24 Contrasts and oblique connections between historical conceptual developments and classroom learning in mathematics[1]

Fulvia Furinghetti
Università di Genova

Luis Radford
Université Laurentienne, Canada

1. INTRODUCTION

Since the end of the 19th century, mathematics educators have made use of the history of mathematics in a variety of ways (Cajori, 1894; Zeuthen, 1902.) For instance, the history of mathematics has been used as a powerful tool to counter teachers' and students' widespread perception that mathematical truths and methods have never been disputed. The biographies of several mathematicians have been a source of motivation for students. By stressing how certain mathematical theories flourished in various countries, the diverse contributions of many cultures to contemporary mathematics become evident. Specialized study groups have emerged in the past years in educational circles. Two of these are the Commission INTER-IREM *Épistémologie et Histoire des Mathématiques* in France, and the *International Study Group on the Relations between the History and Pedagogy of Mathematics* (HPM), which is an affiliate of the International Commission on Mathematical Instruction (ICMI). In addition, regular conferences are organized, such as the European Summer Universities on the History and the Epistemology in Mathematics Education (for proceedings see Lalande, Jaboeuf, & Nouazé, 1995; Lagarto, Vieira, & Veloso, 1996; Radelet-de-Grave, & Brichard, 2001; Furinghetti, Kaijser, & Tzanakis, 2006). Concomitantly, an important number of books are now available to help teachers use the history of mathematics (e.g., Bekken, & Mosvold, 2003; Calinger, 1996; Chabert, Barbin, Guillemot, Michel-Pajus, Borowczyk, Djebbar, & Martzloff, 1994; Dhombres, Dahan-Dalmedico, Bkouche, Houzel, & Guillemot, 1987; Fauvel, & van Maanen, 2000; Katz, 2000; Reimer, & Reimer, 1995; Swetz, Fauvel, Bekken, Johansson, & Katz, 1995). Journals of mathematics education have published special issues on history in mathematics teaching (e.g., *For the Learning of Mathematics*, *Mathematics in School*, *Mathematics Teacher*, *Mediterranean Journal for Research in Mathematics Education, Educational Studies in Mathematics*).

Instead of offering an overview of the different domains where the pedagogical use of the history of mathematics is now ramified, we want, in this chapter, to focus on something that Cajori (1894) initiated many years ago. That is, seeing history not only as a window from which to draw a better knowledge of the nature of mathematics but as a means of transforming the teaching of the subject itself. The specificity of this pedagogical use of history is that it interweaves our knowledge of past conceptual developments with the design of classroom activities, the goal of which is to enhance the students' development of mathematical thinking.

Cajori's 1894 ideas have led us to developments that he could not have suspected. Indeed, Cajori adopted a positivistic view of the formation of knowledge. He saw knowledge as an objective entity that grows gradually and cumulatively. His reading of the history of mathematics was framed by viewing history as an unfolding process guaranteed by the idea of progress—an idea underpinning the Enlightenment philosophy and attitudes toward life from which modern thought arose. Nonpositivistic views about the formation of knowledge were later elaborated by philosophers and epistemologists such as Bachelard (1986), Foucault (1966) and Piaget (1970), among others, and by anthropologists such as Durkheim (1968), Lévy-Bruhl (1928) and Lévi-Strauss (1962), to mention but a few. Bachelard presented an interpretation of the formation of knowledge in terms of ruptures and discontinuities. Piaget was interested in explaining genetic developments in terms of stages and the intellectual mechanisms allowing for the passage from one level to another. Foucault was opposed to the conception of history as a date-labeling practice and investigated the problem of the constitution of knowledge in terms of the conditions of its emergence, which he related to the different spheres of human activity. Bachelard, Foucault, and Piaget had different goals, and thus their projects differed. But what is important for our discussion here is that, contrary to what Cajori and many other positivist thinkers believed, knowledge in general and mathematical knowledge in particular cannot be taken as an unproblematic concept. Behind any concept of knowledge there is an epistemological stance, and this epistemological stance conditions our understanding of the formation of students' mathematical thinking, just as it conditions the interpretation of historical conceptual developments (Grugnetti, & Rogers, 2000; D'Amore, Radford, & Bagni, 2006; Radford, Boero, & Vasco, 2000). Nevertheless, the study of the development of students' thinking and that of the conceptual development of mathematics belong to two different domains—the psychological and the historical, respectively. Each has its specific problems as well as the tools with which to investigate them. Students' conceptualizations can be investigated through classroom observations, interviews, tests, and so forth. The same cannot be done in the historical domain, where historical records are the only available material for study. The difference in methodology in both domains is, in fact, a token of more profound differences. These cannot be ignored in the context of a pedagogical use of the history of mathematics as a useful tool to enhance the development of students' mathematical thinking. Despite their differences, the psychological and historical domains need to be weighed and articulated in a specific way (Fried, 2006; Schubring, 2000). One of today's more controversial themes concerns the terms in which such an articulation must be understood. More specifically, the question is how to relate the development of students' mathematical thinking to historical conceptual mathematical developments. Psychological recapitulation, which transposes the biological law of recapitulation, claims that, in their intellectual development, our students naturally traverse more or less the same stages as mankind once did. Very often, this law has been taken for granted (sometimes implicitly) to justify a link between both domains. In its different variants, however, psychological recapitulation has recently been subject to a deep revision, in part because of the emergence of new conceptions about the role of culture in the way we come to know and think. Schubring (2006) and Radford (1997a) have drawn attention to the role played by conceptions about history in shaping the view of the relation between biological and psychological recapitulation. Anglin (1992) and Grattan-Guinness (1993) have brought to light the issue of the nature of the history of mathematics and its relationship with mathematics. Rubin (2001) has discussed the problem of the teaching of history. The previous problems deserve due reflection since their solutions may affect the use of the history of mathematics in teaching and the interpretation of didactical phenomena.

