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TOWARDS A CULTURAL THEORY OF LEARNING*

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In this paper I sketch a theory of teaching and learning that takes its inspiration from some anthropological and historico-cultural schools of knowledge –the Theory of Knowledge Objectification (TKO). The TKO rests on five main interrelated constructs. The first construct deals with the psychological concept of thinking. Drawing on this concept, the other constructs serve to formulate the problem of learning in a way that does not commit the TKO with rationalist views of cognition and social interaction. The TKO posits the problem of learning as the progressive acquisition of cultural forms of reflection that are objectified as the student engages in joint social activity. Learning, it is argued, arises in the course of sensuous mediated cultural praxes embedded in historically formed epistemes and ontologies.

INTRODUCTION: THEORIES OF TEACHING AND LEARNING

Theories of teaching and learning differ from each other mainly in their conceptions about the learner, the content to be learned, and how learning actually occurs. Most contemporary theories adopt the view according to which the student constructs his or her own knowledge. Although in their account of learning these theories do not exclude the role of the social, often they reduce the social to a kind of external environment to which the cognitive activity of the student has to adapt. Much in vein with Piaget's genetic epistemology, these adaptations are seen as universal regulators with no ties with the individual's sociocultural context (see e.g. Piaget & Garcia, 1989, p. 267). In these theories, the idea of the universal mechanisms of knowledge formation –namely the allegedly logical-mathematical structures of thinking– appear as the warrants of the supposedly universal patterns of conceptual development.

However, at the epistemological level, these theories have been criticized, in part for their commitment to a rationalist view of knowing and cognition (Buck-Morss, 1975; Campbell, 2002; Walkerdine, 1988; Wartofsky, 1983). These theories rest indeed on the idea of an intrinsic rational auto-sustained individual maturing as she interprets and refines the feedback that the environment sends to her. As a result, the idea of the learner that these theories convey is the idea of a self-regulated individual acting in a more and more autonomous form, an idea shaped by the Western concept of the scientist.

At the ontological level, other scholars, working within the framework of Realism, find the idea of universal adaptations insufficient to ensure the convergence between

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the individuals' personal conceptual constructions and a reality that precedes all cognitive activity (see e.g. Thom in Piattelli-Palmarini, 1982).

Without denying the existence of a real world, in an interesting move plainly in accordance with Kant's view of human reason, von Glasersfeld (1995) suggested to give up the idea that the individual's conceptual constructions correspond to the objects of the real word. He put forward a much more modest idea of cognition –one in which our ideas are merely *viable constructs.* According with this theory, the so-called *Radical Constructivism* (RC), these subjective constructs are ready to be changed if compelling evidence suggests so. For many, this move is unconvincing. One the one hand, RC cannot avoid the problem of solipsism (Lerman, 1996). On the other hand, to salvage its underlying extremist subjective epistemology, RC gives up ontology and posits the subjective experiential realm as the limits of reason and knowledge.

At the educational level, Radical Constructivism has also been criticized for failing to account for the dissymmetric distribution of knowledge in the classroom. In a recent plenary lecture, Brousseau (2004) argued that "En didactique, le constructivisme radical est une absurdité". What Brousseau finds absurd in the radical constructivist position is not the claim that legitimate knowledge can only be the result of the individual's own achievement and deeds. On this point, Brousseau, who elaborated his *Theory of Situations* as a response to the general framework of an uncritical learning (learning without meaning), endorses some central tenets of Piaget's constructivism. What he finds erroneous is the idea that the students' constructions necessarily lead to the standard mathematical knowledge (*le savoir savant*). As Brousseau could observe again and again in the classrooms of the Michelet School, the students' subjective conceptual constructs require of an external perspective to, among other things, institutionalize the knowledge arising from classroom mathematical activity. The students cannot be aware of the cultural epistemic status of, say, a method arising as the result of their enquiring activity or, as Brousseau puts the matter, the students may not know that they know. The teacher hence has to highlight those reasonings and methods valued by the mathematicians' community.

