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ORIGINAL ARTICLE

Iconicity and contraction: a semiotic investigation of forms of algebraic generalizations of patterns in different contexts

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Abstract The aim of this paper is to investigate the progressive manner in which students gain fluency with cultural algebraic modes of reflection and action in pattern generalizing tasks. The first section contains a short discussion of some epistemological aspects of generalization. Drawing on this section, a definition of algebraic generalization of patterns is suggested. This definition is used in the subsequent sections to distinguish between algebraic and arithmetic generalizations and some elementary naïve forms of induction to which students often resort to solve pattern problems. The rest of the paper discusses the implementation of a teaching sequence in a Grade 7 class and focuses on the social, sign-mediated processes of objectification through which the students reached stable forms of algebraic reflection. The semiotic analysis puts into evidence two central processes of objectification—iconicity and contraction.

Keywords Algebraic thinking · Contraction · Generalization · Iconicity · Objectification · Meaning · Semiotics · Signs

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> Oh my God! How are you going to find the 100th Term?
> Renée, a Grade 7 student

1 The same and the different

Since concepts neither derive from logical rules—as suggested by rationalism—nor from external impressions—as suggested by empiricism—the origin of all concepts, Vygotsky argued, is to be found in generalizations (Vygotsky, 1986). Without being able to make generalizations, we would be reduced to living in a world of mere particulars *a, b, c*,…. Everything would be different from everything else. Knowledge would be reduced to a perpetual $a \neq b$. To assert that *a* is equal to *b*—or that *a* and *b* are *analogous* as the Greek etymology suggests (Fried, 2006), like when we say that two particular trees are *equal* despite some differences—means to select some features of *a* and *b* and to dismiss some others. This cognitive act entails the formation of a concept—a generalized entity—that does not fully coincide with any of its instances.

The crucial point in an account of generalization is, hence, to account for the manner in which we come to notice the *same* and the *different*. At any rate, this is by no means the result of a contemplative act. As Kant put it:

> I see a fir, a willow, and a linden. In firstly comparing these objects, I notice that they are different from one another in respect of trunk, branches, leaves, the like; further, however, I reflect only on what they have in common … and abstract from their size, shape, and so forth; thus I gain a concept of tree. (Kant, 1974, p. 100)

Our ability to notice differences in things is indeed one of the basic components of cognition. Without it, we would be unable to sort the amazing amount of sensorial stimuli that we receive from the exterior, and the world in front of us would be reduced to an amorphous visual, tactile and aural mass. However, as many of Kant's commentators have pointed out, things are a good deal more complicated than Kant himself suggested. Noticing the differences and similarities that lead to the concept of a tree or any other concept is something that occurs in social activities subsumed in cultural traditions conveying ideas about the *same* and the *different*. Some cultures make finer or different categorizations of trees, plants and colors than others.

Now, since we come into culture in a progressive way, generalization is both something ubiquitous (Mason, 1996) and something that has to be learned. Mathematical generalizations are perhaps the clearest example. As current research conducted in fields like Ethnomathematics and the history of mathematics suggests, there are varied possible ways of dealing with the basic underlying ideas of mathematics—e.g., quantity, shape, space, and time (Crump, 1990; D'Ambrosio, 2006; Gell, 1992; Høyrup, 2007; Perret-Clermont, 2005; Radford, 2008). To reflect mathematically requires one to see particulars as something *general* in a cultural sense.

2 Algebraic pattern generalizations

Despite what the previous section may suggest, my interest in mathematical generalization did not arise out of theoretical concerns. It was the other way around. It arose in the course of a longitudinal classroom-based research that I began conducting in the 1990s. In the classrooms I was working with, the generalization of patterns was (and still is) used as a route to algebra. The main idea was that a certain experience with the numerical exploration of patterns would pave the road to algebraic thinking. Algebra, indeed, was supposed to start with the students' first use of notations.

The use of notations, however, does not seem to be the best way to understand the emergence of algebraic thinking. Algebraic thinking is not about using letters but about thinking in certain distinctive ways. As I have argued elsewhere (Radford, 2006a), Chinese mathematicians thought in algebraic ways without using letters (Martzloff, 1997) and Euclid used letters without thinking algebraically (Unguru, 1975).

This epistemic stance towards algebraic thinking leads us to the following question. If the emergence of algebraic thinking is not characterized by the use of notations, what then is the difference between algebraic and arithmetic pattern generalizations? To answer this question it is worth noting that what we now term algebra emerged as a sophisticated way of calculating on *indeterminate* quantities.[1]

The dividing line between the arithmetic and algebraic generalization of patterns should hence be located in differences in what is calculable within one domain as opposed to the other. While in both domains some generalizations do certainly occur, in algebra, a generalization will lead to results that cannot be reached within the arithmetic domain.

In this same line of thought, in a previous paper (Radford, 2006a), I suggested the following definition. Generalizing a pattern *algebraically* rests on the capability of *grasping* a commonality noticed on some particulars (say p_1, p_2, p_3,…,p_k); extending or generalizing this commonality to all subsequent terms (p_{k+1}, p_{k+2}, p_{k+3}, …), and being able to use the commonality to provide a direct *expression* of any term of the sequence.

There are different aspects involved in this definition. First, a local commonality, say C, is noticed in a few members of the sequence, S. As mentioned previously, this step requires that one make a choice between what counts as the same and the different. Second, this commonality C is then *generalized* to all the terms of the sequence. While the generalized commonality C was what Peirce called an *abduction*—i.e. "a general prediction" (Peirce, 1931–1958, CP 2.270), hence something only plausible—in the last part of the generalization process C becomes the *warrant* to *deduce* expressions of elements of the sequence that remain beyond the perceptual field. Direct expression of the terms of the sequence requires the elaboration of a rule—more precisely a *schema* in Kant's terms (Radford, 2005)—based on indeterminate quantities, in this case *variables*.

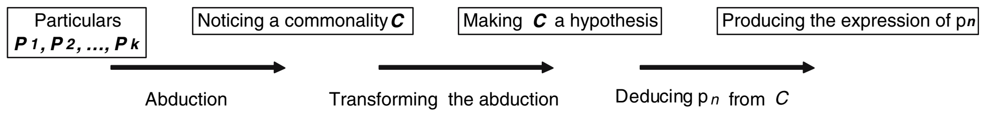
Figure A summarizes the architecture of an algebraic generalization of patterns.