The purpose of this chapter is to discuss in some detail the basic problems referred to in this introduction. In the next section, we deal with psychological recapitulation and mention some of the current arguments against it. In section 3, we examine key ideas about ontogenesis and phylogenesis as found in the works of Piaget and Vygotsky. In section 4, we present some paradigmatic examples of mathematicians who commented on phylogenesis and its relation to ontogenesis. Section 5 focuses on a particular interpretation of the recapitulation law that led to the so-called "genetic approach," which has had a considerable impact on early mathematics education. In section 6, we discuss some episodes that suggest an oblique connection between historical conceptual developments and the learning of mathematics in the classroom. These examples therefore run counter to Cajori's positivist view of knowledge and its recapitulation. Section 7 provides a brief account of a few current approaches in contemporary mathematics education that relate to the history of mathematics regarding either theoretical or practical links between the development of students' mathematical thinking and historical conceptual developments. In the last section, we offer a critical assessment of the law of recapitulation and recommend ideas for conceptual and applied research in the 21st century regarding historical and ontogenetic developments in mathematics education.

2. FROM BIOLOGICAL TO PSYCHOLOGICAL RECAPITULATION

The way in which people perceived psychological recapitulation at the beginning of the 20th century was linked to the way they perceived themselves in the overall view of the world. As long as humans thought of themselves as essentially different from animals and plants, no relation in terms of ancestry between the human and animal kingdoms could be advocated. Thus, in the early 18th century, a common scholarly view to explain the origin of species and to understand the formation of living things was—as indicated in Genesis (see, e.g., Osborn, 1929)—that species came from those beings fortunate enough to survive the deluge by finding refuge on Noah's ark. But with the appearance of the early 19th-century philosophy of nature, humans came to join the greater kingdom of species.

In their broader sense, however, recapitulationist ideas date back to the pre-Socratic thinkers. They did not state them in terms of a condensed process of lower life that culminates with humans. Often their reference point was the cosmos. Thus, Empedocles believed that the growth of the embryo echoes, in a foreshortened way, the cosmogonic process: The embryo is submerged into amniotic fluid, which evokes the originally fluid earth (de Santillana, 1961, p. 114).

During the 18th and early 19th centuries, a vigorous debate separated two opposing schools with regard to the concept of recapitulation. One of them, which became known as *pre-formation theory*, stated that ontogenesis was the unfolding or growing of *preformed* structures, whereas the other school adopted a more dynamic stance, arguing that the embryo was neither the exact miniature of the developed species nor the unfolding of preformed structures, but a being in a state of *development*. The "causes" at the origin of the embryo's unfolding or changes were variously interpreted. Charles Bonnet (1720–1793), usually recognized as one of the leaders of the preformationists, saw change as coming from an affectionate God who had ordered the world according to increasing perfection and progress. Whereas in the early 19th century Naturphilosophen were attributing development to a "natural" final cause, Lamarck and Darwin envisioned a new theory that replaced the philosophical idea of final cause with an efficient cause - individual development. (For a detailed discussion of the ideas of the preformationists and Naturphilosophen, see Gould, 1977.) Indeed, from the mid-19th century onward, the "causes" were seen in the context of the theory of evolution. "Heredity and adaptation are, in fact, the two constructive physiological functions of living things," wrote Haeckel (1912, p. 6), who, in one of the most famous statements ever made in the realm of anthropogenesis (which he called the fundamental law of biogeny), declared that

> The series of forms through which the individual organism passes during its development from the ovum to the complete bodily structure is a brief, condensed repetition of the long series of forms which the animal ancestors of the said organism, or the ancestral forms of the species, have passed through from the earliest period of organic life down to the present day. (pp. 2–3)

Haeckel's law was more than the simple statement of a condensed repetition of steps. What he was suggesting was that the embryos of men and dogs, at a certain stage of their development, are almost indistinguishable. Indeed, to take one of Haeckel's favorite examples, "the human gill slits *are* (literally) the adult features of an ancestor" (Gould, 1977, p. 7).

How, then, was the discussion about the biological growth of humans transferred to the psychological domain? It was Haeckel who, after discussing the nervous system, said, "we are enabled, by this story of the evolution of the nervous system, to understand at length *the natural development of the human mind* and its gradual unfolding" (1912, p. 8, italics as in the original). A sharper formulation was the following: "the psychic development of the child is but a brief repetition of the phylogenetic evolution" (Haeckel quoted by Mengal, 1993, p. 94).

The adoption of the psychological version of biological recapitulation served as a general framework for conceiving the functioning of the child psyche as something traveling along the same path as his or her ancestors. For instance, the child was seen as behaving as humans in previous stages of the chain of evolution (e.g., such as having, in an early stage of his or her development, an "animist" view of nature, that is, that immaterial forces animate the universe).

