RELATIVE MOTION, GRAPHS AND THE HETEROGLOSSIC TRANSFORMATION OF MEANINGS: A SEMIOTIC ANALYSIS

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In this paper, we deal with students' transformation of meanings related to their understanding of Cartesian graphs in the context of a problem of relative motion. The investigation of the students' transformation of meanings is carried out in the course of a process that we term objectification, i.e., a social process related to the manner in which students become progressively conversant, through personal deeds and interpretations, with the cultural logic of mathematical entities. We provide a multi-semiotic analysis of the work done by one Grade 10 group of students and their teacher, and track the evolution of meanings through an intense activity mediated by multiple voices, gestures and mathematical signs.

INTRODUCTION AND THEORETICAL FRAMEWORK

A graph is a complex mathematical sign. It serves to depict, in specific ways, certain states of affairs. Instead of being merely a reproduction of these, a graph supposes a selection of elements: what it depicts is relationships between them. This is why the making of a graph of an elementary phenomenon, such as the motion of an object, is like putting a piece of the world on paper (or electronic medium). But because they are not copies of the phenomena that they depict or represent, making and interpreting mathematical graphs is not a trivial endeavour. A Cartesian graph rests on a sophisticated syntax and a complex manner of conveying meanings.

The investigation of the difficulties surrounding students' understanding of graphs has been an active research area in mathematics education, since the pioneering work of Clement (1989) and Disessa, et al. (1991), informed by Cognitive Science and Constructivism, up to the recent work of e.g. Arzarello and Robutti (2004), Ferrara (2006), Nemirovsky (2003), and Roth (2004), inspired by embodied psychology. This paper wants to contribute to the research on graphs by looking into students' processes of graph understanding. We are interested in particular in researching the way in which students attempt to make sense of graphs related to problems of relative motion–an area little investigated thus far.

Our research draws on a Vygotskian sociocultural perspective in which mathematical thinking is considered a cultural and historically constituted form of reflection and action, embedded in social praxes and mediated by language, interaction, signs and artifacts (Radford, 2006a). A Cartesian graph is an artifact for dealing with and thinking of cultural realities in a mathematical manner. But as mentioned previously this artifact is not transparent: it bears the imprint and sediments of the cognitive activity of previous generations which have become compressed into very dense meanings that students have to "unpack", so to speak, through their personal meanings and deeds.

This process of "unpacking" is the socially and culturally subjective situated encounter of a unique and specific student with a historical conceptual object–something that we have previously termed *objectification* (Radford, 2002, 2006a, 2006b). The construct of *objectification* refers to an active, creative, imaginative and interpretative social process of gradually becoming aware of something and oneself (Radford, 2003). Within this context, understanding the making and meaning of a graph, the way it conveys information, the potentialities it carries for enriching and acting upon our world, rests on processes of objectification mediated by one's voice, others' voices and historical voices (Boero, Pedemonte, & Robotti, 1997). Objectification is indeed a multi-voiced–or what Bakhtin used to call heteroglossic–encounter (Radford, 2000) between an "I", an "Other" and (historical and new) "Knowledge".

The distinctive historically and culturally mediated nature of human cognition is such that, in the objectification of mathematical knowledge, recourse is made to body (e.g. kinesthetic actions, gestures), signs (e.g. mathematical symbols, graphs, written and spoken words), and artifacts of different sorts (rulers, calculators and so on). All these signs and artifacts used to objectify knowledge we call *semiotic means of objectification* (Radford, 2003). In the practical investigation of students' understanding of graphs, we will hence pay attention to the students' discourse, gestures and symbols as they attempt to make sense of a graph.

METHODOLOGY: A MULTI-SEMIOTIC DATA ANALYSIS

Data Collection

Our data, which comes from a 5-year longitudinal research program, was collected during classroom lessons that are part of the regular school mathematics program in a French-Language school in Ontario. In these lessons, designed by the teacher and our research team, the students spend substantial periods of time working together in small groups of 3 or 4. At some points, the teacher (who interacts continuously with the different groups during the small group-work phase) conducts general discussions allowing the students to expose, compare and contest their different solutions. To collect data, we use four or five video cameras, each filming one small group of students.

Data Analysis

To investigate the students' processes of knowledge objectification we conduct a *multi-semiotic data analysis*. Once the videotapes are fully transcribed, we identify salient episodes of the activities. Focusing on the selected episodes, we refine the video analysis with the support of both the transcripts and the students' written material. In particular, we carry out a low motion and a frame-by-frame fine-grained video microanalysis to study the role of gestures and words.