Let us give a short example of algebraic pattern generalization here by referring to a regular Grade 8 mathematics lesson. In this lesson, the students dealt with the sequence shown in Fig. B. After continuing the sequence up to Fig. 5 and calculating the number of circles in Figs. 25 and 100, they had to find a formula for the number of circles in Fig. *n*.

The students worked in small groups and, in one group, a student noticed that the number of circles could be calculated by adding the number of the figure twice and then adding one. To explain her idea to her group-mates, she referred to the first three figures of the sequence: "You would do 1 plus 1, plus 1; 2 plus 2, plus 1; 3 plus 3, plus 1…" Then she concluded: "You add [the number of] the

[1] These calculations were carried out in the context of word-problem solving (concerning Babylonian algebra, see e.g. Høyrup 2002) and pattern investigations (concerning Greek mathematics, see e.g. Manitius 1888, Ver Eecke 1926, and a summary in Radford 2001).

Fig. A The architecture of algebraic pattern generalizations



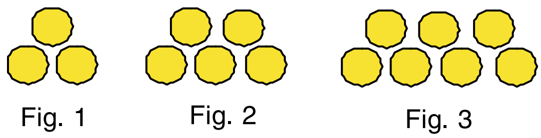


Fig. B A sequence used in a Grade 8 mathematics lesson

figure by itself […], then after this, plus 1", leading the group to the formula $(n + n) + 1$.

We see here that the students inferred a commonality C from a few particular cases. Then, this commonality was (implicitly) generalized to the rest of the terms of the sequence.[2] Next, the *abduction* C became an *hypothesis* and then the rule was calculated. Thus, under the assumption that C is true, the formula $(n + n) + 1$ has to be true, too.

The previous definition of algebraic pattern generalization helps us to distinguish it from other strategies that students often use to deal with patterns. In particular, as we shall see in the next section, it is possible to distinguish between algebraic and arithmetic generalizations of patterns.

3 Arithmetic pattern generalizations

Ferdinand Rivera presented some results at the PME 2006 Conference (Rivera, 2006) from a study about pattern generalization with Grade 6 students (11 years old). In the case in question, the students were presented with a slightly modified version of the sequence shown in Fig. A. The terms started with one circle and increased by two circles. The students had to write a message to an imaginary Grade 6 student clearly explaining what s/he must do in order to find out how many circles there were in any given figure of the sequence.

One of the students noticed that the number of circles in the few given terms increased by 2. The student generalized this commonality to the rest of the terms, but failed at providing a direct rule allowing one to calculate the number of circles in any figure. The student said: "You start at one and keep adding two until you get the right number of circles in all".

Although there is a generalization in this procedure, according to our definition, it is not of an algebraic nature. I will call a pattern generalization of this type an arithmetic one.

Algebraic and arithmetic generalizations are not the only two strategies used by students to deal with patterns. There is also a third strategy, which I will discuss in the next section.

4 Naïve inductions

In the course of several investigations with different cohorts of Grade 7 or Grade 8 students, we have often noticed that, to deal with patterns such as the previous one, the students resort to a method based on guessing the rule. For instance, after noticing that there were 3, 5, and 7 circles in the first three terms of the sequence referred to in Fig. A, one group of students tried the rule or formula "the number plus 2" or $n + 2$, which worked for the first term, but then noticed that it did not work for the second term. They then switched to $2n + 2$, and then to $2n + 1$. Since the last formula led to the expected results when n was successively substituted by 1, 2, and 3, the students concluded that the sought formula was indeed $2n + 1$. The question is: Is this procedure an algebraic pattern generalization?

If we go back to our definition, we see that the students made three consecutive abductions. None of these abductions, however, resulted from inferring a commonality among the first three figures. The abductions were indeed mere guesses. Upon closer inspection, it turns out that the abductions did not lead to a rule produced by a generalization, but a rule obtained by *induction*, i.e. a procedure based on *probable (or likely) reasoning* and whose conclusion goes beyond what is contained in its premises. This is a type of induction that I will qualify as *naïve* to distinguish it from other more sophisticated types of induction.[3]

[2] Although strictly speaking, there is nothing that guarantees that C applies to the other terms, there is no reason to believe the opposite either, i.e. that at a certain point the terms will start behaving differently. The way the question is asked tilts the scale towards the idea that, all things being equal, the commonality should apply to the terms that follow.

[3] The concept of induction has been the object of a vast number of investigations in epistemology and in education; (see e.g. Peirce in Hoopes, 1991, pp. 59–61; Polya, 1945, pp. 114–121; Poincaré, 1968 p. 32 ff.).

5 A broad comparison

The architecture of naïve inductions and arithmetic pattern generalizations are different from the architecture of algebraic pattern generalizations shown in Fig. A.

In arithmetic pattern generalizations, the last arrow is missing, reflecting the fact that the rule was not deduced from the assumption that *C* is true—the calculation with indeterminate numbers was not achieved. It is this characteristic that François Vieta related to the *analytic* nature of algebra to distinguish it from calculation with particular or concrete numbers.

In naïve inductions, the structure is very simple, as shown in Fig. C.

Although the rule may be expressed in the alphanumeric system (i.e. through notations), the generalization is not algebraic. As mentioned earlier in this paper, it is not notations which make thinking algebraic; it is rather the way the general is thought about. One practical result that comes out of this is the following. In the use of patterning activities as a route to algebra, we—teachers and educators—have to remain vigilant in order not to confuse algebraic generalizations with other forms of dealing with the general; we also have to be equipped with adequate pedagogical strategies for making the students engage with patterns in an algebraic sense.

The need to offer students the possibility of becoming engaged with patterns in an algebraic sense led our team to the design of a teaching sequence that we implemented in a Grade 7 class.[4] In the rest of this paper, I shall deal with the first part of the teaching sequence.

6 The teaching sequence

6.1 The general discussion

The teaching sequence started with a general class discussion about the classical toothpick sequence shown in Fig. D.

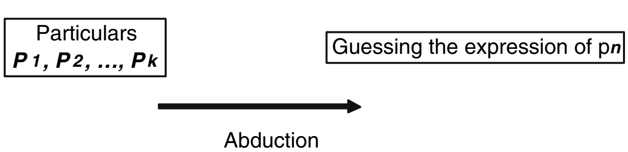


Fig. C The architecture of naïve inductions

[4] The team was comprised of two teachers (Tammy Cantin and Rita Venne-Beaudry), three researchers (Serge Demers, Monique Grenier and Luis Radford) and four research assistants (Isaias Miranda, Emilie Fielding, Mia Coutu and Julie Deault).