Psychological recapitulation endorses a peculiar view of history and development. Concerning development, for Bonnet and the preformists, there was no actual development, strictly speaking, but only growing or unfolding. Environment could not alter the pre-formed structures and their growth. For evolutionary-based recapitulation theories, in contrast, the environment is supposed to play a role in the development of species. The individual is seen as an organism adapting to his or her environment; in the interplay between individual and environment, some biological and psychological functions may develop, whereas others may be lost according to natural selection.

As for history, in contrast to views that conceived of a world that underwent different creations, Bonnet saw the world as created at one time, with its whole history encapsulated within it. History was therefore the unfolding of a predetermined plan. The concept of history was much more problematic for recapitulationists. Indeed, from a theoretical point of view, history and recapitulation become difficult to reconcile because, on one hand, Haeckel's psychological recapitulation supposes that present intellectual developments are to some extent a condensed version of those of the past. On the other hand, natural selection is presented as a function of the environment against which individuals act. For recapitulation to be possible, therefore, such an environment must remain essentially the same, which obviously is not the case. Given that the environment changes, it becomes difficult to maintain that children's intellectual development will undergo the same process as the one experienced by children in the past. The variability that natural selection imposes on the course of events in history conflicts with the idea of recapitulation as a condensed repetition of some intellectual aspects registered in past history. Indeed, this point was recognized as a weakness (see, e.g., Gould's (1979) Lamarckian remarks). Werner (1957), for instance, advocated contextual factors and argued that it is impossible to equate a given intellectual stage of a child in a modern society to the stage an adult could have reached in a ancient society because the respective environments, as well as the genetic processes involved in them, are completely different (see Radford, 1997a). Elias also mentioned the differences that necessarily result as a consequence of variations in cultural settings. Whereas in "traditional" societies children participate directly in the life of adults earlier and their learning is done *in situ* (as apprentices), "modern" children are instructed indirectly in mediating institutions, or schools (Elias, 1991, pp. 66–67). Consider memory, an example that is addressed neither by Werner nor Elias but which conveniently clarifies the previous ideas. As many anthropological accounts clearly show (see, e.g., Lévy-Bruhl, 1928), memory plays a central role in illiterate societies. In contrast, sign systems related to writing in literate societies dispense with memory to a certain and fundamental extent. They create a different way to handle and distribute knowledge and information between the members of the society and shape attitudes about how to scrutinize the future (see Lotman, 1990).

The theoretical difficulties encompassing the crude version of psychological recapitulation encouraged new reflections with the goal of finding more suitable explanations concerning the relationship between phylogenesis and ontogenesis. Contemporary societies organize the mobilization of knowledge in such a way that their individuals are faced with a quicker discovery of ontogenetic objects. The advent of technology, like, for instance, in dynamic geometry environments, provides a clear example of these ideas (see Moreno & Sriraman, 2005). In the next section, we will discuss two different views that have been influential on the use of history in mathematics education.

3. PIAGET AND VYGOTSKY ON ONTOGENESIS AND PHYLOGENESIS

Piaget was interested in understanding the process of the formation of knowledge. To do so, he considered knowledge as something that can be described in terms of levels. One of his aims was to describe those levels, as well as the passage from one level to a more complex one. He said, "The study of such transformations of knowledge, the progressive adjustment of knowledge, is what I call genetic epistemology" (Piaget cited in Bringuier, 1980, p. 7). As a reaction to the simplistic psychological version of recapitulation and the positivist view of knowledge that we mentioned in the introduction, Piaget and Garcia elaborated the concept of *genetic development*. They envisioned the problem of knowledge in terms of the intellectual instruments and mechanisms allowing its acquisition. According to Piaget and Garcia, the first of those mechanisms is a general process that accounts for the individual's assimilation and integration of what is new on the basis of his or her previous knowledge. In addition to the assimilation mechanism, they identified a second mechanism, a process that leads from the *intraobject*, or analysis of objects, to the *interobject*, or analysis of the transformations and relations of objects, to the *transobject*, or construction of structures. This epistemological viewpoint led them to revisit the parallelism that recapitulationists had emphasized. Therefore, Piaget concluded, "We mustn't exaggerate the parallel between history and the individual development, but in broad outline there certainly are stages that are the same" (Bringuier, 1980, p. 48). The two mechanisms were hence considered as invariable, not only in time but also in space. That is, we do not have to specify what they are in a certain geographical space at a particular time because they do not change from place to place or from time to time. They are exactly the same, regardless of the period in history and the place of the individuals.

> In modern mathematics, at the level of algebraic geometry, of quantum mechanics, although it's a much higher level of abstraction, you find the same mechanisms in action—the processes of the development of knowledge or the cognitive system are constructed according to the same kinds of evolutionary laws. (Garcia in Bringuier, 1980, pp. 101–102)

Thus, when Piaget and Garcia investigated the relationship between ontogenesis and phylogenesis, they did not seek a parallelism of contents between historical and psychogenetical developments but of the mechanisms of the passage from one historical period to the next. They tried to show that those mechanisms are analogous to those of the passage from one psychogenetic stage to the next.

The two mechanisms of passage discussed by Piaget and Garcia have a different theoretical background. The second, that of the intra-, inter- and transobjectual relations, obeys a structural conception of knowledge and reflects the role that mathematical and scientific thinking played in Piaget's work. As Walkerdine noted, "In the work of Piaget, an evolutionary model was used in which scientific and mathematical reasoning were understood as the pinnacle of an evolutionary process of adaptation" (Walkerdine, 1997, p. 59). The first one, the assimilation mechanism, has its roots in the conception of knowledge as the prolongation of the biological nature of individuals: "The human mind is a product of biological organization, a refined and superior product, but still a product like another" (Piaget in Bringuier, 1980, p. 108).

