment in which cognitive processes develop through social interaction; its components are: culture, sensory-motor experiences, embodied templates, languages, signs, representations, etc. These elements, merged together, shape a multimodal system through which didactical phenomena are described. An interesting problem consists in studying the relationships between the semiotic bundles and the APC-space.

2. The time variable is important in the description of semiotic bundles, e.g. it is relevant to the diachronic and synchronic analysis. What are the connections between this frame and the didactic phenomena linked to students 'inner times[15]', like those described in Guala & Boero (1999)? There, the authors

15 I thank Paolo Boero for suggesting this problem to me.

list different types of inner times in students' problem solving activities (the 'time of past experience', the 'contemporaneity time', the 'exploration time', the 'synchronous connection time'), which make sense of their mental dynamics. Of course, such activities can be analyzed with semiotic lenses. How do the different inner times enter into a semiotic bundle? Which kinds of conversions or treatments can they generate from one semiotic set to another or within the same semiotic set?.

3. In the processes of students who build new knowledge, there are two dual directions in the genetic conversions within the semiotic bundle: from semiotic sets to semiotic systems (e.g. from gestures to drawings and symbols) or the opposite. The episodes in Penelope's story are an example of the first type, while Vygotsky describes the second type in the internalization processes (a similar example is described in Arzarello & Robutti, to appear). It would be interesting to clarify the nature of this duality of processes.

Aknowledgments

Research program supported by MIUR, the Università di Torino and the Università di Modena e Reggio Emilia (PRIN Contract n. 2005019721).
I thank L. Bazzini, R. Barbero, F. Ferrara, D. Paola, O. Robutti, C. Sabena, and B. Villa, without whose work and encouragement this paper would not exist.

References

Arzarello, F. (in press). Mathematical landscapes and their inhabitants: perceptions, languages, theories, *Proceedings ICME 10*, Plenary Lecture.

Arzarello F., L.Bazzini, Chiappini G.P. (1994), Intensional semantics as a tool to analyse algebraic thinking, *Rendiconti del Seminario Matematico dell'Università di Torino*, 52 (1), 105-125.

Arzarello, F & Edwards, L. (2005). Gesture and the Construction of Mathematical Meaning (Research Forum 2), *Proc. 29th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 122-145). Melbourne, AU: PME.

Arzarello F. & Robutti, O. (2004). Approaching functions through motion experiments. In: R. Nemirovsky, M. Borba & C. DiMattia (eds.), Bodily Activity and Imagination in Mathematics Learning, *PME Special Issue of Educational Studies in Mathematics,* 57.3, CD-Rom, Chapter 1.

Arzarello, F., Ferrara, F., Paola, D., Robutti, O., (2005). The genesis of signs by gestures. The case of Gustavo, *Proc. 29th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 73-80). Melbourne, AU: PME.

Arzarello, F., Bazzini, L., Ferrara, F., Robutti, O., Sabena, C., Villa, B. (2006). Will Penelope choose another bridegroom? Looking for an answer through signs, *Proc. 30th Conf. of the Int. Group for the Psychology of Mathematics Education*. Prague.

Arzarello, F. & Robutti, O. (to appear). Framing the embodied mind approach within a multimodal paradigm, in: Lyn English, M. Bartolini Bussi, G. Jones, R. Lesh e D. Tirosh (ed.), *Handbook of International Research in Mathematics Education* (LEA, USA), 2nd revised edition.

Bara, B.G. & Tirassa, M. (1999). A mentalist framework for linguistic and extralinguistic communication. In: S. Bagnara (ed.), *Proceedings of the 3rd European Conference on Cognitive Science*. Siena, Italy. Roma: Istituto di Psicologia, CNR.

Barbero R, Bazzini L., Ferrara F., Villa B. (in press). The Penelope's story: learning through action, speech and gestures, *Proc. CIEAEM 57*, Piazza Armerina, Italy, July 2005.

Bartolini, M.G. & Mariotti, M.A. (to appear). Semiotic mediation in the mathematics classroom, in: Lyn English, M. Bartolini Bussi, G. Jones, R. Lesh e D. Tirosh (ed.), *Handbook of International Research in Mathematics Education* (LEA, USA), 2nd revised edition.

Bosch, M. & Chevallard, Y. (1999). La sensibilité de l'activité mathématique aux ostensifs, *Recherches en Didactique des Mathématiques*, 19 (1), 77-124.

Bucciarelli, M. (in press). How the construction of mental models improve learning, *Mind and Society*.

Chevallard, Y. (1999), L'analyse des pratiques enseignantes en théorie anthropologique du didactique, *Recherches en Didactique des Mathématiques*, 19/2, 221-266.

Cutica, I. & Bucciarelli, M. 2003. Gestures and the construction of models. *The 2nd International Conference on Reasoning and Decision Making,Reasoning and understanding: Mental models, relevance and limited rationality approaches*, Padova, March, the 17-18, 11.

