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SEMIOTIC REFLECTIONS ON MEDIEVAL AND CONTEMPORARY GRAPHIC REPRESENTATIONS OF MOTION

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Introduction

In this paper I will present an ongoing semiotic investigation of motion and some 14th century attempts at representing it in geometrical terms.

The semiotic approach that I will follow focuses on the manner in which time, space and velocity were signified in certain historical periods. I take as my starting point the idea that the conceptualisation of motion and its key conceptual elements—speed, time and velocity— can only be understood within the scope of the epistemological configurations that make motion thinkable in a certain way.
In other words, any attempt at investigating conceptualizations of time, speed, and velocity needs to take into account the cultural mental structures that underpin them.
These cultural mental structures, or *épistèmes*, to use Michel Foucault's term, are not mere spiritual or intellectual forms evolving on their own. They arise, acquire shape and evolve under the evolution of social practices, as well as their material conditions, forms of production and social relations.

Thus, in the perspective that I am advocating here, to ask questions about time, speed and velocity is also to ask questions about the cultural and social conditions that made these theoretical constructs possible.

Time, Speed and Velocity in the Middle Ages

It is very well known that the investigation of motion became a prominent field of research only in the early years of modernity. However, that does not mean that motion was not an object of reflection in previous historical periods. Indeed, questions about motion can be found in the work of Aristotle and other Greek thinkers. They can also be found, although in a different manner, in the work of medieval scholars and abacists.

In my excursion into the Western pre-modern conceptualizations of motion, I want to start discussing a problem that is historically considered the first in its genre. It is a problem from the 8th century, by Alcuin of York, one of the principal figures of Charlemagne's educational reform. The problem, included in a school textbook —*Problems to Sharpen the Young*— reads as follows:

> There is a field 150 feet long. At one end is a dog, and at the other a hare. The dog chases when the hare runs. The dog travels 9 feet in a jump, while the hare travels 7 feet. How many feet will be travelled by the pursuing dog and the fleeing hare before the hare is seized? (Alcuin, translated from Franci's (2005) edition, p. 68)

The problem, written with a didactic purpose, lets us get a glimpse of the manner in which, at this point in the Middle Ages, speed and time became object of scientific enquiry and mathematical discourse.

The statement of the problem reveals the pregnant phenomenological dimension of a world not yet invaded by clocks that measure time with great digital precision. While speed is measured by "feet"—an already abstract unit that still keeps its embodied form and evokes the spatial relationship between the motion of an individual and its surroundings—time is not explicitly mentioned in the problem. To compare the speed travelled by the dog and the hare, Alcuin resorts to the idea of *jump*. In a jump, the dog travels 9 feet, while the hare travels 7 feet.

How then, without explicitly employing the idea of time, can this problem be solved?

Let us turn to the solution. Alcuin says:

> The length of the field is 150 feet. Take half of 150, which is 75. The dog goes 9 feet in a jump. 75 times 9 is 675; this is the number of feet the pursuing dog runs before he seizes the hare in his grasping teeth. Because in a jump the hare goes 7 feet, multiply 75 by 7, obtaining 525. This is the number of feet the fleeing hare travels before it is caught. (Alcuin, translated from Franci's (2005) edition, p. 68)

The first calculation (i.e., the half of 150) corresponds to the number of jumps. The question is: are these the dog's jumps or the hare's jumps? Jump (*saltu* in the original Latin), is, like foot, an abstract idea: it is neither the dog's nor the hare's. It evokes a phenomenological action that unfolds over a certain *duration*. Time thus appears in the problem only in this oblique way. After each jump, the dog comes 2 feet closer to the hare. Thus, the dog will need $150/2 = 75$ jumps to catch the hare. This number of jumps is multiplied by the 9 feet that the dog goes in a jump—the medieval expression of "speed"— and then by 7, i.e., the number of feet that the hare goes in a jump. The resulting numbers are the speeds travelled by each animal.