Both intellectual mechanisms of knowledge development embody a general conception of rationality that has been contested by some critics who find that what is missing, among other things, is a more vivid role for culture and social practices in the formation of knowledge. For instance, the epistemologist Wartofsky, who has stressed an intimate link between knowledge and the activities from which knowledge arises and is used, said:

> We are, in effect, the products of our own activity, in this way; we transform our own perceptual and cognitive modes, our ways of seeing and of understanding, by means of the representations we make.... […] Piaget's dynamic, or genetic structuralism is important here, of course. His dictum, "no genesis without structure, no structure without genesis," suggests the dialectical interplay of the practical emergence and transformation of structures with the shaping of our experience and thought by structures. But the domain of this genesis I take to be the context of our social, cultural and scientific practice, and not that of biological species-evolution alone.... In a sense, then, our ways of knowing are themselves artifacts which we our-selves have created and changed, using the raw materials of our biological inheritance. (Wartofsky, 1979, p. xxiii)

Vygotsky, in many writings, dealt with the problem of recapitulation and, like Piaget, believed that the understanding of ontogenesis and phylogenesis had to be based on a deep understanding of our biological nature. (This is clear, for instance, in his book *Speech and Thinking*, as well as in the influence he had on his student Luria and the huge amount of physiological research that the latter conducted. See, e.g., Luria, 1966.) Instead of posing the problem of the formation of knowledge in terms of universal and atemporal mechanisms functioning beyond culture, however, he saw the cognitive functions allowing the production of knowledge as inevitably overlapping with the context in which individuals act and live. His basic distinction between lower and higher mental functions is reinforced by the idea that the former belong to the sphere of the biological structure, whereas the latter are intrinsically social. Thus, in a passage from *Tool and Symbol in Child Development*, when discussing the problem of the history of the higher psychological functions, Vygotsky and Luria commented:

> Within this general process of development two qualitatively original main lines can already be distinguished: the line of biological formation of elementary processes and the line of the socio-cultural formation of the higher psychological functions; the real history of child behaviour is born from the interweaving of these two lines. (Vygotsky, & Luria, 1994, p. 148)

The merging of the natural and the sociocultural lines of development in the intellectual development of the child definitely precludes any recapitulation:

> In the development of the child, two types of mental development are represented (not repeated) which we find in an isolated form in phylogenesis: biological and historical, or natural and cultural development of behavior. In ontogenesis both processes have their analogs (not parallels).... By this, we do not mean to say that ontogenesis in any form or degree repeats or produces phylogenesis or is its parallel. We have in mind something completely different which only by lazy thinking could be taken to be a return to the reasoning of biogenetic law. (Vygotsky, 1997, p. 19)

For Vygotsky even the elementary intellectual functions of the individual are intrinsically human, acquired through the activities and actions on which the intercourse between individuals and between people and objects are based. One of the central reasons for this is that human activities are mediated by diverse kinds of tools, artifacts, languages and other systems of signs which, Vygotsky argued, are a constitutive part of our cognitive functions. Most important, these systems of signs, as well as tools and artifacts, are much more than technical aids: They modify our cognitive functioning (for a discussion of this point, see Bartolini-Bussi & Mariotti's, chapter 28, this volume). The knowledge produced by individuals hence becomes intimately related to the activities out of which knowledge arises and the conceptual and material "cultural tool kit" (to borrow Bruner's expression, see Bruner, 1990) with which individuals are equipped. Of course, this does not mean that with every new generation, all knowledge must be constructed anew. As Tulviste (1991) noted, whereas rats are still doing what they did centuries ago, humans have, from one generation to the next, acquired, produced, and passed on their knowledge. During this process, humans have changed their activities and the way in which they think about the world. In Vygotsky's view, knowledge appears as a creative individual and social reappropriation and coconstruction carried out using conceptual and material tools that culture makes available to its individuals. In turn, in the course of this process, the previous tools and signs may become modified and new ones may be created. It is in this sense that tools and concepts have embodied the social characteristics from which they arose, and their insertion into other activities allows for their transformation and eventually their growth. Because activities, sign use, and attitudes toward the meaning of scientific inquiry do not necessarily remain the same throughout time, changes are effected along phylogenetic lines (and the plural, "lines," needs to be emphasized here) serving as the historico-cultural starting point for new genetic developments. Epistemological reflections have then to evidence the relation between cognitive context and action. As Wartofsky pointed out:

> If, in fact, our modes of cognitive practice change with changes in our modes of production, of social organization, of technology and technique, then the connection between cognition and action, between theoretical and applied practice, between consciousness and conduct, has to be shown. (Wartofsky, 1979, p. xxii)

One implication of the previous remarks for the use of the history of mathematics in education is that the study of recapitulation can be advantageously replaced by the contextual study of the social elements in which the historical geneses of concepts are subsumed. A prerequisite for doing this seems to be a clearer understanding of what the anthropologist L. White refers to as the locus of mathematical reality and summarizes by saying that "mathematics in its entirety, its 'truths' and its 'realities', is a part of human culture, nothing more" (White, 2006, p. 307). In this line of thought, the contextual study of the social elements in which the historical geneses of concepts are subsumed can be accomplished through a careful investigation of the cultural symbolic webs shaping the form and content of scientific inquiry and the ways in which mathematical concepts are semiotically represented (Radford, 1997a, 1998, 2000a, 2003a, 2003b, 2006a, 2006b). We return to this point in section 7.