Detlefsen, M., 1986, *Hilbert's Program*, Dordrecht: Reidel.

Dörfler, W. (n.d.) How diagrammatic is mathematical reasoning? Retrieved May 4 2006, from http://www.math.uncc.edu/~sae/dg3/dorfler2.pdf

Duval R. (1993). Registres de représentations sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives*, 5, 37-65. ULP, IREM Strasbourg.

Duval, R. (1999). Conversion et articulation des représentations analogiques. Lille, France: I.U.F.M. Nord pas de Calais.

Duval, R. (2001). The cognitive analysis of problems of comprehension in the learning of mathematics. *Paper presented at the Semiotics Discussion Group of the 25th PME International Conference*, Freudenthal Institute, The Netherlands, July 2001.

Duval, R. (2002). The cognitive analysis of problems of comprehension in the learning of mathematics. *Mediterranean Journal for Research in Mathematics Education*, 1(2), 1-16.

Duval, R. (2006). A cognitive analysis of problems of comprehension in a learning of mathematics, *Educational Studies in Mathematics*, (61), 103-131.

Ernest, P. (2006). A semiotic perspective of mathematical activity: the case of number, *Educational Studies in Mathematics*, (61), 67-101

Frege, G. (1969). *Funktion, Begriff, Bedeutung. Fünf logische Studien*. Göttingen: Vandenhoeck & Company.

Gallese, V. & Lakoff, G. (2005). The Brain's Concepts: The Role of the Sensory-Motor System in Conceptual Knowledge. *Cognitive Neuropsychology*, 22(3/4), 455-479.

Guala & Boero, 1999 Time complexity and Learning, *Annals of the New York Academy of Sciences*, June 30, 1999, 164-7.

Goldin-Meadow, S. (2003). *Hearing gestures: How our hands help us think*. Chicago: Chicago University Press.

Harper, E.. (1987). Ghosts of Diophantus, *Educational Studies in Mathematics*, Vol. 18, (75-90).

Hilbert, D. (1962). *Grundlagen der Geometrie*, Stuttgart: Teubner (original edition in 1899).

Lemaitre, A. (1905). *Observations sur le langage interieur des enfants*, Archives de psychologie, n.4,1-43

Johnson-Laird P.N. 1983. *Mental models: Towards a cognitive science of language, and consciousness*. Cambridge University Press, Cambridge, UK.

Johnson-Laird, P.N. 2001. Mental models and human reasoning. In Dupoux, E. (Ed.), *Language, brain, and cognitive development: Essays in honor of Jacques Mehler*, pp. 85-102. Cambridge, MA: The MIT Press.

Kaput, J., Noss, R. & Hoyles, C.: 2002, Developing New Notations for a Learnable Mathematics in the Computational Era, in: English, L.D. (Ed.), *Handbook of International Research in Mathematics Education*: Lawrence Erlbaum Assoc., NJ, 51-75.

Kita, S. (2000). How Representational Gestures Help Speaking. In McNeill, D. (ed.), *Language and Gesture*, pp. 162-185. Cambridge: Cambridge University Press.

Malcom, N. & von Wright, G.H. (2001). Ludwig Wittgenstein : A Memoir. Oxford: Oxford University Press.

McNeill, D. (1992) *Hand and mind: What gestures reveal about thought*. Chicago: Chicago University Press.

Hartshorne, C. & Weiss, P. (Eds.) (1933), *Collected Papers of Charles Sanders Peirce*, vol. III, Cambridge (MA): Harvard University Press

Kita, S. (2003). Interplay of gaze, hand, torso orientation, and language in pointing. In: S. Kita (ed.), *Pointing. Where language, culture, and cognition meet*. Mahwah: Erlbaum.

Nemirovsky, R. (2003). Three conjectures concerning the relationship between body activity and understanding mathematics. In N.A. Pateman, B.J. Dougherty & J.T. Zilliox (Eds.), *Proc. 27th Conf. of the Int. Group for the Psychology of Mathematics Education* (Vol. 1, pp. 103-135). Honolulu, Hawai'I: PME.

Ogden, C.K. & Richards, I.A. (1923). *The Meaning of Meaning*. London: Routledge & Kegan.

Rabardel, P. (1995). *Les hommes et les technologies, une approche cognitive des instruments contemporains.* Paris: Armand Colin.

Radford, L. (2002). The seen, the spoken and the written. A semiotic approach to the problem of objectification of mathematical knowledge.*For the Learning of Mathematics,* 22(2), 14-23.

Radford, L. (2003a). Gestures, Speech, and the Sprouting of Signs: A Semiotic-Cultural Approach to Students' Types of Generalization. *Mathematical Thinking and Learning*, 5(1), 37–70.

Radford, L., Demers, S., Guzmán, J. and Cerulli, M. (2003b). Calculators, graphs, gestures and the production of meaning, in: N. Pateman, B. Dougherty and J. Zilliox (eds.), *Proceedings of the 27 Conference of the international group for the psychology of mathematics education* (PME27 –PMENA25), University of Hawaii, Vol. 4, pp. 55-62.