These few comments on some current ideas about the learner and how learning occurs provide an idea of some of the theoretical differences in current perspectives in mathematics education. Of course, the differences between theories are subtler as hinted here. My interest is not to delve into these differences. Rather my interest is to recall some of the presuppositions that appear as the focal points from where theoretical differences arise. In the rest of this paper I present some elements of a theory of teaching and learning that takes its inspiration from some anthropological and historico-cultural schools of knowledge. This theory –*The Theory of Knowledge Objectification*– relies on a non-rationalist epistemology and ontology which give rise, on the one hand, to an anthropological conception of thinking, and on the other, to an essentially social conception of learning.

1. A NON-MENTALIST CONCEPTION OF THINKING

1.1 Thinking as a mediated *praxis cogitans*

Typically, thinking is understood as a kind of interior life, a series of mental processes on ideas carried out by the individual. This conception of thinking, as "mental activity" (de Vega, 1986, p. 439), comes from Saint Augustine's interpretation of Greek philosophy at the end of the fourth century, an interpretation that brought about, in particular, a transformation in the original meaning of the Greek term *eidos*. While Homer, among others, used the term *eidos* in the sense of something external rather than mental—"that which one sees," for example, the figure, form or appearance—for Saint Augustine, *eidos* refers to something situated *inside of the individual*. Influenced by this transformation, seventeenth century rationalists such as Descartes and Leibniz believed that mathematics could be practiced even with one's eyes closed, given that the mind does not need the help of the senses or of experience to reach mathematical truths. As Leibniz put the matter, the principles that we need to understand objects or see their properties, the internal rules of reason, are "interior principles" that is, they are within our interior (Leibniz, 1966, pp. 34-37). Anthropologists such as Geertz have demonstrated the limitations of the conceptualization of ideas as "things in the mind" or of thinking as an exclusively intracerebral process. Geertz (1973, p. 76) claims that "The accepted view that mental functioning is essentially an intracerebral process, which can only be secondarily assisted or amplified by the various artificial devices which that process has enabled man to invent, appears to be quite wrong." He argues that "the human brain is thoroughly dependent upon cultural resources for its very operation; and those resources are, consequently, not adjuncts to, but constituents of, mental activity. (Geertz, *ibid*.).

The conception of thinking as a kind of interior life has had a great influence in the investigation of cognition in mathematics education. Written questionnaires, interviews, and drawing exercises have often been used to get a glimpse of what is going *in* the head. To avoid the pitfalls of this mentalistic approach, some theories have simply discarded any psychological considerations. They have made "l'économie du sujet."

The Theory of Knowledge Objectification (TKO) takes off from a non-mentalist position on thinking and intellectual activity. This theory suggests that thinking is a *praxis cogitans*, that is, a social practice (Wartofsky, 1979). To be more precise, thinking is considered to be *a mediated reflection on the world in accordance with the form or mode of the activity of individuals*.

The mediating nature of thinking refers to the role, in the Vygotskian sense, played by artefacts (objects, instruments, sign systems, etc.) in carrying out social practice. Artefacts are not merely aids to thinking (as cognitive psychology would have it) nor simple amplifiers, but rather constitutive and consubstantial parts of thinking. We think with and through cultural artefacts, so that there is an external region which, to

paraphrase Voloshinov (1973), we will call the *zone of the artefact*. It is within this zone that cultural subjectivity and objectivity mutually overlap and where thinking finds its space to act and the mind extends itself beyond the skin (Wertsch, 1991).

The reflexive nature of thinking means that the individual's thinking is neither the simple assimilation of an external reality (as the Empiricists and Behaviorists propose) nor an *ex nihilo* construction (as certain constructivist schools would have it). Thinking is a *re-flection*, that is, a dialectical movement between a historically and culturally constituted reality and an individual who refracts it (as well as modifies it) according to his/her own subjective interpretations, actions and feelings.

One of the roles of culture is to suggest to students ways of perceiving reality and its phenomena, literally, ways of setting one's sights (*manières de viser*), as Merleau-Ponty (1945) would say, or ways of intuiting, as Husserl (1931) might have it. In a more general fashion, the *re-flexivity* of thinking, from the phylogenetic point of view, consists in individuals giving rise to thinking and to the objects that thinking creates. However, at the same time, from the ontogenetic point of view, the individuals' thinking is, from the outset, subsumed by their cultural reality and by the historically formed concepts that they encounter in their environment. This is why, we originate thinking, but at the same time become subsumed by it. (Eagleton, 1997, p. 12)