The data that will be discussed here comes from a Grade 10 lesson about the interpretation of a graph in a technological environment based on a graphic calculator TI 83+ and a probe–a Calculator Based Ranger or CBR (a wave sending-receiving mechanism that measures the distance between itself and a target). The students were already familiar with the calculator graph environment and the CBR. In previous activities, they had dealt with a fixed CBR and one moving object. In the activity that we will discuss here, the students were provided with a graph and a story. The graph showed the relationship between the elapsed time (horizontal axis) and the distance between two moving children (vertical axis) as measured by the CBR (see Figure 1). The students had to suggest interpretations for the graph and, in the second part of the lesson (not reported here), to test it using the CBR. Here is the story: "Two students, Pierre and Marthe, are one meter away from each other. They start walking in a straight line. Marthe walks behind Pierre and carries a calculator plugged into a CBR. We know that their walk lasted 7 seconds. The graph obtained from the calculator and the CBR is reproduced below."

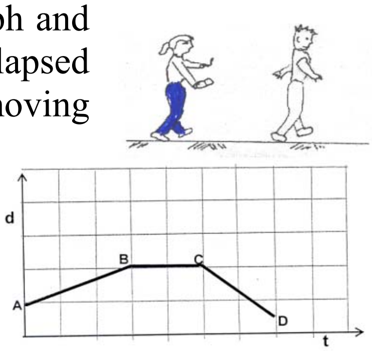


Figure 1. The Pierre and Marthe story, story illustration, and the graph given to the students.

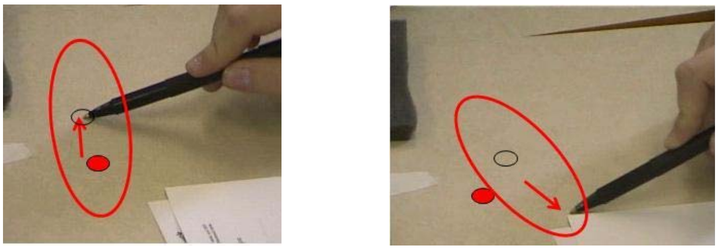
RESULTS AND DISCUSSION

We will focus on one 3-student group and present some excerpts of the students' processes of understanding, with interpretative commentaries on the progressive manner in which objectification was accomplished. The students were Maribel (M), María (MJ) and Carla (C). The students discussed the problem for a few minutes. In Line 1 (L1), Maribel gives a summary of the group discussion:

1 M: (*Moving the pen on the desk, she says*) He moves away from Marthe for 3 seconds and then (*moving the pen further along the desk; Picture 1*), he stops, so he might have like dropped something for 2 seconds, and (*moving the pain back this time; Picture 2*) he returns towards Marthe.

2 C: Well, even though he moves away, but he returns back to…I don't know.

3 MJ: Well, if she walks with him, so, it [the graph] doesn't really make sense!



Pictures 1 (left) and 2 (right). Maribel moving the pen on the desk to signify Pierre stops (segment BC) and Pierre returns towards Marthe.

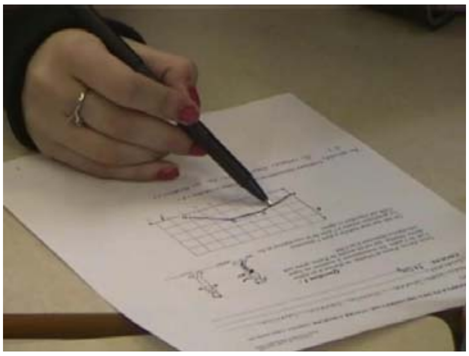
The students' first interpretation rests on the idea of "absolute motion". The segments AB, BC, and CD are interpreted as Pierre moving away, stopping, and coming back. Although the students' current interpretation is not yet aligned with the expected mathematical interpretation, we can see that the students' current interpretation has been forged through a complex coordination of perceptual, kinaesthetic, symbolic, and verbal elements. Maribel's dynamic pointing gestures are not merely redundant mechanisms of communication, but key embodied means of knowledge objectification. Through these gestures and their synchronic link with movement verbs ("to move away", "to come back"), Maribel offers an attempt at making sense of the graph. It is at the end of this episode that María reminds her group-mates that Marthe is moving too, so that, according to the current interpretation, the graph "doesn't really make sense!" Twenty seconds later, Maribel offers a refined interpretation that tries to address the issue raised by María:

4 M: Well technically, he walks faster than Marthe… right?

5 MJ: She walks with him, so it could be that […] She is walking with him, so he can walk faster than her (*she moves the pen on segment AB; see Picture 3*). [He] stops (*pointing to points B and C*)…

6 M: No, there (*referring to the points B and C*) they are at the same distance…

7 C: (*After a silent pause, she says with disappointment*) Aaaaah!



Picture 3. MJ moves the pen from A to B, meaning Pierre's motion (L5).