Radford, L. (2006). The Anthropology of Meaning, *Educational Studies in Mathematics*, (61), 39-65.

Robutti, O. (2005). Hearing gestures in modelling activities with the use of technology, in F. Olivero & R. Sutherland (eds.), *Proceedings of the 7th International Conference on Technology in Mathematics Teaching*, University of Bristol, 252-261.

Simpson, S.G: (1999). *Subsystems of second order Arithmetic*. New York: Springer.

Sfard A. & McClain, 2002, Analyzing Tools: Perspectives on the Role of Designed Artifacts in Mathematics Learning, *The Journal of the Learning Sciences*, Volume 11, Numbers 2&3, Special Issue, 153-162.

Steinbring, H. (2005). *The Construction of New Mathematical Knowledge in Classroom Interaction*. New York: Springer.

Steinbring, H. (2006). What makes a sign a mathematical sign? An epistemological perspective on mathematical interaction, *Educational Studies in Mathematics*, (61), 133-162.

Vérillon, P., & Rabardel, P. (1995). Artefact and cognition: a contribution to the study of thought in relation to instrumented activity. *European Journal of Psychology in Education,* vol. IX, n°3.

Vygotsky, L.S. (1978). *Mind in society: The development of higher psychological processes.* Cambridge, MA: Harvard University Press.

Vygotsky, L. S. (1997). *Collected works*, Vol. 4 (R. Rieber, Ed.). New York: Plenum.

Vygotsky, L.S. (1986), *Thought and Language*, Cambridge (MA): MIT Press.

Wertsch, J. V., & Stone, C. A. (1985). The concept of internalization in Vygotsky's account of the genesis of higher mental functions. In J.V. Wertsch (Ed.), *Culture, communication and cognition: Vygotskian perspectives.* Cambridge: Cambridge University Press.

Ferdinando Arzarello
Dipartimento di Matematica
Università di Torino
Italia

E-mail: ferdinando.arzarello@unito.it

Conclusiones y perspectivas de investigación futura

Bruno D'Amore

No cabe duda de que la semiótica, disciplina surgida de un género de estudio del todo diverso (véase la *Introducción* de Luis Radford en esta misma revista), ha conquistado un lugar importante en los estudios de Didáctica de la Matemática.

Respecto a su ingreso en dicho campo, la visión semiótica inició solidificando sus diversos aspectos con trabajos que explicaban el pasaje del concepto a sus representaciones, para después abrir su camino en direcciones diferentes, como lo demuestra la amplia colección de estudios que en esta publicación aparecen. Ahora, el desafío consiste en tratar de entender hacia qué tendencia se moverá la investigación en el futuro. Para poder plantear algunas hipótesis, considero útil un ulterior análisis de la historia reciente y de las mismas bases culturales.

Una problemática importante –y todavía central– es la tocante a la representación de los objetos matemáticos. Por lo general, en Didáctica de la Matemática decimos "pasar de un concepto a sus representaciones"; sin embargo, ¿qué es un *concepto*? La pregunta aún continúa siendo fundamental. En D'Amore, 2006 (pp. 205-220) intenté plantear las bases para responder a dicha cuestión, aparentemente ingenua; empero, lo que se llega a constatar, con certeza absoluta, es que la *definición* revela, por muchos motivos, una complejidad inmensa.

Entre las dificultades que presenta la definición, está que en la idea de *concepto* intervienen muchos factores y causas. Para decirlo brevemente (y, por tanto, en modo incompleto), no parece correcto afirmar que *un concepto matemático es aquel que se halla en la mente de los científicos que a este tema han dedicado su vida de estudio y reflexión*. Parece más correcto señalar que hay una fuerte componente *antropológica*.

Así, en la *construcción de un concepto* participarían tanto la parte institucional (el Saber) como la personal (de quien tiene acceso a tal Saber, que implica no sólo el científico). Esta propuesta la han expuesto diferentes autores; yo me limito a sugerir el trabajo de Godino y Batanero, 1994, porque hace hincapié en la importancia del debate en el cual estoy tratando de inserirme, al tratar las relaciones entre significados institucionales y personales de los objetos matemáticos.

Distinguir el *concepto* de su construcción no es fácil y, quizá, no es ni posible ni deseable, ya que un concepto se halla continuamente en fase de construcción; aquí estriba su parte más problemática, pero también la más rica de su significado. Podríamos llamar a tal construcción *conceptualización*, y reflexionar sobre qué es y cómo se da. En el intento por clarificar dicho argumento, muchos investigadores han propuesto hipótesis y teorías que no detallaré; basta recordar las contribuciones –muchas veces en franca oposición–

de Vygotski, Piaget, Gal'perin, Bruner o Gagné. Para una rápida recapitulación, puede consultarse D'Amore, 2006.