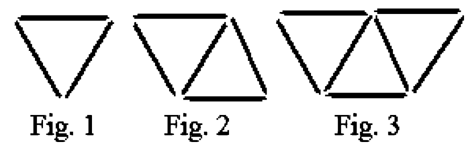


Fig. D The introductory example discussed by the teacher and the students

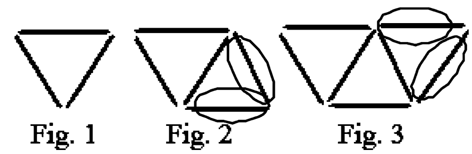


Fig. E The teacher circled some parts of two elements of the sequence

The students determined (without difficulty, of course) that there were 3, 5, and 7 toothpicks in Figs. 1, 2, and 3. Then, the teacher asked the class what to do in order to find a rule that would allow one to find the number of toothpicks in "big" figures such as Fig. 100. The students suggested that the number of toothpicks always increases by two. The teacher translated this numeric abduction into a geometric one: on the blackboard, she circled the two rightmost toothpicks in Figs. 2 and 3 (see Fig. E).

Then, she made it apparent for the students that Fig. 1 can also be seen in a similar manner (see Fig. F) and wrote, above the terms, the constant number between them, indicating, with an arrow, the utmost left toothpick of Term 1 and the number of toothpicks under each term, as shown in Fig. F.

She then asked the class the question about the number of toothpicks in Fig. 100. The answer came from the back of the class. One student replied:

1. Student: 201 toothpicks.
2. Teacher: O.K., can you explain how you arrived at 201?
3. Student: The first figure starts with 1; then you add two each time. So I did a hundred times two, that's two hundred *(inaudible)*.
4. Teacher: (*Noticing that many students did not follow the idea, she added*) If you add two to each of the

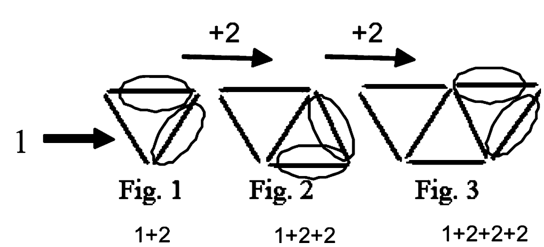


Fig. F Through a complex use of signs, the teacher emphasized one of the possible ways in which the terms of the sequence could be read figures, and if I ask you to go to the 100th figure, how many + 2s will you do?

5. Another student: A hundred!
6. Teacher: (*Synthesizing the idea*) Plus two, plus two, plus two, plus two, a hundred times, it's like saying a hundred times two.

I have presented the teacher's efforts at making apparent for the students the kind of reasoning that was expected from them in some detail. In a pilot project conducted in 2001 in a Grade 7 class, we noticed that, for this and for other similar patterns (overall those defined only numerically, without geometric support) the vast majority of students resorted to naïve inductions. The first part of the lesson was hence devoted to a general discussion of a particular problem allowing the teacher to pinpoint certain crucial elements that we wanted the students to notice. By circling certain toothpicks in the particular figures on the blackboard, a commonality was emphasized; by counting the toothpicks in a specific way ("1 + 2", "1 + 2 + 2", "1 + 2 + 2 + 2") the teacher paved the way for the formation of the rule underlying the generalization. In line 2 of the previous excerpt, the teacher asked for an explanation. In line 4, in an explicit way, she transformed the abduction into a hypothesis (see the conditional term "if" at the beginning and middle of the utterance). Then, in line 6, she linked the repeated addition to the multiplicative sense of the calculations on which the generalization is based.

The design of the teaching sequence and its implementation in the classroom were backed by the Theory of Knowledge Objectification (Radford, 2006b). According to this theory, the basic problem of learning does not have to do with letting the students construct their own knowledge. It is not a question of the students being incapable of constructing knowledge. On the contrary, students are tremendously creative. However, nothing guarantees that their idiosyncratic procedures and ideas will necessarily converge with the cultural ones conveyed by the mathematics curriculum. The central educational problem is rather to have the students making sense of sophisticated cultural ways of reflecting about the world—in this case, algebraic ways of acting and reflecting—that have been constituted over the course of centuries.

Making sense of cultural ways of mathematical reflection does not consist, of course, of a mere *transmission* of knowledge. In the previous passage, the teacher did not transmit the generalizing method. The sophisticated form of generalization was noticed by the class in a *joint* student-teacher interaction that was much more than an exchange of points of view. This kind of interaction has a profound epistemic value: it is a complex unitary and dialectical process carried out by the teacher *and* the students in which learning becomes entangled with teaching (Bartolini Bussi, 1998, p. 69). It includes cognitive aspects such as formulating (lines 1, 3), reformulating (line 4), noticing (Figs. E and F), interpreting (lines 1, 2), emphasizing (line 6), etc. All attempts at transmitting knowledge are doomed to failure, for as Vygotsky and his collaborators insisted again and again, learning is a genetic process that develops in a *progressive* manner (Vygotsky, 1986, p. 146). The Theory of Knowledge Objectification thematizes this genetic process as an active and creative process of noticing and making sense of the conceptual object that is the goal, in Leont'ev's (1978) sense, of the activity. Naturally, the teacher can "see" this conceptual object; the students cannot. The teacher's pedagogical actions are precisely framed by this *asymmetrical* epistemic relationship (Radford, 2006c).

I call *objectification* the process of making the objects of knowledge apparent. Since these objects are *general*, they cannot be fully exhibited in the concrete world. Teachers and students then resort to signs, body, tools—in short to all sorts of artifacts that I have termed elsewhere *semiotic means of objectification* (Radford, 2003a). Thus, the elliptic signs used by the teacher to encircle specific toothpicks in the figures, the digits, the numerical expressions, the pointing gestures, the linguistic terms—like the traveling metaphor of "*going to* the 100th figure"—are all semiotic means of objectification.

Let us return to the classroom. Since the teacher wanted to make sure that the students really understood, the calculation of the number of toothpicks in Fig. 100 was followed by other questions. Working in small groups of three or four members, the students were then asked to deal with three similar problems. The first one was framed in the context of an amount of money that increased at a constant rate from an initial stipulated amount of money. The initial amount of money played the same role as the left toothpick indicated by the arrow in Fig. E. The next two problems presented a challenge: the initial object was not given in the statement of the problem; the students had to find it. In addition to this, in these two problems the sequences were purely numeric. In the first one, the constant difference between consecutive terms was a decimal number; in the second one, a negative number. Our choice of decimal and negative numbers resulted from our intention to discourage the students from using naïve inductions (see Radford, 2006a, footnote 2, p. 17 that makes reference to the pilot study).