The graph interpretation has changed: In L4, Maribel introduces the two-variable comparative expression "X walks faster than Y". In L5, María reformulates Maribel's idea in her own words while producing a more sophisticated interpretation. Indeed, L5 contains three ideas: (1) Marthe walks with Pierre; (2) Pierre walks faster than her, and (3) Pierre stops. Although improved, the interpretation, as the students realize, is not free of contradictions. Even if, at the discursive level, Marthe is said to be walking (L5), segment AB is still understood as referring to Pierre's motion (see Picture 3). However, segment BC is interpreted not in terms of *motion* but of *distance* (L6). Moreover, it is interpreted as the distance between Pierre and Marthe. So, while segment AB is about Pierre's motion, segment BC predicates something about both children. The oddity of the interpretation leads to a tension that is voiced by Carla in Line 7 with an agonizing "Aaaaah!" The partial objectification bears an untenable incongruity.

The students continued discussing and arrived at a new interpretation: Pierre and Marthe maintained a distance of 1 meter apart throughout, but they could not agree on whether or not this interpretation was better than, or even compatible with, Maribel's interpretation (L4). Having reached an impasse, the students decided to call the teacher (T). When he arrived, María explained her idea, followed by Maribel's opposition: It is this opposition that is expressed in L8:

8 M: No, like this (*moving the pen along segment AB*) would explain why like, he goes faster, so it could be that he walks faster than her…

9 T: Then if one is walking faster than the other, will the distance between them always be the same?

10 M: No, (*while moving the pen along AB, she says)* so he moves away from the CBR and then….What happens here (*pointing to segment BC*), like?

11 MJ: He takes a brake.

12 T: So, is the CBR also moving?

13 M: Yes.

In L9, the teacher rephrases in a hypothetical form the first part of Maribel's utterance (L8) to conclude that, under the assumption that Pierre goes faster, the distance cannot be constant. Although inconclusive from a logical point of view, the teacher's strategy helps move the students' discourse to a new conceptual level. Maribel's L10 utterance shows, indeed, that the focus is no longer on relative speed but on an emergent idea of relative distance. The gesture is the same as María's in Picture 3, but its content is different. However, as shown in L10, the students still have difficulties providing a coherent global interpretation of the graph. How to interpret BC within the new relative motion context? Drawing on Maribel's utterance (L10), the teacher suggests a link between Marthe and the CBR, but the idea does not pay off as expected. He then tries something different:

14 T: OK. A question that might help you… A here (*he writes 0) at the intersection of the axes and moves the pen along the segment 0A*) What does A represent on the graph? (*he moves the pen several times between 0 and A; see Picture 4*)

15 MJ: Marthe.

16 T: This here is 0? We'll only talk about the distance. OK? (*He moves the pen again as in Line 14*)

17 MJ: 1 meter.

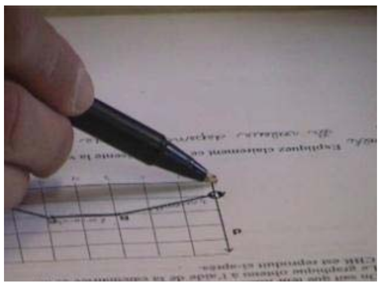
18 T: It represents 1 meter, right? … 1 meter in relation to what?

19 M: The CBR.

20 T: OK. So, does it represent the distance between the two persons?

21 M: So this (moving the pen along the segments) would be Pierre's movement and the CBR is 0.

22 MJ: (*Interrupting*) First he moves more…



Picture 4. The teachers moves the pen back and forth between 0 and A.

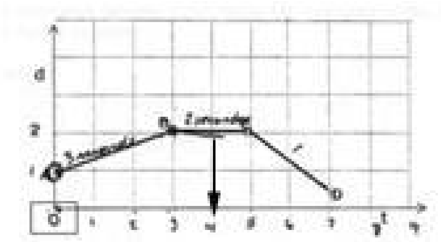
Capitalizing on the emerging idea of relative distance, the teacher's strategy now becomes to call the students' attention to the meaning of a particular segment–the segment 0A. He captures their attention in three related ways: writing (by writing 0 and encircling the point A); gesturing (by moving the pen between 0 and A back and forth); and verbally (L14). In L15, point A is associated with Marthe. In L16, he formulates the question in a more accurate way, and takes advantage of the answer to further emphasize the idea of the relative meaning of the distance. Line 21 includes the awareness that the CBR has to be taken into account, while L22 is the beginning of an attempt at incorporating the new significations into a more comprehensive account of the meaning of the graph.