Adentrarse en esta aventura nos conduce, por lo menos, a darnos cuenta de un hecho: la segunda pregunta, *¿qué es* o *cómo se llega a la conceptualización*?, es un misterio. La cuestión pasa a través de un recorrido por los famosos *triángulos* (hay bibliografía específica en D'Amore, 2006):

- El de Charles Sanders Peirce (1839-1914), publicado en 1883: *intérprete*, *representante*, *objeto*
- El de Gotlob Frege (1848-1925), publicado en 1892: *Sinn* [sentido], *Zeichen* [expresión], *Bedeutung* [indicación]
- El de C. K. Ogden e I. A. Richards, que quería ser un compendio de los otros dos, y apareció en 1923: *referencia*, *símbolo*, *referente*
- El de G. Vergnaud (1990), por el cual un concepto C es la terna (S, I, *S*), donde S es el referente, I el significado y *S* el significante

Queda claro que *apropiarse* de un concepto, independientemente de lo que esto signifique, necesita siempre de algo más que *nombrarlo* (la cuestión se originó por lo menos en la Edad Media, apunta D'Amore, 2006) y *representarlo*, lo cual nos lleva a la famosa *paradoja* de Duval, 1993 (p. 38).

Kant, en la *Crítica de la razón pura*, señala que el conocimiento es resultado de un contacto entre un sujeto que aprende y un objeto de conocimiento. Él recurre a una comparación: así como el líquido adopta la forma del recipiente que lo contiene, las impresiones sensoriales adoptan las formas que le imponen las estructuras cognitivas. Pero para que eso suceda (y es la bien conocida *hipótesis fuerte* de Kant) se requieren de formas innatas de sensibilidad, como espacio, tiempo, causalidad, permanencia del objeto y uso de experiencias precedentes.

El conocimiento no es una simple representación de la realidad externa, sino el resultado de la interacción entre el sujeto que aprende (sus estructuras cognitivas) y sus *experiencias sensoriales*. Además, el sujeto que aprende abandona la típica pasividad (cartesiana o lockiana), pues construye y estructura sus experiencias; de este modo, participa activamente en el proceso de aprendizaje y lo transforma en una verdadera y propia *construcción*. Un objeto de conocimiento, al entrar en contacto con un sujeto que aprende, se modifica y reconstruye por los instrumentos cognitivos del sujeto.

Pero, ¿de dónde provienen esos instrumentos cognitivos que sirven para transformar las experiencias del sujeto? La epistemología del aprendizaje de Kant, para usar una terminología moderna, se refiere a un aprendiz adulto, dotado de un lenguaje desarrollado, con capacidad de abstracción y de generalización. Aquí es pertinente la siguiente pregunta: ¿cómo cambia todo esto si hablamos de aprendizaje en ambiente escolar, de aprendices no adultos (niños, adolescentes o jóvenes) y a las primeras armas, con lenguajes aún en elaboración?

No es del todo absurdo pensar que la epistemología constructivista de Piaget, formulada en los años treinta[1], surgió por la necesidad de dar respuesta a este

1 Estoy pensando en Piaget (1937), por ejemplo.

problema. Por tanto, el *saber adquirido* puede verse como el producto de la elaboración de la experiencia con la que entra en contacto el sujeto que aprende. Y esta elaboración consiste no sólo en la interacción entre el individuo y su ambiente, sino también en el modo como aquél interioriza el mundo externo. Independientemente de las peculiaridades de tales *actividades*, el sujeto que aprende debe comprometerse en algo que necesariamente lo lleva a simbolizar. Esta es una necesidad típicamente humana, ya que es una elaboración (con características internas o sociales, e incluso ambas) organizada alrededor de o en los sistemas semióticos de representación.

Se puede agregar que el *conocimiento es la intervención y el uso de los signos*. Así, el mecanismo de producción y de uso, subjetivo e intersubjetivo, de estos signos, y el de la representación de los *objetos* de la adquisición conceptual, resulta crucial para el conocimiento.

Todo eso había sido ya previsto en el programa de la epistemología constructivista, enunciada por Piaget y Garcia (1982), particularmente en el capítulo IX. Al hablar sobre la experiencia del niño, indican que las situaciones que él encuentra son generadas por su entorno social y los objetos aparecen situados en contextos que les dan el significado específico. Por tanto, este niño no asimila objetos puros, sino las situaciones en las cuales los objetos tienen roles específicos; a medida que su sistema de comunicación se hace más complejo, la experiencia directa de los objetos queda subordinada al sistema de interpretaciones suministrado por el entorno social.

No hay duda de que el conocimiento en la escuela y su aprendizaje como construcción se hallan condicionados por situaciones específicas de la institución. Por ende, el aprender en la escuela *¡no es el aprender total!* Los problemas del aprendizaje matemático en la escuela, aún antes de ser de orden epistemológico, pertenecen a un ambiente sociocultural.