Problems like this became popular later on. There is a problem in a 14th century Italian manuscript, composed by Piero dell'Abacco, that reads as follows:

> A fox is 40 steps ahead of a dog, and every 3 steps of those of the dog are as long as 5 of the fox. I ask in how many steps the dog will reach the fox. (dell'Abacco, translated from Arrighi's (1964) edition, p. 78)

As in Alcuin's problem, time is mentioned implicitly through *motion*. However, in dell'Abacco's problem, there is no such idea as *jump*: the addressee is left to imagine that *when* the fox travels 3 of his steps, the dog travels 5 of his. Distance, hence, is not measured by the same standard. In symbolic terms what we have is: 3D= 5F.

Dell'Abacco's solution is as follows: "Do in this way: if 3 is worth 5, how much is 5 worth? Multiply 5 by 5, which is 25, and divide by 3, you will have 8 1/3. Now you may say: for each 5 of those (steps) of the dog, you have 8 1/3 (steps) of the fox; so the dog approaches the fox 3 1/3 (steps). In how many steps will he (the dog) reach her (the fox) by (covering) 40 steps? Then say: if 5 are worth 3 1/3, for 40, how many will I have? Multiply 5 by 40, which is 200, and divide by 3 1/3. Bring (i.e., reduce) to thirds, thus multiply 3 by 200, which makes 600, and divide by 3 (and) 1/3 , that is 10/3 and then divide 600 in 10, it gives 60. And the dog will do 60 steps before it reaches the fox. And it is done. And the proof is that in 60 steps the fox goes 60, and the dog in 60 steps is worth 100 [i.e., 60 steps of the dog are worth 100 steps of the fox], because three of his (dog's steps) are worth 5 (of the fox); therefore 60 steps (of the dog) are worth a good 100 (of the fox). It is done." (Arrighi, 1964, p. 78).

Thus, in the absence of a precise quantification of time, the solution proceeds by a comparison of travelled steps, i.e., 3D=5F, where times remains implicit: we have to know nonetheless that while the fox goes one step the dog goes one step as well. From this equality, Piero deduces that 5D= 8 1/3F. Thus, when the dog takes 5 dog-steps, the distance between the fox and the dog diminishes by 3 1/3 fox-steps. Knowing that they are apart 40 fox-steps, and continuing using the rule of three, Piero concludes that the dog will need to go 60-dog steps to reach the fox.

Hence, it does not come as a surprise that in many Medieval and early Renaissance mathematical problems that were accompanied by drawings, time remains expressed in the perceptual motion of the moving objects (see Figure 1).

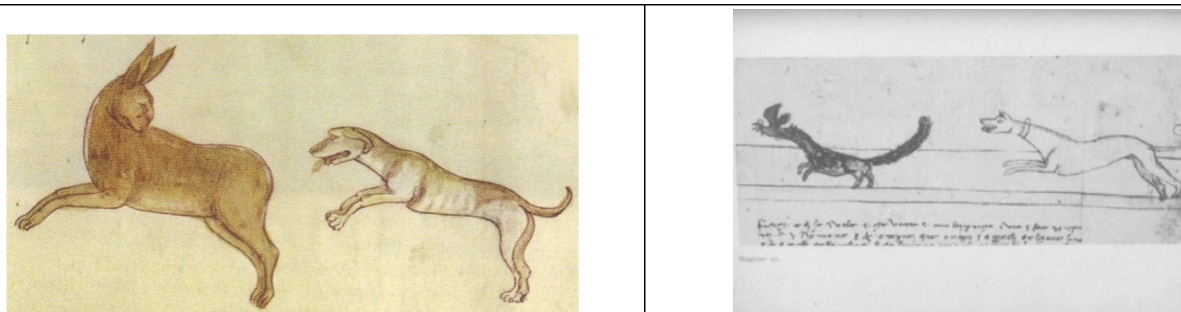

Figure 1. Left, the drawing accompanying Alcuin's problem. Right, the drawing accompanying dell'Abacco's problem.