6.2 The first problem

Here is the statement of the first problem:

Marc saves $3 a week. He started off with $12.

We asked the students how much money Marc would save by the end of the third, fifth and 100th weeks. The students were asked to explain their procedure. Two other tasks were: to explain in words how to calculate the amount of money saved by the end of any week and to find an algebraic formula giving the amount of money saved after *n* weeks.

The interaction analysis that follows focuses on one of the small groups of students: Mélanie, Judy, Kathy, and Renée.

As anticipated, the first part of the problem did not present major difficulties:

7. Mélanie: O.K. you go 100 times…
8. Judy: So that would be like 3 times 100 […] 12 + 3 times 100, because you start with $12.

But when the students had to explain how to calculate the amount of money by the end of *any* week in words, some difficulties appeared:

9. Mélanie: So we will say: you have to add 12 + 3, and then, you add the number of weeks.

To talk about *any* week—as opposed to a *particular* week, such as week 3 or week 10—requires that students deal with *indeterminate quantities*. It entails an abstraction that may present some difficulties for them (Radford, 2003a). For instance, here, the multiplicative structure of the rule objectified in the classroom discussion (lines 1–6) vanishes and is replaced by an additive one (line 9). Judy intervened and suggested the following algebraic generalization:

10. Judy: You multiply 3 by the number of weeks and then you add 12.

Although writing the rule in notations was not obvious for all the students, the difficulties were overcome:

11. Mélanie: I don't understand the thing with *n*, it's the only thing that I don't understand.
12. Kathy: Me neither.
13. Judy: *n* is like a box with a question mark inside it, because you don't know what it is.
14. Renée: 3 + 12 whatever, 3 + 12 times *n*.
15. Judy: The formula would be 12 + 3 times *n* (*she writes 12 + 3n*).

In line 13, Judy suggests a metaphor for dealing with the meaning of *n*. This is followed by Renée's formula, which mixes up the role of numbers 12 and 3—another example of the difficulties arising from the abstraction of the context (compare lines 14 and 15).

Although the understanding of the students in the group is not equal, the process of objectification is well on its way. The next problem was decidedly more challenging for the students.

6.3 The second problem

In this problem, the students were presented with the four first terms of a numeric sequence, as shown below:

0.42	0.75	1.08	1.41
Term 1	Term 2	Term 3	Term 4

They were asked to find Terms 5, 6, and 7 and to answer similar questions as in the previous problem.[5]

The students noticed without difficulty that the terms increase by 0.33 and used this increase to calculate the following three terms one after the other (see Fig. G).

When they tackled the question of calculating Term 100, they said:

16. Judy: Yes, O.K. (*reading the problem*) Find Term 100 in this sequence. Explain your procedure. Oh my!
17. Mélanie: Well, it would be like 2.40 times…
18. Judy: No, 33 times 100 […] No, it's 42 plus 33 times 100. Wait a minute, I'm going to try something…
19. Renée: It's 33 times 100.
20. Mélanie: We're at 2.40 for Term 7. And Term 8 would be 3…we have to do 33 times 33 plus, 33 divided by something.
21. Renée: 33 times 100.
22. Mélanie: No, that would be too big of a number […]
23. Renée: 33 plus something times 100.

The students have a sense that a multiplication has to be carried out, but 2.40 (the 7th term) cannot be one of the multiplied terms, as suggested by Mélanie in line 17. We see some of the difficulties arising from a situation in which the numbers do not get their meaning from a concrete context (as was the case in the previous problem). The meaning must now result from abstract positional relations between numbers.

[5] In the previous problems, there was essentially just one form of expressing the general term of the sequence. Thus, as discussed previously, in the first problem given to the students, the general term was 12 + 3*n*. In the second and following problems, the *n*th term of the sequence can be expressed by an infinite number of polynomials. Generally speaking, a sequence $p_1, p_2, p_3,\ldots,p_k$ of k terms ($k \geq 1$) can be interpolated by polynomials of degree $k - 1$, k, $k + 1$, etc. One of the reviewers suggested that we should have used sequences having one solution only. I do not think so. As we shall see, the fact that there were an infinite number of solutions to express the general term in the second and following problems never occurred to the students, nor did it become a problem during their first contacts with algebra. Sensitivity to this problem can only arise if one already knows a great deal about polynomials, which of course was not the case with our Grade 7 students. In using patterns as a route to algebraic generalizations, our focus was on linear patterns.

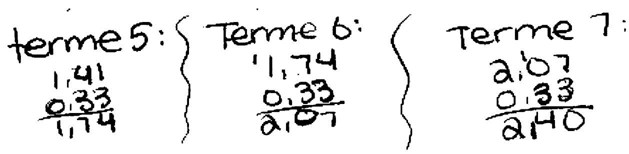


Fig. G The successive calculation of Terms 5, 6, and 7. From Kathy's worksheet

The students continued discussing for a while. They agreed that the multiplication should be 100 × 0.33, i.e. 33. They still had to add the starting point:

24. Mélanie: Now, you've got 33, then you add your [point] 42, so 30.42.
25. Judy: No, we already calculated Term 1, if we do 0.33 times 100.
26. Mélanie: I don't understand what you're trying to tell me.
27. Judy: Look, if we do 0.33 times 100, we've done Term 1 up to Term 100, but we haven't found the one before.
28. Renée: You have to do + 33 to give…, that's true.
29. Kathy: Wouldn't you start with your 42?

In lines 25 and 27, Judy appropriately pointed out that when they multiplied 0.33 times 100, Term 1 was already considered in the calculations and that consequently 0.42 was not the starting point.

Therefore, Judy suggested calculating a particular term and using it to test the rule:

30. Judy: If we do a term that we haven't found yet, but that isn't too big … We can go like Term 7 plus 0.33, plus 0.33 which gives us an answer. We can know what the good answer is and if we've got the good procedure for finding the answer.

But this amounts to giving up the algebraic character of the generalization and falling into the unstable and unpredictable realm of naive inductions! It amounts to recapitulating …

The teacher decided to intervene.