Si aceptamos que todo conocimiento (matemático, en particular) refleja al mismo tiempo una dimensión social y una personal, la escuela no es una excepción; incluso, en ella queda institucionalizada esa doble naturaleza. Durante el aprendizaje de las matemáticas se introduce a los estudiantes en un mundo nuevo, tanto conceptual como simbólico –sobre todo, representativo–, que no es fruto de una construcción solitaria, sino de una verdadera y compleja interacción con los miembros de la microsociedad, de la cual forma parte el sujeto que aprende: los propios compañeros, los maestros y la *noosfera* (a veces borrosa, otras evidente).

Es mediante un continuo debate social que el sujeto que aprende toma conciencia del conflicto entre *conceptos espontáneos* y *conceptos científicos*. Así, enseñar no consiste sólo en el intento de generalizar, amplificar, volver más crítico el *sentido común* de los estudiantes, sino se trata de una acción más bien compleja, como nos ha enseñado Vygotski en *Pensamiento y lenguaje* (1962), cuando afirma que un concepto es algo más que la suma de ciertos vínculos asociativos formados por la memoria, pues consiste en un auténtico y complejo acto del pensamiento al que se puede llegar sólo cuando el desarrollo mental del niño ha alcanzado el nivel requerido. Sin embargo, el desarrollo de los conceptos presupone el de muchas funciones intelectuales (atención, memoria lógica, abstracción, capacidad de comparación y diferenciación); la experiencia ha demostrado que la enseñanza directa de los conceptos es imposible y estéril.

En matemáticas, la asimilación conceptual de un objeto pasa necesariamente a través de la adquisición de una o más representaciones semióticas (Chevallard, 1991; Duval, 1993, 1999; Godino y Batanero, 1994), lo cual nos obliga a aceptar la afirmación de Husserl, pero centrada por Duval hacia la Didáctica de la Matemática, que *no existe noética sin semiótica*.

Como sugiere Duval, la construcción de los conceptos matemáticos depende, estrechamente, de la capacidad de usar *más* registros de sus representaciones semióticas:

- De *representarlos* en un registro dado
- De *tratar* tales representaciones en un mismo registro
- De *convertir* tales representaciones de un registro dado a otro

El conjunto de estos tres elementos, al igual que las consideraciones de los párrafos anteriores, evidencian una profunda relación entre noética y constructivismo. Así, la *construcción del conocimiento en matemáticas* se puede pensar como la unión de tres *acciones* sobre los conceptos: la expresión misma de la capacidad de *representar* los conceptos, de *tratar* las representaciones obtenidas en un registro establecido y de *convertirlas* de un registro a otro.

Todo esto constituye, en mi opinión, sólo el punto de partida para especificar y explicar históricamente la importancia que la Didáctica de la Matemática reconoció a los estudios sobre la semiótica, en el momento en que ingresaron a su campo de investigación. Hoy se prefiere seguir una vía de carácter no nominalista, que podríamos llamar de pensamiento entendido como praxis reflexiva sensorial-intelectual, apoyada en sistemas semióticos de significado cultural. Según esta línea, trazada por Luis Radford, estos sistemas semióticos, construidos socialmente por los individuos a partir de su realidad concreta, transformados activamente de generación en generación, "naturalizan" la realidad de los individuos, enmarcan lo que se entiende por evidencia, argumentos convincentes, demostraciones, etc. y subtienden las reflexiones que los individuos hacen de su mundo.

Pero, volvamos a la pregunta inicial. ¿Qué dirección tomarán estos estudios en el futuro? Podemos ver ya importantes señales, que emergen en las páginas que aquí quisimos recoger. Quizás una gran influencia tendrán particularmente los estudios sobre la comunicación, sobre las acciones de las comunidades de práctica, las reflexiones sobre la dimensión ontogenética, así como la contribución de análisis críticos de temas que han fundado nuestra disciplina y que ya se delinean como evoluciones de un futuro próximo.

En este número especial de la revista *Relime*, reunimos a varios especialistas con el fin de presentar el estado del arte de las diversas tendencias que conforman, actualmente, el estudio de la semiótica en nuestro sector. Algunos de estos trabajos contribuyen a dar una respuesta adecuada a muchas de las preguntas precedentes.

La respuesta a la primera pregunta, *¿qué es un concepto?*, plantea problemas teóricos. Seguir profundizando en ellos parece ser un campo donde la semiótica puede dar importantes resultados en un futuro cercano. Varios textos aquí reunidos sugieren que las respuestas a esta pregunta, y a las que planteé en el curso de este artículo, deben incluir el aspecto institucional (Godino y colaboradores), pero también el contexto cultural (Radford, Cantoral y colaboradores) y cognitivo (Arzarello, Radford, Duval, Otte, Arzarello).

Es así como Godino y sus colaboradores presentan una actividad concreta del EOS en el análisis de textos escolares, en el cual utilizan los criterios de idoneidad tanto *epistémica* como *cognitiva*; un análisis de este tipo puede tener repercusiones profundas de carácter institucional.

Cantoral y sus colaboradores abordan la socioepistemología, mediante la cual la actividad matemática se sitúa en un contexto cultural de práctica social.