As these examples intimate, there was, in the Middle Ages, a certain interest in motion problems arising in pedagogical environments. Although these problems were used as educational means, they nonetheless uncover manners in which space, time and motion were conceptually related.

The implicit nature of time in the previous problems should not come as a surprise, if we bear in mind that issues like the elucidation of time, determination of its nature and forms for calculating it were not pressing in the sociocultural context of medieval activities. People lived subjected to the cycle of seasons. Church time marked the canonical hours: *prime* was around the beginning of the day or first hour; *terce* was the middle of the morning; *nones* was midday; *vespers* corresponded to the middle of the afternoon and *compline* meant the end of the day.
The unit of labour time was the day, defined by sunrise and sunset. Since the hours depended on sunrise, the length of the hours changed with the seasons. Labour time, notes Le Goff,

> was still the time of an economy dominated by agrarian rhythms, free of haste, careless of exactitude, unconcerned by productivity—and of a society created in the image of the economy, sober and modest, without enormous appetites, undemanding, and incapable of quantitative efforts. (Le Goff, 1980, p. 44)

But in the 12^{th} Century the cultural mental structures shifted or evolved into new decisive directions, when merchants required a finer measure of time for their activities. The duration of a sea trip or the journey from one place to another became important elements to take into account. Merchants became aware that to make money, time had to be used in a rational way. A sense of strict and minute planning emerged in order to master the interval between conceiving a venture and executing it to make the greatest profit possible. Church time and its canonical hours were, of course, ill-suited to cope with the increasing complexity of the new economic life. The day as a time unit was not appropriate either, for it was clear that important decisions taken at a certain point of the day rather than a bit later could make and unmake fortunes. (Le Goff, 1980, p. 35).
However, as Epstein notes in his enlightening study of the sense of time in medieval Genoa, calendars supplied the year, month, and day, but not the hour. Epstein asks: "How did people know what time it was?" (Epstein, 1988, p. 242) Epstein's analysis points to the impossibility of being able to tell the time exactly. People just did not know. Before, mechanical clocks, sundials and the ringing of bells seem to have been the principal methods used to mark the time of day. Time was hence something mediated by aural and visual experiences. At any rate, time was not mediated, sensed and expressed numerically. Epstein's research, based on 3,902 Genoese notarial acts written in 1201, 1203, 1205, 1206, 1207, 1210, and 1213, shows that notaries started recording the hour in the legal documents, but still in the only approximate way that was possible at this historical period. Thus, in a notarial act concerning the declaration by a witness of a deathbed testament, dated April 22 1201, the lawyer Guglielmo da Sori wrote, "inter terciam et nonam"—i.e., "between terce and nones" or between the middle of the morning and midday.

Numerical Time and Speed

We have just mentioned that time was not exactly quantified but only given in an approximate manner. In fact, according to Epstein, there were twenty-nine forms for expressing time (some notaries would write, between "Terce and Nones" "a little after Terce", etc.).
It is hence surprising to see that, in his *Liber Abaci*, finished in 1202—that is to say, exactly at the same period in which Guglielmo da Sori and other notaries start putting approximate hours in their legal documents—Fibonacci (or Pisano) dealt with a problem that reads as follows:

> Two ships are some distance apart, which journey the first can complete in 5 days, the other in 7 days; it is sought in how many days they will meet if they begin the journey at the same time. (Pisano, 1202/2002, p. 280)

What is surprising in this problem is not its thematic content, for as I mentioned earlier, the expansion of maritime commerce was becoming part of the everyday life of Italian ports. Rather, it is that, after a series of calculations, Fibonacci arrives at the following answer: 2 and 11/12 days (which he writes as $\frac{11}{12}2$). To what part of the day does the 11/12 fraction correspond exactly?