31. Teacher: (*Pointing to the toothpick sequence on the blackboard*) Knowing that you're starting with one toothpick, you add 2 of them, you add 2 of them, you add 2 of them. Have you determined how the terms increase?
32. Judy : Plus 33.
33. Teacher: 0.33, O.K. What's your starting point?
34. Judy : 0.42.
35. Teacher: Pardon?
36. Judy: 0.42.
37. Teacher: You have your [first] figure, but that's not your starting point. You told me (*referring to the toothpick sequence*) that [here] it was 1 because you had 1 [toothpick] at the start.
38. Judy: So our starting point would be 0.01 […] [The first term] would then be equal to 0.34
39. Renée: [The first term,] it's 42.
40. Judy: We would have to use another number […] So, we would have to do 0.09 plus 0.33. So it would be 0.09 plus 0.33 […] Because [point] 42 minus 0.33 gives 0.09.

6.3.1 *Iconicity*

The teacher's intervention was directed towards getting the students to pay attention to the introductory toothpick problem in order to see how the starting point was identified. This is an objectifying process based on iconicity—i.e. a manner of noticing similar traits in a previous procedure. Although the recourse to this strategy kept the students moving along the lines of knowledge objectification, the students still had a long way to go. It is in the genetic nature of knowledge objectification to go from the fuzzy identification of what Husserl (1970) called an objectivity (e.g., an object, a procedure, a reasoning, a state of affairs, etc.) to a clearer picture of it. Objectification is indeed made up of various layers of intelligibility, and iconicity is one of the processes which allows us to make the progressive transition from fuzziness, to one of those clearer and more intelligible layers of objectification. A token of the fuzziness through which the students were still reflecting while on their way to making sense of a more sophisticated manner of reflection-an algebraic one-can be seen in the students' need to test the stated rule:

41. Judy: Here we can try if it's right. We have Term 7 that is 2.40; Term 8, that is + 0.33; then Term 9 that would be + 0.33 again. (*She tests the formula and seeing that it works, says*) That's it! It would be 0.9 + 0.33 times our number.

Despite all the "good" reasons to believe in the formula, they needed to test it. The formula's epistemic certainty did not result from the apodeictic or indisputable assurance of the calculations, but from its validity as tested on one known case. Conviction will come later on. It will come with the increased awareness of the meaning of the numeric terms involved in the calculations and the students' growing familiarity with the targeted cultural mode of mathematical thinking. Ontogenetically speaking, mathematical certainty does not come from within mathematics only, but also from the relationship between the knowing subject and the object of knowledge, as mediated by social activity.

6.3.2 Contraction

The earlier intense discussion resulted in the objectification of the calculations to find some particular terms of the sequence. Asking the students to *explain to someone* how to do the calculations for *any* term of the sequence introduces two new elements. On the one hand, it introduces the encounter with a generic Other.[6] On the other hand, the students now have to deal with a *particular* but *unspecified* number, requiring a higher level of abstraction (Radford, 2003a)—the level of variables (Bardini, Radford, & Sabena, 2005). It also puts the students in a situation where they have to *summarize* their previous mathematical experience. Summarizing is another genetic process of knowledge objectification. It leads one to making a choice between what counts as relevant and irrelevant; it leads to a *contraction* of expression and a concomitant deeper level of consciousness and intelligibility:

42. Judy: O.K. well, you do, you multiply 0.33 by the number of the term that you want to find. Then, you add 0.09 then that gives you your answer.
43. Mélanie: People are going to wonder how we found 0.09.
44. Judy: You say that we did [point] 42 minus [point] 33.

This contraction process is even more visible in the next passage, where Judy suggests the symbolic formula:

45. Judy: It would be like 0.09 + 0.33 times *n*.

We shall now turn to the third problem and continue focusing on the process through which the students objectify the generalizing rule.

6.4 The third problem

In this problem, the students were presented with the four first terms of a decreasing numeric sequence, as shown below:

11	9	7	5
Term 1	Term 2	Term 3	Term 4

The students noticed that the terms diminished by two and continued the sequence up to Term 7. Then they had to find Term 100. Quickly, Judy sketched the general structure of the rule:

46. Judy: It's something, minus 2 times 100. Don't you understand what I said?
47. Mélanie: Yes […] We know that the terms decrease by 2.

The "something" (line 46) that has to be placed at the beginning of the formula is the starting point of the sequence, which, as the students realized in the previous example, should not be confused with the first term. But, when the calculations were eventually carried out, the negative numbers became the focus of attention and the application of the intuited general rule to a particular case became problematic:

In Judy's group, the discussion focused on the size of the term (*Is it reasonable to expect something around -200?*):

48. Renée: It's supposed to give us a negative, oh yes.
49. Mélanie: We know that it's going to be –100 or –200 or something else like that. It's pretty hard when it's negatives […] We're already at –1. So, –1 –2 would be –3, right ? Then, –5, –7, –9, –11, –13, –15, –17, –19, –21, –23, so, if we kept going for a long time, we would be able to get to [Term] 100 […]
50. Judy: Yes. The number that we're going to find has to be odd, because they're all odd numbers.
51. Mélanie: Yes, O.K. […] we know that it's going to be negative odd, right, right? […] We look, like, clueless. Look at the board there.
52. Judy: Look at this here, then at this here [i.e. the problem on the blackboard], it's mixing me up, because it's not the same thing. [….] (*talking about the sequence on the blackboard*). We start with a small number and we go up.
53. Mélanie: [Here] (*referring to the decreasing sequence*) we start with a big number and then we go down.

The loss of control over the calculations led the students, in line 49, to come back to the iterative procedure of subtracting 2 to the terms. In their attempt to regain control over the situation, the students noticed that the result is a negative odd number, but, of course, this information was not of much help (lines 50–51).

Mélanie and Judy, then, suggested to their group-mates that they do something that they did not do in the previous problem—resort to the iconicity of procedures in search of the *same* thing (or what Vergnaud (1990) would term the *invariants*).

Following Mélanie's suggestion (line 51), Judy turned to the blackboard and scrutinized the traces of the previous activity on the toothpick problem. She then suggested interpreting the sequence as a toothpick decreasing sequence whose first term is 11:

[6] Otherness is in fact a crucial aspect of knowledge objectification, as Hegel remarked in another context (Hegel, 1977).