4. INTERPRETATION OF RECAPITULATION LAW BY MATHEMATICIANS

During the period when the treatises of Zeuthen and Cajori appeared, the history of mathematics was growing as a scientific discipline. The first journals dealing exclusively with the history of mathematics were appearing at that time. We have extensive evidence that mathematicians were looking at the history of mathematics with great interest for two main reasons. The first one was functional to scientific research in general: the history of a given theory was considered a model inspiring and supporting further studies in the field. In mathematics, the much quoted motto by Abel, "Learn from the masters!" echoes this conviction, as well as the following passage by Eugenio Beltrami, a mathematician well known for his model of a non-Euclidean geometry (see Clebsch, 1873, p. 153, our translation):

> Students should learn to study at an early stage the great works of the great masters instead of making their minds sterile through the everlasting exercises of college, which are of no use whatever, except to produce a new Arcadia where indolence is veiled under the form of useless activity.

The second reason why the mathematicians were looking at the history with great interest was linked to the developments of mathematical research in those years, when axiomatization and the foundational works were undertaken. These themes were turning mathematicians' attention to reflections on the nature of mathematics and on the activity of doing mathematics. The history of mathematics was considered a field that offered inspiration for discussing these kinds of problems. In this context, we will consider some interpretations of recapitulation law made by important mathematicians.

In the first issue (1899) of *L'Enseignement Mathématique*, an important journal devoted to the teaching of mathematics, the eminent mathematician Henri Poincaré clearly stated his position on the relations between conceptual and historical developments:

> Without a doubt, it is difficult for a teacher to teach a reasoning that does not satisfy him completely.... But the teacher's satisfaction is not the sole purpose of teaching... above all one should be concerned with the student's mind and of what we want him to become.
>
> Zoologists claim that the embryonal development of animals summarizes in a very short time all the history of its ancestors of geologic epochs. It seems that the same happens to the mind's development. The educators' task is to make children follow the path that was followed by their fathers, passing quickly through certain stages without eliminating any of them. In this way, the history of sciences has to be our guide. (Poincaré, 1899, p. 159; our translation)

Poincaré gave examples of concepts to be taught at an intuitive stage before presenting them rigorously. Among these examples were fractions, continuity and area. As far as we know, Poincaré never used his ideas on the efficacy of recapitulation law directly with teachers. This makes Poincaré's position different from that of Felix Klein, another supporter of the use of the history of mathematics in teaching. In contrast, Klein applied his ideas in courses for prospective teachers and in related texts that he wrote.

Klein supported the German translation of the famous book *A Study of Mathematical Education* by Benchara Branford (1921) in which, according to Fauvel (1991, p. 3), the theory of recapitulation "reached its apogee." This can be considered evidence of Klein's agreement with the recapitulation law (Fauvel, 1991, p. 3). Nevertheless, from what Klein wrote in his articles and books (see Klein, 1924), we understand that the application of the law was not advocated in a literal sense. As in the case of Poincaré, his opinion on the use of history was born of his wish to abolish the use of mathematical logic and the excesses of rigor advocated by some of his colleagues. Klein was interested in the dichotomy of "intuition versus rigor" and, as far as school is concerned, was in favor of intuition. He singled out the history of mathematics as being the suitable context for bringing intuition back into the teaching and learning process:

> I maintain that mathematical intuition ... is always far in advance of logical reasoning and covers a wider field.... I might now introduce a historical excursus, showing that in the development of most of the branches of our science [mathematics], intuition was the starting point, while logical treatment followed. This holds in fact, not only of the origin of the infinitesimal calculus as a whole [this issue was discussed at the beginning of Klein's paper] but also of many subjects that have come into existence only in the present [19th] century. (Klein, 1896, p. 246)

Klein claimed that in school, as well as in research, the phase of formalization must be preceded by a phase of exploration based on intuition.

We find an analogous statement in a secondary school geometry book written by the famous Italian mathematician, Francesco Severi, which clearly refers to school practice:

> We need to take inspiration from the principle that in learning new notions, the mind tends to follow a process analogous to that according to which science has developed. One who is aware of the value of foundation theories [in Italian, *critica dei principi*] does not make the dangerous mistake of giving to elementary teaching a critical and excessively abstract style. It is necessary to know foundation theories for personal intellectual maturity; but in elementary teaching they are not to be considered as a pedagogical means. (Severi, 1930, p. IX; our translation)

Both Klein and Severi do not clearly state what "intuition" means for them, but both state what intuition is opposed to: rigor, excessive abstraction and formal logic used at the beginning of the presentation of a mathematical notion. (It may be interesting to note that Severi, famous during the first half of the 20th century, is one of the scholars of the Italian school of algebraic geometry who based his results on intuition to such a degree that these were published without being carefully verified by a mathematical proof, as reported by Hanna (1996).

An explicit reference to the recapitulation problems was made by Thom (1973) in his plenary talk delivered in 1972 at the second ICME congress in Exeter.