This and other similar problems may be better conceptualized as part of a process that served to create the conditions for the emergence of a new concept of time and the new devices to measure it, i.e., the mechanical clocks, which appeared several decades later (see Thorndike, 1941).

Perhaps the most important point here for our discussion is that the new concept of time that was being carved at this historical period was the result of changes in forms and contents in human activities and their ensuing forms of economic production and social relations. It was a process that affected much more than the conceptualization of time. In the course of such a cultural process, late Medieval and early Renaissance individuals also carved new forms of spatial pictorial representations. These pictorial representations, which made the Renaissance a highly visual culture, were to have an influence in entertaining the possibilities of describing motion in graphical terms, even if much more research must be carried out in order to spell out the details.

In the same way as time became minutely measurable, so too did space, through the invention of perspective. Individuals were no longer organized in terms of size, which reflected the hierarchy of social ranks assigned to them by God, but through quantitative measurable relationships as seen by an abstract eye. This was the emergence of a new form of dynamic *narrative* in which the individual was displayed in concrete spatial and temporal settings.

Through numbers, speed and time became *systems*. As such, they made possible the creation of unifying systems of knowledge representation.

Unifying Systems of Knowledge Representation

To deal with the idea of unifying systems of knowledge representation (of which Cartesian graphs are an example) let me solve one of the previous problems through our modern symbols and concepts. Let us consider Piero dell'Abacco's problem and assume that the fox travels 5 fox-steps per unit of time; then, the dog travels 25/3 fox- steps per unit of time. Referring to the spatial place where the *dog* was at the beginning of its race, the distance travelled by the dog (expressed in fox-steps) is:

$$d_d = 8\tfrac{1}{3}t$$

Referring to the *previous* spatial point (i.e., the *dog*'s initial place), the distance (expressed in fox-steps) travelled by the *fox* is:

$$d_f = 40 + 5t$$

The point at which the dog catches the fox is characterized by the equality:

$$8\tfrac{1}{3}t = 40 + 5t$$

And when we solve this equation, what we get is not the distance travelled by one or the other, but *time*—the time that both animals have been in motion: $t = \frac{40}{3\frac{1}{3}} = 12$.

The sense of our modern solution is quite different from Alcuin's and dell'Abacco's. We, more or less consciously, forget what $8\frac{1}{3}t$ and $5t$ mean and mechanically subtract them to get $3\frac{1}{3}t$; that we then equate to 40. From there, we find t, the numerical value of this elusive concept that was not even mentioned in either dell'Abacco's statement of the problem nor in its solution. Now we substitute the value of t in the first equation and get $d_d = 8\frac{1}{3} \times 12 = 100$ fox-steps (or 60 dog-steps).

But there is also something very different in the modern solution. The motion of both the dog and the fox were referring to a *same spatial point* (the initial position of the dog in the race). In Medieval problems, the motion of the moving objects involved remained without being described within a unifying system of reference. Calculations are made by *comparison* of "speeds" and not by *integration* of data into a same totality.
From a semiotic point of view, there is a striking similarity between the mathematical problems and the paintings and drawings of the Middle Ages. Their respective signs revolve around a main "subject" without being linked by a truly functional unifying system of representation. For instance, the main subject (e.g. the saint) is surrounded by objects in a juxtaposed manner. Each object contributes to the whole meaning of the drawing or painting by addition of its particular meaning (see Figure 2, left).
The order of the signs in paintings and mathematical texts was profoundly transformed with the concurrent invention of the technique of perspective and algebraic symbolism (Radford, 2006). The new cultural forms of knowledge representation continued to privilege a certain subject, but now there was a relational link ensuring the relationship between a chosen central object and other objects (see Figure 2, middle and right).