Radford basa su aporte en la idea de *praxis reflexiva* y expone una *teoría cultural de la objetivación*. Tal propuesta tiene una doble valencia: la cultural (de análisis crítico de posiciones, en algunos casos ampliamente compartidas) y la cognitiva.

Duval insiste en la importancia del análisis semiótico complejo en el ámbito matemático y cognitivo. Vuelve a los orígenes de la semiótica con el fin de sugerir motivaciones para el análisis de los signos, así como de las relaciones de semejanza, referencia, causalidad y oposición. Esta modalidad de afrontar la problemática es útil tanto para el desarrollo de las matemáticas como para el análisis de su aprendizaje.

Otte propone que la explicación es consubstancial de la exhibición de signos y sentido, ya que no hay diferencia entre idea y símbolo a pesar de lo que sostienen el idealismo filosófico y el mentalismo cognitivista, lo cual ejemplifica al tratar el tema de la demostración en matemáticas.

Arzarello muestra en primer lugar un análisis crítico e histórico sobre la idea misma de semiótica. Parte de su fundamento teórico y propone diversas interpretaciones y luego enfoca a la semiótica como aproximación modal, que también ofrece análisis de eventos sucedidos en el aula.

La semiótica que nos interesa, de manera específica, atañe al uso de signos y al desarrollo conceptual en el salón de clases. Muchos de los artículos aquí reunidos atienden este aspecto.

Así, Koukkoufis y Williams emplean la teoría de la objetivación para estudiar la manera en que generalizan jóvenes alumnos.

Adalira Sáenz-Ludlow enfoca su atención, fuertemente teórica, en una idea muy concreta, la de *riqueza matemática del alumno*, y en la influencia de los maestros en el discurso matemático.

Gagatsis y sus colaboradores dan a conocer estudios críticos sobre los cambios de representación de objetos relacionados con el concepto de función.

Bagni ofrece un estudio experimental hecho con alumnos de secundaria que intentan dar sentido a frases paradójicas.

D'Amore propone un ejemplo de aula donde se presenta un cambio de sentido frente a diferentes representaciones del mismo objeto, conseguidas por tratamiento semiótico.

Este número especial de *Relime* se inspira en las discusiones colectivas precedentes que menciona Luis Radford en su *Introducción*. Quiere ser una modesta contribución analítica y problemática al tema de la semiótica en el ámbito de la Educación Matemática.

Me agrego a los agradecimientos de Luis, extendiéndolos a nuestros autores y a todos los lectores.

Referencias

D'Amore, B. (2006). *Didáctica de la matemática*. Bogotá, Colombia: Magisterio. [Primera edición en italiano, 1999, Bologna: Pitagora].

Duval, R. (1993). Registres de représentations sémiotique et fonctionnement cognitif de la pensée. *Annales de Didactique et de Sciences Cognitives 5*, 37-65.

Duval, R. (1999). *Semiosis y pensamiento humano. Registros semióticos y aprendizajes intelectuales*. Cali, Colombia: Universidad del Valle. [Primera edición en francés 1995, Berne: Peter Lang].

Godino, J. D. & Batanero, C. (1994). Significado institucional y personal de los objetos matemáticos. *Recherches en Didactique des Mathématiques 3*, 325-355.

Piaget, J. (1937). *La construction du réel chez l'enfant.* Neuchâtel: Delachaux et Niestlé.

Piaget, J. & Garcia, R. (1983). *Psychogenèse et histoire des sciences*. Paris, France: Flammarion.

Vergnaud, G. (1990). La théorie des champs conceptuels. *Recherches en Didactique des Mathématiques 19*, 133-169.

Vygotsky, L. (1962). *Thought and language*. Cambridge, USA: MIT Press.

Traducción de Martha Isabel Fandiño Pinilla

Bruno D'Amore
Dipartimento di Matematica
Università di Bologna
Italia

E-mail: damore@dm.unibo.it

SUGERENCIAS PARA LA PREPARACIÓN DE ARTÍCULOS

En el caso de reportes de estudios experimentales, de casos, de observación, etnográficos, etcétera, recomendamos que los escritos contengan:

1) Resumen del trabajo en no más de 10 renglones en español, y no más de 5 palabras clave. Todo esto, junto con el título debe ser incluido con su traducción al inglés, francés y portugués.

2) Una exposición del problema de investigación (su pertinencia y relevancia en el tema que se aborda).

3) Indicaciones globales acerca de la estructura teórica del reporte.

4) Justificación de la metodología usada.

5) Desarrollo de algunos ejemplos y análisis de resultados.

6) Referencias bibliográficas.

Si se trata de ensayos teóricos y filosóficos, nuestra recomendación es la siguiente:

1) Iniciar con una exposición del problema de investigación (su pertinencia y relevancia en el tema que se aborda).

2) Ofrecer indicaciones sobre la estructura teórica o filosófica en la cual se desarrolla el tema del artículo.