> Pedagogy must strive to recreate (according to Haeckel's law of recapitulation—ontogenesis recapitulates phylogenesis) the fundamental experiences which, from the dawn of historical time, have given rise to mathematical entities. Of course this is not easy, for one must forget all the cultural elaborations (of which axiomatics is the last) which have been deposited on these mathematical objects, in order to restore their original freshness. One must forget culture in order to return to nature. (p. 206)

Behind this claim lay the rise of modern mathematics and the stormy quarrels between the supporters of its introduction in the school and those who were afraid of the dangers that would come of such a step.

5. THE GENETIC APPROACH

Using the history of mathematics in teaching does not necessarily entail a direct assumption of the recapitulation law; it also may be used in the so-called *genetic approach* to teaching. The term "genetic" is an ambiguous one because it is used with different meanings. In particular, in the foundation literature, the term *genetic method* is used in contrast to *axiomatic method*. David Hilbert probably introduced this term, which was popularized by Edward V. Huntington. Before Hilbert, we find other uses of the word "genetic." Immanuel Kant stated that all mathematical definitions are genetic; after Kant, the term "genetic definition" is present in all major treatises on logic.

In addition to its use among mathematicians and philosophers, we find the word "genetic" in other fields of research. Piaget and Garcia used it in their epistemological studies. As for mathematics education, Ed Dubinsky, who dealt with genetic decomposition, used the word.

Here we are concerned with the word "genetic" as it is used in connection with history. In the 1920s, the idea of a genetic principle was taking shape, as evidenced by the work of N. A. Izvolsky.[2]

Gusev and Safuanov (2000) reported Izvolsk's complaints against traditional teachers and textbooks on the grounds of their legendary indifference to explanations about the origins of geometrical theorems. Izvolsky argued that when attempts to link knowledge to its historical roots are made, students see geometry in a different way. Moreover, students themselves sometimes guess that a given theorem was not originated by a mere wish on the part of the teacher or the authors of the textbook, but by questions which arose in previous tasks. It so happens that students try to imagine the origin of a theorem. According to Izvolsky, even if their hypotheses are not correct from a historical point of view, this approach to the teaching of geometry is valuable (Gusev, & Safuanov, 2000, p. 22).

The idea of a genetic approach later took a definite form in a work by Otto Toeplitz which he wrote to describe a method of presenting analysis to university students.[3] The following passage from (Toeplitz, 1963) illustrates the ideas underlying the genetic method:

> Regarding all these basic topics in infinitesimal calculus which we teach today as canonical requisites, e.g., mean-value theorem, Taylor series, the concept of convergence, the definite integral, and the differential quotient itself, the question is never raised "Why so?" or "How does one arrive at them?" Yet all these matters must at one time have been goals of an urgent quest, answers to burning questions, at the time, namely, when they were created. If we were to go back to the origins of these ideas, they would lose that dead appearance of cut and dried facts and instead take on fresh and vibrant life again.[4]

Burn (1999) explained Toeplitz's ideas in the following way:

> The question which Toeplitz was addressing was the question of how to remain rigorous in one's mathematical exposition and the teaching structure while at the same time unpacking a deductive presentation far enough to let a learner meet the ideas in a developmental sequence and not just in a logical sequence. While the genetic method depends on careful historical scholarship it is not itself the study of history. For it is selective in its choice of history, and it uses modern symbolism and terminology (which of course have their own genesis) without restraint. (Burn, 1999, p. 8)

It is not by chance that this alternative approach developed in the domain of teaching calculus. It is in this domain that the notion that learning mathematics takes place in a sequence predetermined by mathematical logic has shown its pedagogical limitations. Indeed, when organized around their logical basis, the definitions of the main concepts of calculus (integrals, limits, derivatives) are abstract, and therein lies the burden of formal rules and theorems. Students have difficulty grasping the corresponding meanings. At present there are projects (not based on history) that take these difficulties into account and organize the teaching of calculus according to different patterns. An example is the Harvard project based on giving an informal, operative approach to concepts (see Hughes-Hallett et al., 2005). Since his early studies on the teaching/learning of calculus, Tall has followed a similar approach, strongly supported by information technology (see Tall, 2003.)

What Toeplitz (1963) proposed is realistic and may be considered a compromise between the two ways of thinking about teaching mathematics, the logical versus the developmental. The goal of Toeplitz's historically based approach was to provide a progressive process of understanding that the student performs through a sequence of steps. Because Toeplitz's aim was to provide teaching materials that facilitate the learning of calculus, the main concern of the author was not to teach history, but to find learning sequences. Burn (1993, 1999, 2005) elaborated on these ideas. Through his work he is sustained by the view "that learning or growth in mathematics consists of a transition from experience of the particular, through pattern recognition or problem-solving, to perception of a generic" (Burn, 2005, p. 269). The historical development is suitable to his purpose

> not because history is an infallible guide to student development today, but because where current psychological research does not map out a path consonant with student intuitions, historical enquiry can reveal actual steps of success in learning. There is no attempt in this paper to rewrite history, only a desire to use history to identify possible developmental steps. (Burn, 2005, p. 271)

Like in Toeplitz's work, the presentation of historical materials is not shaped according to recapitulationist principles because it uses modern symbols, verbal expressions and cultural tools that are different from those of past authors.