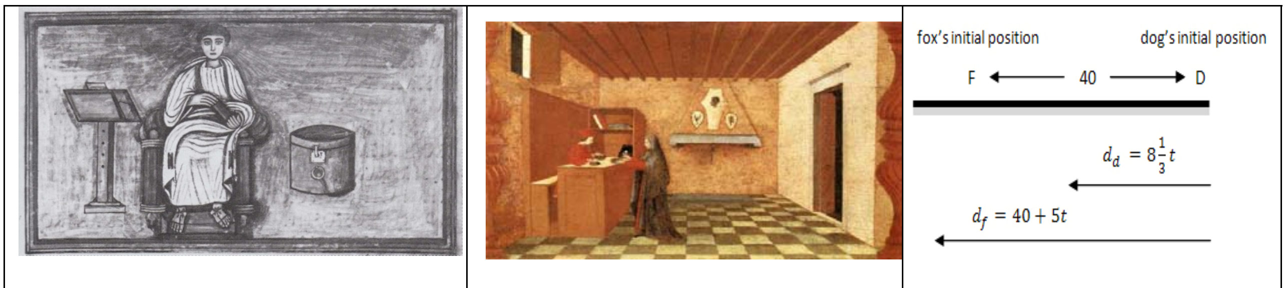

Figure 2. Left, a drawing from a 5th c. Manuscript (Brown, 1990, p. 19); objects are juxtaposed. Middle, Uccello's painting *The selling of the host* (ca. 1468); objects are related in a unifying perspective system. Right, contemporary formulas of the distance in an integrated system where reference to a *same spatial point* and *common time* become the explicit organizing element.

The emergence of a Cartesian system of coordinates and its central point (0, 0) was one of the most sophisticated ways in which to express the complex set of relations between the objects described in the situation at hand. This was one of the crucial developmental steps in the mathematical study of motion. The creation of such a system required, in particular, a mathematical refinement in the conceptualisations of speed and time.

The antecedents of Cartesian graphs were certainly the configurations used by Oresme in the 14th century and by some of his followers. I want to finish my presentation by making some comments on Oresme's idea of configuration.

Oresme's Configurations

In his Tractatus de configurationibus qualitatum, composed in the middle of the 14th century while he was at the College of Navarre, Nicholas Oresme introduced the idea of configuration, as a conceptual tool to deal with questions about motion, among others. Working within the medieval Aristotelian framework of natural philosophy in order to offer alternative accounts of magic and astrological explanations of phenomena, he distinguished between qualities and their intensities. A quality is something that remains the same over a certain period of time. Motion, as the quality of something changing place, could be understood through its *internal* configuration, accounted for by its intensity or "quantitative" dimension: the velocity that the moving object has at its various moments during its trajectory.

Oresme used perpendicular segments to represent this internal configuration. In dealing with motion, by analogy to the lines of latitude and longitude used by cartographers in drawing maps, Oresme used a horizontal segment (the longitude) to represent the duration in time of the quality (the degree of velocity) and a vertical segment (the altitude) to represent the velocity at that precise moment (Figure 3).

Figure 3. Spatial representation of extension and intension.

Oresme distinguished between qualities of uniform intensity and those of uniformly non-uniform intensity. He represented a quality of uniform intensity by a rectangle (Figure 4, left). This rectangle is its *configuration.* Similarly, he represented a quality of uniformly non-uniform intensity starting from zero intensity as a right triangle (Figure 4, right), which is its configuration.

The area of these configurations is the space travelled by the object. In our modern symbols,