54. Mélanie: I don't understand, I don't understand, I don't understand. I understand the problem, it's just…
55. Renée: It's too complicated.
56. Judy: O.K. (She draws a figure made of 11 toothpicks as shown in Fig. H), look here. We have three little sticks.
57. Mélanie: You're going to mix us up with the little sticks. No offense but, you're going to mix us up.
58. Judy: Look! 3, 4, 5, 6, 7, 8, 9, 10, 11. (She counts the little sticks that she has drawn on the page.) O.K. so this (she indicates the figure she has drawn) could be our Term 1.
59. Mélanie: [You have to do] like term 100!
60. Judy: It's the same as that, but there [*referring to the problem on the blackboard*] we just add some more, and here we add less.

Iconicity rests on making a distinction between the same and the different. Naturally, for the students, the central problem was how to recognize those characteristics that they were looking for in the many signs that were on the blackboard. As the dialogue shows, instead of looking at what is the same, the students focused on what was different (increasing versus decreasing sequences; see lines 52 and 53). Being able to discriminate between the same and the different is, as mentioned in Sect. 1, a crucial step towards generalization. In lines 56 and 58, Judy suggested drawing a toothpick figure having 11 toothpicks. They would then proceed to remove two toothpicks at a time. This is certainly an interesting and creative effort at reconceptualizing the problem, but what could they do to enter into the negatives in this way? The teacher went to see the group and asked whether or not they had calculated Terms 5–7:

61. Judy: I found up to [Term] 14.
62. Teacher: (*Asking in an ironic tone*) You're not going to get to [Term] 100 like that?! (*The students smile.*) […] O.K., you just have to go back to what you just did. In the problem, even if there is a minus 2, it doesn't mean that the process for arriving at the rule is different. It's the same thing. So here, I'm asking you to find the 100th term. So, what do you do each time?

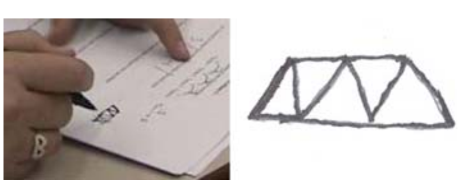


Fig. H Left, Judy draws Term 1. Right, a cleaned, calqued version of Term 1

63. Judy: Minus 2, so that would be like minus something, minus 2 times 100 […]
64. Teacher: O.K. (*seeing that Kathy does not seem convinced continues*). If it was + 2 then, it's the same thing […] It's the same thing: 100 times minus 2. […] minus 2 plus minus 2 plus minus 2 plus minus 2 plus minus 2 plus minus 2 …
65. Mélanie: It's like negative 200.

In line 62, the teacher emphasizes the fact that having a positive or negative constant difference between consecutive terms in a sequence does not affect the way the repeated addition must be carried out. To accentuate this key aspect, she says three times: "It's the same thing" (lines 62 and 64). And to convey the idea of a repeated addition of numbers, she utters the sentence "minus 2 plus minus 2 plus minus 2 plus minus 2 plus minus 2 plus minus 2". To highlight the dissymmetrical role of numbers and actions, the term "plus" is pronounced at a cadence that distinguishes it from the occurrences of the term "minus 2". Figure I shows the waveform of prosodic analysis conducted with Praat—a voice analysis dedicated software (Boersma & Weenink, 2006).[7] The waveform shows an excerpt of the teacher's utterance and indicates two things. First, that the utterance, although comprising 17 words, was uttered very quickly; second, that the cadence of the utterance was made in such a way as to clearly distinguish two chunks of information: the number (minus 2) and the repeated action of adding. The short duration of the utterance intimates the idea that the focus of attention is not on the successive results obtained if the calculations were to be carried out term after term; rather, the focus of attention should be put on a *synthetic* and *global* result. The cadence of the utterance shows the role of *rhythm* as a semiotic means of objectification. Rhythm allows the teacher to highlight the monotony of additive actions, thereby calling the students' attention to the sought-for meaning, i.e. that the sequence of actions can be *contracted* in a multiplicative form.

What remained to be tackled was the other part of the rule, the part involving the starting point. The teacher continued:

66. Teacher: Yes, you started off well; it's just that you're missing something. Look at Term 1 (She points to Term 1; see Fig. J), what does 11 represent?
67. Judy and Mélanie: Term 1.

[7] *Prosody* refers to all those features to which speakers resort in order to mark the ideas conveyed in conversation in a distinctive way. Typical prosodic elements include intonation, prominence (as indicated by the duration of words) and perceived pitch. Some works on prosody include Bolinger (1983), Goodwin, Goodwin & Yaeger-Dror (2002), Roth (2007) and Selting (1994).

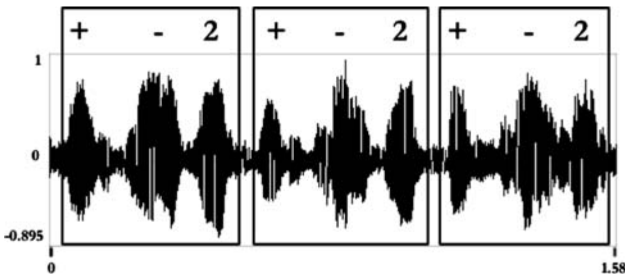


Fig. I A Praat edited prosodic analysis of a 1.58 s excerpt of the teacher's utterance. The waveform indicates the temporal and rhythmic organization of the utterance: "plus minus 2 plus minus 2 plus minus 2"

68. Teacher: And how did you arrive at 11? (*The students explain.*) […] Let's go back to [problem 2] again. (*She goes back to problem* 2 *and pointing to Term* 1 *says*) 0.42 is your first term (see Fig. J). You told me that the difference was 0.33 (see Fig. J) […] But there (pointing to 0.09) 0.09, 9 hundredth, where does it come from?
69. Judy: We did 0.42 minus 0.33 which gave us 0.09 (see Fig. J).
70. Teacher: […] We have the same genre of problem here (*She indicates a place on Renée's sheet*). 11, how did you arrive at 11? What happened there?
71. Judy: 13 – 2.
72. Mélanie: O.K., so we know it's 13.
73. Judy: […] So, 13 plus minus 2 times 100 (*she writes* 13 + – 2 × 100).

The iconic process of objectification launched from lines 66 to 70 allows the students, with the help of the teacher, to identify the similar features in the generalizing rule (line 71) and attains a clear formulation by lines 72 and 73. Figure J highlights the prominent coordination of gestures, words, and symbols in what we have termed elsewhere a *semiotic node* (Radford et al., 2003b), i.e. a segment of the students' semiotic activity where knowledge becomes objectified. The next day, when the activity was resumed, the students produced the formula $13 + -2 \times n$.