1.2 The anthropological dimension of thinking

In the preceding section, it was said that thinking should be considered as a mediated *re-flection* of the world, in keeping with the form or mode of the activity of individuals. What this means is that the way in which we come to think about and know objects of knowledge is framed by cultural meanings situated beyond the very content of the activities in whose interior the act of thinking itself occurs. These cultural meanings act as mediating links between individual consciousness and objective cultural reality and they make themselves into prerequisites and conditions for individual mental activity (Ilyenkov, 1977, p. 95). These cultural meanings suggest courses of action to our cognitive activity and give it a certain *form*. It is for this reason that thinking is not something that we simply begin to do in a more or less unpredictable way and during which we suddenly come across a good idea. Even though it is true that practical sensual activity, mediated by artefacts, enters into the thinking process, in its very content, the way in which this occurs is subject to the cultural meanings in which the activity is being maintained. Here is an example. The difference between the thinking of a Babylonian scribe and that of a Greek geometer cannot be reduced only to the kinds of problems with which they were respectively occupied, or to the artefacts they used to think mathematically, or the fact that the former was reflecting in a context tied to political and economic administration, whereas the latter was thinking within an aristocratic and philosophical context. The difference between the thinking of the Babylonian mathematician and that of the Greek one has to do with the fact that each one of these forms of thinking is

underpinned by a particular *symbolic superstructure*. This symbolic superstructure, which elsewhere we have called a *Semiotic System of Cultural Signification* (Radford 2003a), includes cultural conceptions surrounding mathematical objects (their nature, their way of existing, their relation to the concrete world, etc.) and social patterns of meaning production. The thinking of the Babylonian scribe is framed by a realist pragmatism where mathematical objects such as "rectangle," "square," and so forth –objects which the Greek geometer of Euclid's time conceptualized in terms of Platonic forms or Aristotelian abstractions – acquire their meaning.

In their interaction with activities (their objects, actions, division of labour, etc.) and with the technology of semiotic mediation (the zone of the artefact), the *Semiotic Systems of Cultural Signification* give rise, on the one hand, to forms or modes of activities, and, on the other hand, to specific modes of knowing or *epistemes* (Foucault, 1966). While the first interaction gives rise to the particular ways in which activities are carried out at a certain historical moment, the second interaction gives rise to specific modes of knowing which allow for the identification of "interesting" situations or problems and which demarcate the methods, reasoning, evidence, etc. that will be considered culturally valid[1].

From our perspective, cultural diversity in the form of human activity explains the diversity of forms that mathematical activity takes on, something which is demonstrated to us by history. Rather than seeing these historical forms as "primitive" or "imperfect" versions of a kind of thinking that is marching towards a perfected form as represented by current mathematical thought (ethnocentrism), the anthropological dimension of the theory of objectification considers these forms as belonging to human activity and thus resists privileging western rationalism as rationalism *par excellence*.

The manner in which the Babylonian scribe, the Greek geometer and the Renaissance abacist end up thinking about and knowing objects of knowledge, the way in which they approach their problems and consider them to be solved, all are framed by the very mode of the activity and the corresponding cultural episteme (Radford, 1997, 2003a, 2003b).

2. THE EPISTEMOLOGICAL AND ONTOLOGICAL BASES OF THE THEORY OF KNOWLEDGE OBJECTIFICATION

Any didactic theory, at one moment or another (unless it voluntarily wants to confine itself to a kind of naïve position), must clarify its ontological and epistemological position. The *ontological* position consists in specifying the sense in which the theory approaches the question of the nature of conceptual objects (in our case, the nature of mathematical objects, their forms of existence, etc.). The *epistemological* position consists in specifying the way in which, according to the theory, these objects can (or cannot) end up being known.

Contemporary didactic theories that start from an application of mathematics, gradually adopt—even if it is not mentioned explicitly— a realist ontology and approach the epistemological problem in terms of abstractions. Naturally, the situation is not that simple, as Kant himself recognized. As for Realism –which, in an important way, is the Platonist version of the instrumental rationalism (Weber, 1992) which emerged during the Renaissance– the existence of mathematical objects precedes and is independent from the activity of individuals. Like the Platonist, the Realist believes that mathematical objects exist independently of time and culture. The difference is that, while Platonic objects do not mix with the world of mortals, the objects of the Realist govern our world. According to realist ontology, this explains the miracle that is the applicability of mathematics to our phenomenal world (Colyvan, 2001). Naturally, in order to achieve this, Realism makes a leap of faith that consists in believing that the ascent from abstraction to objects is certainly possible. The faith which Plato placed in reasoned social discourse (*logos*) and which Descartes placed in cogitating with oneself are subjected to scientific experimentation by Realism.