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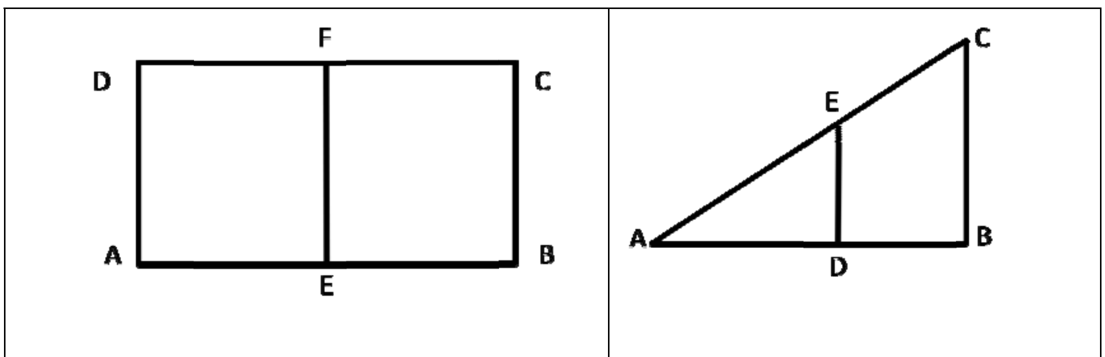

Figure 4. Configurations of qualities having uniformly and non-uniform intensities.

Oresme was aware of the conceptual difficulties in using such geometric representations. As Clagett (1964, p. 309) notes, "He was essentially Aristotelian in believing in the continuity rather than atomicity of physical entities". In fact, Oresme used these representations as aids of the *imagination*. He observed:

> Although indivisible points or lines are non-existent, still it is necessary to feign them mathematically for the measures of things and for the understanding of their ratios. Therefore, every intensity which can be acquired successively ought to be

imagined as a straight line perpendicularly erected on some point or space of the intensible thing, e.g., of a quality. (Oresme in Clagett (1964), p. 305)

A bit later, he says, "the measure of intensities can be fittingly imagined as the measure of lines, since an intensity could be imagined as being infinitely decreased or infinitely increased in the same way as a line." (Oresme in Clagett (1964), p. 306).

He also notes that "The consideration of these lines is of assistance and leads naturally to the knowledge of any intension" and "A 'line of intension' of this sort … is not extended outside of the point or subject in actuality (*secundum rem*), but only in the imagination (*secundum ymaginationem*). It could be imagined as going in any direction you wish, but it is more convenient to imagine it as standing perpendicularly on the subject which is informed with this quality." (Oresme in Clagett's (1959), p. 349).

Configurations, of course, had their own limitations. As Clagett notes, Oresme "thought it a strong possibility that such geometrical entities were mere fictions, and that the use of them sometimes led to improbable or impossible conclusions concerning nature." (Oresme in Clagett (1964), p. 309).

The previous remarks suggest then that, to a certain extent, configurations could be seen as precursors of graphs. The use of configurations, however, was not intended to describe positions of objects. As Durand notes, "'figure' constituted a graphic pattern which aided the student to visualize the order and intensity of properties in natural objects, and the character of the process by which they underwent variation." (Durand, 1941, p. 178). The main variables were time and velocity or acceleration. But, as Wartofsky (p. 1968, p. 442) remarks, "only one of these coordinates (time) is quantitative here, in the strict sense of being numerable ('dimensive' in the medieval terminology), whereas the other (velocity) is conceived of in terms of intensity of a quality (i.e., of 'more' and 'less'.)". However, this "numerability" of time is merely theoretical, for it faces the same practical difficulties exhibited by the Genoese notaries I mentioned before. Within this Aristotelian framework, time derives its order and continuity from those of space via those of movement. Time is the measure of time, but there were no practical ways of calculating it. Time, like other quantifiable variables, remains expressed within the theoretical apparatus of rations and proportions. Hence, its expressibility is always relational. Represented as part of a segment, an instant is always an instant measured by a segment in relationship to another segment and it may sometimes be measurable, sometimes incommensurable.
In a similar manner, intensions, even if they can be represented linearly, do not have an intrinsic or abstract unit. Velocity does not have a metrical substratum that would allow one to make additive calculation with them.

We can see how these concepts of time and velocity still resist being thought of as our x-y axes in modern Cartesian graphs. In order for time, space and velocity to become candidates of modern Cartesian axes, it is necessary that they undergo a conceptual process of 'metrisation', that is to say, a process of abstraction in the course of which they become numbers *per se* — numbers in a modern sense.

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