3) Exposición detallada de la posición del autor dentro del tema o los temas de exposición.

4) Implicaciones o consecuencias de la investigación en el área.

5) Incluir referencias bibliográficas.

Los artículos serán evaluados por tres investigadores reconocidos y con experiencia dentro del área. Específicamente se tomará en cuenta la atención a los criterios anteriores, así como a la claridad de la presentación e interés para la comunidad de matemática educativa.

GUÍA PARA LA PRESENTACIÓN DE ARTÍCULOS
(Relime acepta artículos en español y en portugués para su publicación)

Características de los artículos:

1) Extensión máxima de 30 páginas a espacio sencillo. En caso de que su propuesta exceda esta extensión, el Comité de Redacción podrá considerar la posibilidad de publicarlo siempre que tal propuesta no ocupe todo el espacio disponible en un número de la revista; en otro caso, su artículo será remitido para su consideración en otra de las publicaciones del Clame (libros, ensayos, etcétera).

2) Todos los artículos deberán ser trabajos inéditos de investigación. No se aceptarán traducciones previamente publicadas en su lengua original.

3) Los trabajos deberán acompañarse de:

* Portada que indique el título del texto (no mayor de ocho palabras), con su traducción al inglés, francés y portugués, nombre completo del autor y del centro de adscripción.

* Sinopsis o resumen, no mayor de 10 renglones, que explique el tema, los objetivos y la metodología del texto, que plantee las conclusiones principales y que incluya la traducción del resumen a fin de que aparezca en castellano, portugués, francés e inglés.

* Palabras clave, no más de 5 que recojan las ideas centrales y su traducción al inglés, francés y portugués.

* Datos generales del autor (notas curriculares, dirección, número telefónico particular y laboral y de ser posible número de fax y cuenta de correo electrónico).

4) Se solicita a los autores enviar los trabajos en CD o disquete para PC (3.5”) utilizando el procesador de texto Word 6.0 o superior, tipo de letra Times New Roman y tamaño de 12 puntos. Para las expresiones matemáticas se solicita usar texto con el siguiente formato; las literales en itálica, números y signos matemáticos en letra normal, tal como se muestra en el ejemplo:

$$f\,(x) = 3x + 1$$

O bien, utilizar el editor de ecuaciones de Word, respetando el mismo formato. Adicionalmente se enviarán cuatro copias impresas del mismo trabajo, tres de las cuales habrán de omitir los datos del autor.

5) Con el propósito de facilitar el proceso de publicación, los trabajos se escribirán de la siguiente manera:

* Usar márgenes de una pulgada (2.5 centímetros) en ambos lados de la página tamaño carta.
* Mecanografiar o imprimir de un solo lado del papel.
* No usar sangrías.
* Colocar un solo espacio después de punto y seguido.
* No usar guiones de separación de palabras.
* Diferenciar bien los títulos de los subtítulos.

* Las palabras, frases o señalamientos especiales que se deseen destacar llevarán cursivas o negrillas (no usar subrayado).

* Bibliografía, referencias y notas: solicitamos emplear el estilo de la APA (Publication Manual of the American Psychological Association, 4th ed., 1994) para las citas de pie, notas, referencias textuales y bibliografía.

Ejemplos:

Revista especializada:

Sepúlveda, G. (1989). El paradigma de la educación actual. *La Educación 104,* 57-68.

Brousseau, G. (1980). Pròblemes de l’enseignement des décimaux. *Recherches en Didactique des Mathématiques 1* (1)**,** 11-59.

***Revista de interés general* (*publicación periódica*):**

Vélez, E. (1984, mayo). Logros ocupa-cionales del bachiller colombiano: El caso de la Cohorte de 1978. *Educación, Formación Profesional y Empleo* (pp. 167-194). Bogotá, Colombia: SENA.

Libro:

Lakatos, I. (1977). *Proofs and refutations, the logic of mathematical discovery*. Cambridge, USA: Cambridge University Press.

Capítulo de libro:

Artigue, M. (1992). Functions from an algebraic and graphic point of view: Cognitive difficulties and teaching practice. En E. Dubinsky & G. Harel (Eds.), *The concept of function: aspects of epistemology and pedagogy* (pp. 109-132). Washington, DC, EE. UU.: Mathematical Association of America.

Dauben, J. (1984). El desarrollo de la teoría de conjuntos cantoriana. En I. Grattan-Guinness (Ed.), *Del cálculo a la teoría de conjuntos, 1630-1910. Una introducción histórica* (pp. 235-282). Madrid, España: Alianza Editorial. (Versión original en inglés publicada en 1980).

Organizaciones y documentos:

UNESCO (1983). Anuario Estadístico. París: UNESCO.

Memorias de Reuniones y Simposia:

Cantoral, R. (1995). Matemática, matemática escolar y matemática educativa. En R. Farfán (Ed.), *Publicación de la Novena Reunión Centroamericana y del Caribe sobre Formación de Profesores e Inves-tigación en Matemática Educativa* (volumen 1, pp. 1-10). La Habana, Ministerio de Educación, Cuba: Cinvestav.