The teacher's intervention can be better understood as an intervention occurring in the Vygotskian *Zone of Proximal Development* (Vygotsky, 1986). The students could not have completed the objectification of knowledge by themselves. However, they were not far away from it. The teacher's punctual and precise intervention helped the students to recognize the different elements of the generalizing rule and formulate it by themselves.

Let us quickly see what happened in another group formed by three students: Vicky, Paul and Sylvain.

As in the first two problems, in problem 3, Vicky's group underwent a process of objectification similar to that in Judy's group: the students suggested $-2 \times 100 + 11$. After a comparison with the previous problem and with the help of the teacher, they finally realized that the term to be added to -2×100 was not 11 but 13. When they had to explain the rule, they summarized it as follows:

74. Paul: You have to do the difference, the difference between the…
75. Vicky: (*Interrupting*) The difference (*she looks at problem number* 2) times the number of the term, plus your first term.
76. Paul: The first term before. The term before your first term.

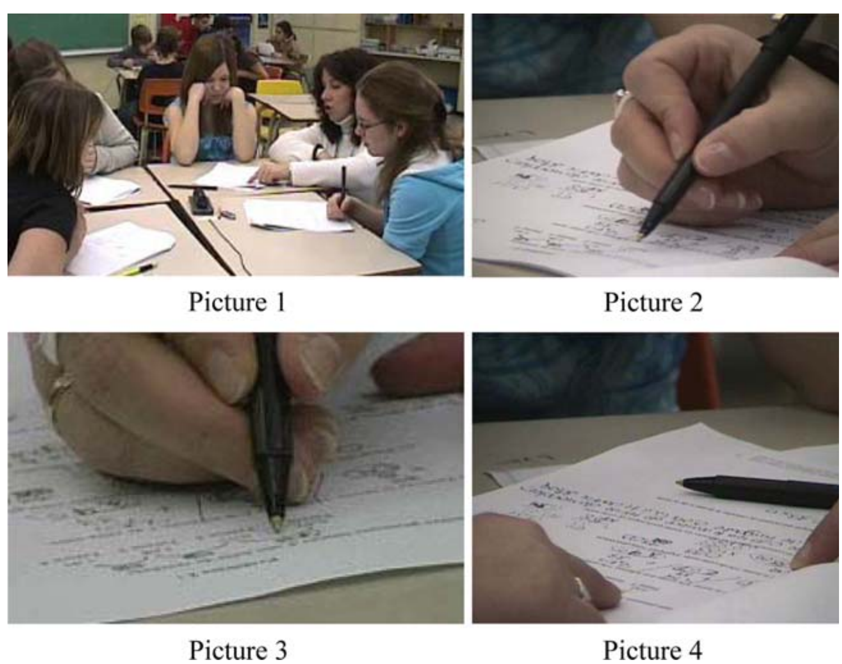


Fig. J Picture 1, the teacher points to Term 1 in Problem 3. To emphasize similarities, in pictures 2 and 3, the teacher makes reference to Problem 2 and points to 0.42 (Term 1) and to the constant difference (0.33), respectively. In Picture 4, Judy, right after having pointed to 0.42, points to 0.33 and says "which gave us 0.09" (line 69)

77. Vicky: Yeah. (*Continuing her explanation*) and the first, the first term.
78. Paul: Your term before Term 1. (*He makes gestures on the table*.)
79. Vicky: Your pre-term.
80. Paul: (*He laughs.*) Your pre-term, O.K. […]
81. Vicky: Plus, what are we going to call the thirteen then?
82. Sylvain: The term before the first term.

In Problem 1 (i.e. the "Money-Problem") the use of letters was not clear for this group. Sylvain strongly disagreed with Paul about using a letter for the number of weeks:

83. Sylvain: (*Talking to Paul*) you and your letters…! It's the number of weeks! We can't put a letter! What does a letter mean?!

In problem 3, the passage to the symbolic expression did not pose difficulties. Sylvain said:

84. Sylvain: O.K., so minus 2, times *n*, plus 13.

The teacher was busy discussing with another group at the back of the classroom. Vicky and her group-mates were about to start the next problem on the activity sheet. It was clear that after reflecting and becoming engaged in the classroom discussion and then in three similar problems, the constitution of the algebraic generalizing rule had become relatively stable for this group. But I wanted to see to what extent the objectification had been achieved. I took off the headphones that I wear behind the camera and went to talk to Vicky's group, which happened to be the group I was videotaping:

85. Luis: So, do you know what to do now in order to find any given one [term] in a sequence of numbers like these (*pointing to the activity sheet*)?
86. Vicky and Paul: Yes.
87. Luis : Let's suppose that I gave you some kind of sequence of numbers, alright? What would you do?
88. Paul: You have to find the one before.
89. Luis: Tell me all of the steps.
90. Vicky: O.K., we would do, uh…the difference (*She points to the difference* "–2" *in her sheet*) times the number of the term, plus the term before the first term, like the term that we started with.

In line 90, Vicky summarizes the steps to be followed in a sentence. The steps to be followed have become clear. The schema has been objectified to an unprecedented layer of generality. There are no numbers or examples in the rule. The traces of concrete sensuous activity can only be distinguished through an incidental pointing gesture indicating the particular constant difference "–2" that, as particular, remains now outside the realm of speech. Precise general terms now replace the place that particulars occupied in the students' prior mathematical experience. A remarkable contraction focused on the *same* and the *relevant* has been achieved. The students now recognize that the nature of the terms of the sequence (natural numbers, decimals, negative numbers, etc.) does not matter. The nature of numbers belongs to the different; the same belongs to actions carried out on remarkable numbers: the starting point or "the term before the first term", as Vicky says; the (constant) difference, the (indeterminate) number of the term.

As confidence is gained, testing the rule becomes an unnecessary procedure. Neither Vicky's nor Judy's group felt the need to test the rule in problem 3. The teaching-learning sequence secured a space for algebraic pattern generalization to become a way of reflecting on lineal sequences.

7 Synthesis and concluding remarks

In this paper, I argued that algebraic pattern generalizations are not characterized by the use of notations but rather by the way the general is dealt with. I suggested that algebraic pattern generalizations entail (1) the grasping of a commonality, (2) the generalization of this commonality to all terms of the sequence under consideration and (3) the formation of a rule or schema that allows one to determine any term of the sequence directly. However, from an ontogenetic viewpoint, the linking of these three central elements is not without difficulties. Thus, generally speaking, numeric patterns are reputedly difficult not because of the difficulties that the students encounter in grasping the commonality, but because the students tend to fail at using it to form a direct and meaningful rule. As Rossi Becker and Rivera comment, the students

> usually employ trial-and-error and finite differences as strategies for developing closed forms or partially correct recurrence relations with hardly any sense of what the coefficient and the constant in the linear pattern represent. (Rossi Becker & Rivera, 2006, p. 96)

One of the dangers of using patterns as a route to algebra is indeed the fact that students often make recourse to trial and error or naïve inductions instead of resorting to algebraic generalizations (MacGregor and Stacey, 1993; Castro Martínez, 1995).