The ontological and epistemological position of the theory of objectification moves away from Platonist and realist ontologies and from the Platonists' and Realists' conception of mathematical objects as eternal objects preceding the activity of individuals. By distancing itself from an idealist ontology, the theory also distances itself from the idea that objects are the product of a mind that works folded in onto itself or according to the laws of logic (the Rationalist Ontology). The theory of objectification suggests that mathematical objects are historically generated during the course of the mathematical activity of individuals. More precisely, mathematical objects *are fixed patterns of reflexive activity (in the explicit sense mentioned previously) incrusted in the ever-changing world of social practice mediated by artefacts*.

The conceptual object "circle", for example, is a fixed pattern of activity whose origins cannot be found in the intellectual contemplation of the round objects which the first individuals would have encountered in their surroundings, but rather must be found in the sensual activity that led said individuals to notice the emergent object:

> People could see the sun as round only because they rounded clay with their hands. With their hands they shaped stone, sharpened its borders, gave it facets. (Mikhailov, 1980, p. 199)

This sensual experience of labour has remained fixed in language which encapsulates original meanings, such that

> the meaning of the words "border", "facet", "line" does not come from abstracting the general external features of things in the process of contemplation (Mikhailov, ibid.)

but rather comes from the activity of labour that has been taking place since the origins of humanity. Far from surrendering itself completely to our senses, our

relationship with nature and the world is filtered through conceptual categories and cultural significations which make it so that

> man could contemplate nature only through the prism of all the social work-skills that had been accumulated by his predecessors. (Mikhailov, ibid.)

3. LEARNING AS THE CULTURAL OBJECTIFICATION OF KNOWLEDGE

In the previous sections we have seen how human activity, from the phylogenetic point of view, can generate conceptual objects, which in turn are transformed as a result of the activities themselves. From the ontogenetic point of view, the central problem is to explain how acquisition of the knowledge deposited in a culture can be achieved: this is a fundamental problem of mathematics education in particular and of learning in general.

As mentioned in the Introduction, classical theories of mathematical education posit the problem in terms of a construction or re-construction of knowledge on the part of the student. The idea of the construction of knowledge originates with the epistemology elaborated by Kant in the eighteenth century. For Kant, the individual is not only an introspective thinker whose mental activity, if it is well carried out, will bring him mathematical truths as upheld by the rationalists (Descartes, Leibniz, etc.); nor is he only a passive individual who receives sensory information in order to formulate ideas, as proposed by the Empiricists (Hume, Locke, etc.). For Kant, the thinker is a being in action: the individual is craftsman of his/her own thinking (Radford, 2006a). Through these ideas Kant expressed, in a coherent and explicit way, the epistemological change that had been gradually taking place since the appearance of manufacturing and the emergence of capitalism in the Renaissance and that Arendt (1958) summarizes in the following way: the modern era is marked by a displacement in the conception of the meaning of knowledge; the central problem of knowledge lies in a movement that goes from 'the what' (the object of knowledge) to 'the how' (the process), in such a way that, unlike medieval man, modern man can only understand that which he himself has made.

According to the theory of objectification, learning does not consist in constructing or reconstructing a piece of knowledge. It is a matter of endowing the conceptual objects that the student finds in his/her culture with meaning. The acquisition of knowledge is a process of active elaboration of meanings. It is what we will later call a process of objectification. For the moment, we need to discuss two important sources for the elaboration of meanings that underlie the acquisition of knowledge.