Tesis:

Flores, R. (1992). *Sobre la construcción del*

concepto de convergencia en relación al manejo heurístico de los criterios. Tesis de maestría no publicada, Cinvestav, México.

Referencias Electrónicas

Publicación periódica en línea:

Candela, A. (1999). Prácticas discursivas en el aula y calidad educativa. *Revista Mexicana de Investigación Educativa 4*(8), 273-298. Obtenido en junio 7, 2004, de http://www.comie.org.mx/revista/Pdfs/Carpeta8/8invest3.pdf

Documento en línea:

PISA (2003). *Aprender para el mundo de mañana: Resumen de resultados.* Obtenido en abril 4, 2005, de http://www.ince.mec.es/pub/pisa2003resumenocde.pdf

Artículos de Internet basados en una fuente impresa:

Parnafes, O. & Disessa, A. (2004). Relations between types of reasoning and computational representations. [Versión Electrónica]. *International Journal of Computers for Mathematical Learning 9*(3), 251-280.

Documento sin fecha ni autor identificado:

Si el autor de un documento no está identificado, comience la referencia con el título del documento:

Calculadoras y Educación (s.f.). Obtenido en enero 12, 2000, de http://www.agapema.com/period/dijo.htm

Documento disponible en la página web de una Universidad o Departamento:

Chou, L., McClintock, R., Moretti, F., & Nix, D. H. (1993). *Technology and education: New wine in new bottles: Choosing pasts and imagining educational futures.* Obtenido en agosto 24, 2000, del sitio web de Columbia University, Institute for Learning Technologies: http://www.ilt.columbia.edu/publications/papers/newwine1.html

Referencia de texto:

Las referencias de texto irán al final de la oración, frase o texto como se muestra en los siguientes ejemplos:

- Una fuente con dos autores: (Gubar & Hebin, 1980).
- Una fuente con más de tres autores: (Agard et al., 1990).
- Una referencia a varios trabajos, separarlas citas con punto y coma: (Gómez, 1992; Cordero, 1995; García, 1994).

Citas de Notas:

El formato para citar una fuente será en forma de referencia de texto. Es posible incluir notas al pie de página para proveer información adicional de una referencia textual o aclarar una idea del texto. Si se utilizan notas al pie de página, el autor deberá colocarlas en la parte inferior de la página de referencia. Si se utilizan notas finales, se colocarán al final del artículo siguiendo una numeración ascendente.

Información complementaria

1) No se devolverán los disquetes ni los artículos originales.
2) El editor se reserva el derecho de hacer algunas modificaciones necesarias para mantener el estilo de la publicación.
3) Los textos pueden ser publicados en otro órgano editorial previo permiso expreso, por escrito, y haciendo referencia explícita de la fuente.
4) Los autores recibirán gratuitamente 2 ejemplares del número en que se haya publicado su artículo.
5) No se realizarán pagos a los autores por los artículos que se publiquen en **Relime**.

Para mayores informes, puede visitar la página web : http://www.clame.org.mx/relime.htm o bien el correo electrónico :

relime@clame.org.mx

Suscripción a Relime

Datos personales

Nombre ______________________ ______________________ ______________________
nombres apellido paterno apellido materno

Dirección particular ____________________________ _____ ______________________
calle número colonia

__________________ __________________ __________________ ________________
ciudad, estado, código postal país

Tel. ____________ Fax____________ E-mail ______________________________________

Datos de su institución

Escuela o Institución donde trabaja ____

Dirección ________________________________ ________ ________________________
calle número colonia

__________________ _________________ ___________________ _________________
ciudad, estado, código postal país

Teléfono particular _____________________ Fax __________________________

Datos sobre sus actividades

Si es estudiante,
nombre de la licenciatura, ingeniería o posgrado que estudia

____

Institución donde estudia ____Semestre _______

Si es profesor,
nivel que imparte____________________
Nombre de la(s) asignatura(s) que imparte ___________________ _____________________

________________ _________________Nivel educativo donde trabaja__________________

Si es investigador,
Institución a que pertenece ____

Línea de investigación que trabaja ____

Nivel educativo más frecuente donde hace sus investigaciones _______________________

Costo del volumen 9 (tres numeros)
Suscripción institucional: 120 dls
Suscripción individual: 50 dls
Suscripción para miembros de Clame: 20 dls
Costo por un numero: 15 dls

Estos precios no incluyen costos por envio ni cargo por depósito en cuenta bancaria. Para cotización actual, favor de pedir informes a suscripción-relime@clame.org.mx

Revista Latinoamericana de Investigación en Matemática Educativa Número Especial.

Se imprimió en los talleres de
Impresora MMC
5a Cerrada de Barranca, Manzana 4, Lote 5
Col. El manto, Iztapalapa. México, DF.

Septiembre de 2006.
Tiraje: 2000 ejemplares más sobrantes

ISSN 1665-2436