The teaching sequence partially described in this paper was an attempt at providing the students with a teaching-learning environment in which they could reflect *algebraically* on pattern generalizations (limiting such a reflection to linear polynomials). The design and implementation of the teaching sequence was based on the Theory of Knowledge Objectification. As formulated within this theory, the central educational problem is to offer the students the possibility of securing, as much as possible, particular historically constituted modes of thinking. Now, fluency with the latter cannot result from mere transmission. The Theory of Knowledge Objectification thematizes the process of becoming fluent with historically constituted modes of thinking as social processes of *objectification*. By processes of objectification, it is meant the active and creative social processes of sense-making in the course of which students notice and come to master, through practice, cultural mathematical forms of reflection and action and, at the same time, achieve the development of deeper layers of subjectivity and consciousness.[8]

The teaching sequence discussed in this paper was oriented towards the students' attainment of elementary algebraic forms of thinking and was based on three problems of increasing complexity. We paid particular attention to the way in which the noticed commonality (the constant increase or decrease between consecutive terms) was transformed from abduction to hypothesis and how this hypothesis could give rise to the *deduction* of a schema or rule. By varying the context of the problems, we wanted to investigate the manner in which the students' reasoning evolved towards stable forms of algebraic reflection.

Two distinctive processes of objectification played a crucial role in the evolution of students' mathematical experience: iconicity and contraction.

Iconicity is not merely the contrast between two given conceptual forms. It is the process through which the students draw on previous experiences to orient their actions in a new situation. In other words, iconicity is based on the projection of an earlier experience onto a new one-a projection that works on the progressive identification of the *similar* and the *different* and that makes possible, through a back and forth movement, the emergence of the second conceptual form (here a generalizing procedure). In some cases, iconicity may remain in the background of discourse, as appears to be the case in the link between the discussion summarized in Fig. F and line 3 on the one hand, and line 8, on the other. In other cases, resorting to iconicity seems much more difficult, as in problem 2. In line 31, the teacher did indeed have to call the students' attention to the procedure followed in the toothpick pattern (Fig. D). In line 37, there is an evocation of the relevant elements in the original context and the projection of these onto the new context of the decimal sequence. It is in the course of this projection that the numbers of the new context acquire a new meaning: the decimal number 0.09 is the iconic projection of the leftmost toothpick of the first term of the sequence and enters the universe of discourse with the same attributes as the former: as the *starting point* of the sequence.

Iconicity reappeared in the third problem, this time under the students' initiative. In a very creative act, Judy interpreted the new Term 1 as made up of toothpicks (see Fig. H). This is a way of establishing similarities between two different contexts. However, the idea, as we saw, was received with skepticism and then forgotten. The teacher intervened and, in line 62, suggested that similarities were to be found elsewhere. Similarities were to be found-not in the elements of the two sequences-but in the repeated additions, regardless of the fact that in the first case, the number to be added was positive and in the second case, negative.

But iconicity is not a mere matter of logic. It is first of all a matter of making apparent the relevant similar. And, because similar things are similar in some respects (and hence different in others), to emphasize the similarity of repeated additions, the teacher made recourse to prosody. She used rhythm to project the idea of repeated addition in the new context, treating the disturbing constant difference "–2" as an ordinary number (see Praat prosodic analysis in Fig. I), much as the number "2" was used in the toothpick context.

Contraction-the second main process of objectification in the analysis-makes it possible to cleanse the remnants of the evolving mathematical experience in order to highlight the central elements that constitute it. Contraction is indeed a necessary condition of knowledge attainment. We can easily imagine the difficulties that we would experience if we needed to pay attention to each and every detail of our surroundings and the experience we make of them. We would need to attend to an amazing number of things that go beyond of the threshold of consciousness (Nørretranders, 1998). Contraction is the mechanism for reducing attention to those aspects that appear to be relevant. This is why, in general, contraction and objectification entail forgetting. We need to forget to be able to focus. This is why to objectify is to see, but to see means at the same time to renounce seeing something else.

Contraction depends on the semiotic systems through which objectification occurs. There are in fact two types of

[8] One of the reviewers found my reference to different theories (epistemology, semiotics, psychology, and phenomenology) unnecessary, if not unpleasant, in my account of the students' processes of generalization. As all teaching and learning phenomena, the students' processes of generalization are very complex. A single viewpoint (be it epistemological, psychological, or semiotic) is certainly insufficient for offering a satisfactory scientific explanation. The traditional psychologism from which Mathematics Education emerged in the 1970s is no longer a possibility. Cognition is much too complicated a phenomenon; understanding it requires that we make recourse to several viewpoints. My paper is an attempt at going beyond self-contained disciplines. It is about a coherent multidisciplinary approach. That which my reviewer sees as a move into the darkness of theorizing, I see as the emergence of light.

contraction: within a same set of semiotic systems (which Arzarello (2006) calls a *Semiotic Bundle*) and between semiotic systems. In the semiotic system of language and gestures, contraction (produced e.g., during the summary of a procedure) leads to a shorter statement having fewer and better articulated words, accompanied perhaps by shorter or more precise gestures. Lines 42, 73, and 75–77 are examples of the first type of contractions. The symbolic formulas produced by the students are examples of the second type: they are contractions of experiences lived and objectified through speech and gestures now being expressed in the alphanumeric semiotic system of algebra. As fluency is gained, contraction increases. The students' overt actions and sensuous experiences become embedded in the way signs are used to reflect and carry out mathematical activities.

I do not think that iconicity and contraction exhaust all the types of processes of objectification that take place during the students' attainment of the historically constituted algebraic mode of thinking. There may be some other subtle types of objectification that only further analyses can reveal. All in all, the analysis presented here suggests that one of the central problems posed by algebraic generalization is the fact that it imposes the economy of particular actions. In turn, this economy creates a distance between the original context and the new one—a new plane of mathematical generality and subjective consciousness—where signs and numbers acquire a non-contextual, relational mode of signification.

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