An older example of the use of the genetic method (intertwined with a naïve heuristic approach) is in the treatise on geometry by Alexis-Claude Clairaut (1771). The preface of his book is an early example of predidactic literature (see Barbin, 1991). He wrote:

> Even if geometry is abstract in itself, we nonetheless must agree that the difficulties suffered by beginners come mostly from the way it is taught in usual treatises. They always start with a great deal of definitions, questions, axioms, and preliminary principles, which only seem to promise dry issues for readers.... To avoid this dry quality that is naturally linked to the study of geometry, some authors put examples after each proposition to show it is possible to do them; but in this way, they only prove the usefulness of geometry without making it any easier to learn. Because each proposition is presented before its use, the mind reaches concrete ideas after having toiled with abstract ideas. Having realized this fact, I intended to find out what may have given birth to geometry and tried to explain principles with the most natural methods, which I suppose were adopted by the first inventors, while trying to avoid the wrong attempts they had necessarily made. (Clairaut, 1771, pp. 2–4; our translation)

According to Glaeser (1983), Clairaut contributed greatly to the introduction of the genetic method. Glaeser commented on Clairaut's work with the following observations: "Having given up both dogmatic exposition, and the following of the true historical development of discovery, this method consists in imagining a road that learned peoples 'could have followed!' Thus this is pedagogy-fiction" (Glaeser, 1983, p. 341, our translation). If we compare the passage from Toeplitz's book and Clairaut's passage, we see a strong coincidence of intentions and didactic observations (i.e., the common idea of the "dryness" of mathematical content from the learner's perspective).

Toeplitz's work on the genetic method was probably rediscovered some decades later, as observed by Kronfellner (2002), because of the need to create a counter-current to the New Mathematics movement and its focus on rigor.

Freudenthal (1973) provided an interpretation of the genetic method through his method of "guided reinvention:"

> Urging that ideas are taught genetically does not mean that they should be presented in the order in which they arose, not even with all the deadlocks closed and all the detours cut out. What the blind invented and discovered, the sighted afterwards can tell how it should have been discovered if there had been teachers who had known what we know now.... It is not the historical footprints of the inventor we should follow but an improved and better *guided* course of history. (Freudenthal, 1973, pp. 101, 103; our italics)

An example of application based on Freudenthal is provided by the project Reinvention of Algebra, which uses "informal, pre-algebraic methods of reasoning and symbolizing as a way to facilitate the transition from an arithmetical to an algebraic mode of problem solving" (see Von Amerom, 2001, p. 239).

All in all, the genetic approach continues to be used in recent research (see e.g., Farmaki & Paschos, 2007; see also Törner & Sriraman, 2005), reflecting, one way or another, the view that mathematical knowledge has a history and that its appropriation by the students entails a genetic process, a process that requires the students' active engagement with the objects of knowledge.

6. UNPACKING A HISTORICALLY AND CULTURALLY CONSTRUCTED KNOWLEDGE: EPISODES IN THE CLASSROOM

In our previous chapter on recapitulation (see Furinghetti, & Radford, 2002, p. 642), we reported the claims made by teachers who were required to express their opinions about the potential benefits of using the history of mathematics in teaching:

- The students' development of concepts follows the historical sequence,
- The historical genesis of the concept may help teachers understand the genesis of the concept in students' minds,
- If I present the students with how algebra developed in history, they feel differently about their difficulties in learning it.

These claims are paradigmatic of three ways in which teachers look at history in the teaching of mathematics. The third claim refers to what we may term the "consoling" function of history that is epitomized by the following sentence written by a teacher describing his work on medieval problems about probability (see Paola, 1998, for a fuller account): "The incursion into history had the goal of giving dignity to the mistake made by students [in solving a problem of probability solved also by Luca Pacioli]: it was not a trivial mistake if a mathematician made it" (p. 33).

The first and the second claims echo the idea of parallelism and deserve further discussion from the perspective of recapitulation theory. One main origin of these claims stems from the way teachers get in touch with the history of mathematics. Usually they do not read primary sources and their knowledge of history comes from manuals about the history of mathematics or university courses. These sources of historical knowledge may convey a view of the development of mathematics focused on the final result of the development, that is, on a polished theory in which attempts and failures have been dropped as steps deprived of knowledge value. Borrowing the metaphor quoted in Pizzamiglio (1999) we may say that this mode of viewing mathematical development echoes theatrical tragedy: it encompasses the inexorability of a consequential order, as that pursued in a systemic exposition of knowledge. This opposes a genre like epic poetry where all possible results as well as the complex and contended realizations of some of them are considered. In the courses for prospective teachers described in Furinghetti (2007), we encouraged the participants to use the history of mathematics to produce teaching sequences. We found two modes working for teachers: the evolutionary and the situated. The first mode was mainly observed in those who rely on historical manuals: the participants were inclined to identify the elements that made a certain stream of thinking dominant and their teaching sequences focused on the polished theory. The other mode affecting prospective teachers' products was brought out mainly by reading original passages from selected historical authors. Understanding was focused on the historical context and the focus was on the roots of concepts.