3.1 The knowledge deposited in artefacts

One of the sources of the acquisition of knowledge results from our contact with the material world, the world of cultural artefacts which surrounds us (objects, instruments, etc.) and in which is found the historically deposited knowledge from the cognitive activity of passed generations. Although it is true that some animals are

able to use artefacts, nevertheless, for animals, artefacts do not end up acquiring a durable meaning. The wooden stick that a chimpanzee uses in order to reach a piece of fruit looses its meaning after the action has been executed (Köhler, 1951). It is for this reason that animals do not preserve artefacts. Furthermore—and this is a fundamental element of human cognition—unlike animals, the human being is profoundly altered by the artefact: by making contact with it the human being restructures his/her movements (Baudrillard, 1968) and new motor and intellectual skills are formed such as anticipation, memory and perception (Vygotsky and Luria, 1994).

The world of artefacts appears, then, to be an important source for the process of learning, but it is not the only one. Objects cannot make clear the historical intelligence that is imbedded in them. This requires that they be used in activities as well as in contact with other people who know how to "read" this intelligence and help us to acquire it. Symbolic-algebraic language would otherwise be reduced to a group of hieroglyphics. The intelligence that said language carries would not be noticed without the social activity that takes place in the school. It is this social dimension which constitutes, for the theory of objectification, the second essential source for learning.

3. 2 Social Interaction

Even though the importance of the social dimension has been underlined by a great number of recent studies on classroom interaction, there are subtle differences with regards to its cognitive contribution (Yackel and Cobb, 1996; Sierpinska, 1996; Steinbring, Bartolini Bussi and Sierpinska, 1998). Often, interaction is considered as a negotiation of meanings or as a simple environment that offers the stimuli of adaptation that are required for students' cognitive development. The problem is that the classroom is not a merely material space where the students find an environment to adapt themselves; it is not only a matter of "external" conditions to which the subject must accommodate his/her activity. The crucial point is that the classroom is a symbolic space; it is a space where conceptual objects, activities and the material means that mediate them are endowed with scientific, aesthetic, ethical values, etc. that end up affecting the actions that individuals carry out and the reflections that these actions necessitate. As was mentioned in the first part of this article, the actions that individuals carry out are submerged in cultural modes of activity. It is for this reason that the classroom cannot be viewed as an enclosed space, folded over against itself, where knowledge rules are negotiated. In fact, these rules have a whole cultural history behind them and therefore pre-exist the interaction that takes place in the classroom.

According to the perspective that we are suggesting, interaction plays a different role. Rather that performing a merely adaptive function—a catalyzing or facilitating one—according to the theoretical perspective that we are sketching, interaction is consubstantial to learning.

Therefore, we see that there are elements that play a basic role in the acquisition of knowledge and that these are the material world and the social dimension. The allocation of meaning that rests on these dimensions has a profound psychological importance inasmuch as it is both an awareness of cultural concepts as well as the process of development of the specific capacities of the individual. It is for this reason that, according to our perspective, learning is not merely appropriating something or assimilating something; rather, it is the very process by which our human capacities are formed.

3.3 Learning activity

A central element of the concept of activity is its objective (Leont'ev, 1978). Even though the objective may be clear for the teacher, generally speaking, this is not necessarily the case for the students. If the objective were to be clear to them, then there would be nothing left for them to learn. Within the didactic project in the class, the teacher proposes a series of mathematical problems to the students so that a given objective can be achieved. Solving these problems becomes an end that directs the actions of the students. However, from the perspective of the Theory of Knowledge Objectification, doing mathematics cannot be reduced to solving problems. Without devaluing the role of problems in knowledge formation (see, for example, Bachelard, 1986), for us, problem solving is not the end but rather one of the means for achieving the type of *praxis cogitans* or cultural reflection that we call mathematical thinking. So that, behind the objective of the lesson, there lies a greater and more important objective—the generally held objective for the teaching and learning of mathematics—namely, the elaboration on the part of the student of a reflection defined as a common and active relationship with his/her cultural-historical reality.

In other words, learning mathematics is not simply learning *to do* mathematics (problem solving), but rather it is learning *to be* in mathematics. The difference between doing and being is immense and, as we shall see later, it has important consequences not only for the designing of activities but also for the organization of the class itself and the roles that students and teachers play within it.

3.4 The objectification of knowledge

The greatest objective of the teaching of mathematics is that the student learn to reflect according to certain historically constituted cultural forms of thinking that distinguish it from other types of reflection (for example, those of a literary or musical kind) inasmuch as in mathematical reflection, the individual's relationship with the world emphasizes ideas regarding form, number, measurement, time, space, etc. It is this emphasis which distinguishes mathematical thinking from other kinds of thinking.