The students thus entered into a new phase of knowledge objectification. They continued discussing in an intense way. Here is an excerpt:

23 C: He moves away from her, he stops then comes closer.

24 M: But she follows him… So, he goes faster than she does, after, they keep the same distance apart.

In L23, Carla still advocates an interpretation of the graph that suggests a fragile understanding of relativity of motion. In the first part, she makes explicit reference to Marthe ("He moves away from her"), but in the second and third part of the utterance, Marthe remains implicit. In L24, Maribel offers an explanation that overcomes this ambiguity. Even though the segment AB is expressed in terms of rapidity, the previously reached awareness of the effect of rapidity in the increment of distance makes the interpretation of BC coherent. The recapitulation of the students' efforts is made by Maribel, who, before the group start writing their interpretation, says: "Maybe he [Pierre] was at 1 meter (*pointing to A*) and then he went faster; so now he is at a distance of 2 meters (*moving the pen in a vertical direction from BC to a point on the time axis, see Picture 5and Picture 6*); and then they were constant and then (*referring to CD*) they slowed down. Would that make sense?"



Picture 5 and Picture 6. Maribel makes a vertical gesture that goes from BC to the time axis (we have indicated this gesture by an arrow). This gesture is a generalization of the teacher's gesture (Picture 4).

The students succeeded in refining their objectification, although some edges still remained to be polished. In the interpretation of CD, Maribel did not specify in which manner they slowed down. Was it Pierre who slowed down? Was the reduction of distance the effect of Marthe increasing her speed? Was it something else? These questions were discussed in the final general classroom discussion. In writing their answer, this group, however, realized that something important was missing. Naturally, writing requires one to make explicit, and thereby objectify, relationships that may remain implicit at the level of speech and gestures. Maribel's activity sheet contains the following answer: "Pierre moves away from Marthe by walking faster for 3 seconds. He is now 2 meters away from her. They walk at the same speed for two seconds. Pierre slows down for two seconds so he gets closer to Marthe".

CONCLUDING REMARKS

In this paper, we dealt with the students' transformation of meanings related to their understanding of graphs. The investigation of the transformation of meanings was carried out in the course of a process that we have called *objectification*, i.e., a social process related to the manner in which students become progressively aware, through personal deeds and interpretations, of the cultural logic of mathematical entities–in this case, the complex mathematical meanings that lie at the base of the ways in which Cartesian graphs are used to describe some phenomena and convey meanings.

Our data suggests that one of the most important difficulties in understanding the graph was overcoming an interpretation based on a phenomenological reading of the segments in terms of absolute motion, and attaining one that put emphasis on relative relations. Instead of representing the state of an object in reference to a fixed point, points in the second case represented and came to signify relationships between them and a moving point. As we saw, the logic of interpreting a Cartesian representation of relative motion became progressively apparent for the students through intense activity mediated by multiple voices, gestures and mathematical signs. The phenomenological interpretation of the graph was replaced by one centred on relative distances. Crucial in this endeavour was the teacher's intervention. The teacher was indeed able to create a successful *zone of proximal development* that afforded the evolution of meanings both at the discursive and gestural levels. Thus, after his intervention, in the same way that words became more and more refined, so too did gestures: while the students' first gestures were about Pierre's motion, their last gestures were related to distances in a meaningful relational way.

We want to submit that the successful creation of a *zone of proximal development* was due to the teacher's ability to find a common conceptual ground for the evolution of the students' meanings. The teacher brought out the students' meanings from behind, as it were, and helped them push their meanings beyond their initial locations. The coordination of words with the sequence of similar gestures and signs in the Cartesian graph (Pictures 4) helped the students understand the meaning of the segment 0A in the context of the problem. The segment 0A entered the universe of discourse and gesture, and its length started being considered as the initial distance between Pierre and Marthe at the beginning of their walk. Without teaching the meaning directly, the teacher's interactional analysis of the meaning of segment 0A was understood and generalized by the students in a creative way (Picture 5).

Borrowing a term from M. M. Bakhtin, we want to call the transformative process undergone by the students' meanings as *heteroglossic*, in that heteroglossia, as we intend the term here, refers to a locus where differing views and forces first collide, but under the auspices of one or more voices (the teacher's or those of other students'), they momentarily become resolved at a new cultural-conceptual level, awaiting nonetheless new forms of divergence and resistance.

Endnote

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