The theory of objectification nevertheless does not see learning as a simple imitation or participation consistent with a pre-established practice, but rather sees it as the fusion between a subjectivity which seeks to perceive the cultural modes of reflecting

and the conceptual objects such a reflection is about. In order to get to know objects and products of cultural development, it is "necessary to carry out a determined activity around them, that is to say, a kind of activity that produces its essential characteristics, embodied, 'accumulated' in said objects." (Leontiev, 1968, p. 21).

Teaching consists of generating and keeping in movement contextual activities which are situated in space and time and which are heading toward a fixed pattern of reflexive activity incrusted in the culture. This movement, which could be expressed as the movement from process to object (Sfard, 1991; Gray and Tall, 1994) has three essential characteristics. First, the object is not a monolithic or homogenous object. It is an object made up of layers of generality. Second, from the epistemological point of view, these layers will be more or less general depending on the characteristics of the cultural meanings of the fixed pattern of activity in question (for example, the kinaesthetic movement that forms a circle; the symbolic formula that expresses it as a group of points at an equal distance from its centre, etc.). Third, from the cognitive point of view, the layers of generality are noticed in a progressive way by the student. The learning process consists in finding out how to take note of or how to perceive these layers of generality. Just as learning is a re-flection, to learn presupposes a dialectical process between subject and object mediated by culture; a process during which, through his/her actions (sensory or intellectual) the subject takes note of or becomes aware of the object.

Objectification is precisely this social process of progressively becoming aware of the Homeric *eidos*, that is, of something in front of us—a figure, a form—something whose generality we gradually take note of and at the same time endow with meaning. It is this act of noticing that unveils itself through counting and signalling gestures. It is the noticing of something that reveals itself in the emerging intention projected onto the sign or in the kinaesthetic movement which mediates the artefact in the course of practical sensory activity, something liable to become a reproducible action whose meaning points toward this fixed eidetic pattern of actions incrusted in the culture which is the object itself.

4. THE CLASSROOM AS A LEARNING COMMUNITY

4.1 Being-with-others

The classroom is the social space in which the student elaborates this reflection, defined as a common and active relation with his/her historical-cultural reality. It is here that the encounter between the subject and the object of knowledge occurs. The objectification that allows for this encounter is not an individual process but a social one. The sociability of the process, nevertheless, cannot be understood as a simple business interaction during which each player invests some capital in the hopes of ending up with more of it, or as a kind of game between adversaries (as in the Theory of Situations). Here, sociability means the process of the formation of consciousness

which Leontiev characterized as *co-sapientia*, that is to say, as knowing in common or knowing-with others.

Naturally, these ideas imply a re-conceptualization of the student and his/her role in the act of learning. Insofar as current theories in mathematics education draw on the concept of the individual as formulated by Kant and other Enlightenment philosophers, education justifies itself by guaranteeing the formation of an autonomous subject (understood in the sense of being able to do something for oneself without the help of others). Autonomy is, in effect, a central theme of modern education that has served as a basis for the theorizing of socio-constructivism (see, for example, Yackel and Cobb, 1996) and the Theory of Situations (Brousseau, 1986; Brousseau and Gibel, 2005, p. 22). The rationalism that weighs on this concept of autonomy comes from its alliance with another key Kantian concept: that of liberty. There can be no autonomy without liberty and, for Kant, liberty means the convenient use of Reason according to its own principles so that "it is through reason that we get an insight into principles" (Kant, 1900, p. 34).

Since the Enlightenment did not put forward the possibility of there being a multiplicity of reasons, but rather postulated that western reason was The Reason, community coexistence implies respect for a duty which, in the end, is nothing but a manifestation of that universal reason, whose epitome is mathematics. It was this supposed universality of reason that led Kant to fuse together the ethical, political and epistemological dimension and to affirm that "to do something for the sake of duty means obeying reason." (Kant, 1900, p. 37).

For the Theory of Knowledge Objectification, classroom functioning and the role of the teacher are not limited to trying to achieve autonomy. It is more important to learn how to live in the community that is a classroom (in its fullest sense), to learn to interact with others, to open oneself up to understanding other voices and other consciousnesses, in brief, *to be-with-others* (Radford, in press).

Just as "the social is irreducible to individuals, however numerous they might be" (Todorov, in Bakhtine, 1984, p. 19), sociability in the classroom means a coming together through links and relations that are prerequisites for that kind of reflection that we mentioned earlier, defined as common and active and which is elaborated by the student along with his/her historical-cultural reality. This sociability not only leaves its mark on the conceptual content being pursued but is furthermore an integral part of it.

The intrinsic social nature of knowledge and mathematical thinking has brought us then to conceiving of the classroom as a learning community whose functioning is oriented toward the objectification of knowledge. Its members work in such a way that: the community allows for the personal achievement of each individual; each member of the community has his/her place; each member is respected; each member respects others and the values of the community; the community is flexible in its ideas and its forms of expression; the community opens up space for subversion in

order to insure: modification, change and its transformation. Being a member of the community is not something that comes as a matter of course. In order to be a community member, students are encouraged to: share in the objectives of the community; involve themselves in the classroom activities; communicate with others. The abovementioned guidelines are not simply codes of conduct. On the contrary, they are indexes of forms of being in mathematics (and, as a consequence, of knowing mathematics) in the strictest sense of the term.

CONCLUDING REMARKS

Some theories in mathematics education have intentionally excluded the psychological aspects of learning and have occupied themselves with mathematical situations that can favour the emergence of precise mathematical reasoning. Such is the case for the Theory of Situations. On the contrary, other theories have fixated themselves on the mechanisms of the negotiation of meaning in the classroom and the way in which this negotiation explains the construction of representations that the student makes of the world. Such is the case of socio-constructivism. The intellectual debt that the Theory of Knowledge Objectification owes to these two theories is immense and our reference to them should not be seen in a negative light. These theories are sustained by fundamental principles and clear modes of operation that confer upon them an impeccable solidity. Nevertheless, the TKO takes off from other principles. On the one hand, it bases itself on the idea that the psychological dimension of learning has to be an object of study in mathematics education. On the other hand, it suggests that the meanings circulating in the classroom cannot be confined to the interactive dimension that takes place in the class itself; rather, they have to be conceptualized according to the context of the historical-cultural dimension. Therefore, the Theory of Knowledge Objectification proposes a didactic anchored on principles according to which learning is viewed as a social activity (*praxis cogitans*) deeply rooted in a cultural tradition that precedes it. Its fundamental principles are articulated according to five interrelated concepts[2]. The first of these is a concept of a psychological order: the concept of *thinking*, elaborated in non-mentalist terms. The second concept of the theory is of a socio-cultural order. This is the concept of learning. The third concept of the theory is of an epistemological nature and deals with those super-epistemic aspects that frame learning in the form of *semiotic systems of cultural signification* –cultural systems that "naturalize" the ways that one questions and investigates the world. The aforementioned concepts come to be completed by a fourth concept of an ontological nature—that of mathematical objects, which we have defined as *fixed patterns of reflexive activity incrusted in the ever-changing world of social practice mediated by artefacts*. To render the theory operational in its ontogentic aspect, it was necessary to introduce a fifth concept of a semiotic-cognitive nature—that of objectification, or a subjective awareness of the cultural object. In this context, and in light of the previous fundamental concepts, learning is defined as the social process of

objectification of those external patterns of action fixed in the culture. Although space constraints did not allow me to illustrate here the students' processes of objectification, these processes have been study in detail in my classroom research (see e.g. Radford, 2003c, 2006b; Radford et al. 2004, 2006; Sabena et al. 2005).

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End Notes

1. Henceforth, it is not only the action which constitutes the schema of the concept (Piaget)—or its seal or emblem (Kant)—but also the meaning of the action in a precise moment of the socio-cultural activity within which the action occurs (Radford, 2005).

2. I do not have room here to state the way in which these principles frame the fundamental didactic problems of the theory. I can only mention that the problem of learning, as a *practical* problem, is one of the central research problems of the theory (see the references to our classroom-based work). This central problem is considered as deeply rooted in the problem of the student's formation of his or her consciousness –something that happens as the student objectifies the conceptual content that orients the activity and that the theory posits as something happening in the interweaving of the subjective, social and cultural dimensions of knowing and doing.

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