Other reasons for the idea of parallelism hinted at by the teachers' answers come from reflections on their own experience in the classroom. Von Amerom (2001, p. 239), in her comment about the work on reinventing algebra, observed that "knowledge of the historical development of algebra has led to a sharper analysis of student work and the discovery of certain parallels between contemporary and historical methods of symbolizing." Not only teachers, but also researchers, have intimated possible connections between historical developments and students' ways of approaching concepts, in particular, as far as students' difficulties and mistakes are concerned (see Bagni 2000a, 2000b, 2005, 2006; Dubinsky et al. 2005a, 2005b; McGinn & Boote, 2003)

In the following, we report episodes from the classroom that could contribute to raising teachers' perceptions of oblique connections between historical and contemporary conceptualization. As we shall see, these connections cannot be seen as a form of recapitulative parallelism but rather as a kind of *unpacking* of a historically and culturally constructed knowledge (Radford, & Puig, 2007).

6.1. First episode

The first episode focuses on a Grade 9 pupil (Giulia) who is studying algebra in school; in particular, she knows how to solve first degree equations, but is not used to translating word problems into algebraic language. The following problem, taken from the medieval treatise of arithmetic by Paolo dell'Abbaco (1964, p. 44), was proposed to her:

> A gentleman asked his servant to bring him 7 apples from the garden. He said: "You will meet 3 doorkeepers and each of them will ask you half of all the apples plus two taken from the remaining apples." How many apples must the servant pluck if he wishes to have 7 apples left? (our translation)

The problem was presented to the pupil in the original version which is written in an old- fashioned Italian with words and a syntax no longer used in the current language. The following solution provided by Paolo dell'Abbaco was not shown to the pupil:

> Act in this way and say: if the servant wants to have 7 [apples] left, how many apples is it suitable that he has at the last door? And afterwards say: if he wants to have 18 left at the second door, it is expedient that he has 40; and afterwards if he wants to have 40 at the third door it is expedient that he has 84. You have passed three doors and starting with 84, 7 are left. It is done. (our translation)

Paolo dell'Abbaco's solution is given through arithmetic calculations without explanations, since, as is often the case in medieval treatises, the solution does not intend a methodological generalization as we understand it today (Radford, 2003b). The process of arriving at a solution carried out by the pupil, fully described in Furinghetti (2007) consists of the following steps: reading the text of the problem and showing bewilderment because of the unusual form of the problem and the task. After reading the text more carefully, some of the noteworthy steps were:

- drawing a figure and using also the symbolic mode (use of the letter "*x*" to indicate variables as well as iconic representation of three gentlemen)
- an arithmetic attempt with the number 50 (chosen by chance)

$$\frac{50}{2} + 2 = 27$$
$$50 - 27 = 23$$
$$\frac{23}{2} + 2 = 13.5$$
$$23 - 13.5 = 9.5$$

- shifting from arithmetic to algebra. This shift happens by looking at the number 50 in the arithmetic expression as a *generic number*. Radford and Puig (2007) noted that an algebraic equation is in fact like a *diagram* in that an equation exhibits, through algebraic signs, the *relations* between the involved quantities. Here, the arithmetic expressions also work as a diagram in which the relations between the involved quantities emerge. The pupil writes a system of equations with unknowns and auxiliary unknowns

$$\frac{x}{2} + 2 = a$$
$$x - (\frac{x}{2} + 2) = y$$
$$\frac{y}{2} + 2 = b$$
$$y - (\frac{y}{2} + 2) = z$$
$$\frac{z}{2} + 2 = c$$
$$z - (\frac{z}{2} + 2) = 7$$

- scaffolding by the teacher to arrive at the solution

The solution of the ninth-grade pupil was compared with the solutions produced by 15 experts (seven university students and eight prospective teachers with strong mathematics backgrounds). All the students except two used algebraic techniques; one tried to solve by using numerical attempts on a wrong formula, and one used an arithmetic technique. Among those who used algebra, we find solutions involving some algebraic signs, but the way of reasoning is arithmetic-like: the solver starts from the final data (the seven apples left) and goes back to the amount of apples. Below, the arithmetic solution is reported; it is remarkable that the solver, now a prospective teacher, was a very clever university student in mathematics who wrote her dissertation on algebra.

> I find the apples required before passing through the last door. Since the doorkeeper asks half plus 2 apples, the 7 apples are half of the amount less 2. Then, before the last door, the gentleman has 18 apples. I note that 18 is (7×2)+4, then, I deduce that before the second door, the gentleman has (18×2)+4 = 40 apples and thus he must pluck (40×2)+4 = 84 apples.

The use of arithmetic by this student is not due to a lack of knowledge but to the search for a context from which it will be possible to endow the problem-solving process with meaning. The reader may note that her solving process is similar to that of Paolo dell'Abbaco.

6.2. Second episode

The following episode refers to an activity of problem solving carried out in a classroom of students aged about 15, working in groups of three and familiar with performing exploration activities. A set of problems centered on proof was given to the students. The students knew that their performance would not be assessed with a mark. They were only asked to engage in solving the problems as diligently as possible and to write out all their thoughts during the solution. We also asked them to write down the difficulties encountered and if they had enjoyed the problems.[5] For the purposes of this chapter, it is fruitful to focus on the work of a group of three boys (Andrea, Luca, Simone), which hints at interesting connections with historical aspects. The problem was the following:

> Given a cube made of little cubes all equal, take away a full column of little cubes. The number of the remaining little cubes is divisible by six. Try to explain why this happens.

The students started by drawing a cube. First, they tried a cube formed by 3^3 little cubes and went on by alternating explorations of an inductive type (the cases of cubes formed by 4^3, 5^3, etc. little cubes) with reflections on the particular case of the cube they had drawn. The drawing acted as a generic example. The exploration of particular cases even went on after the determination of the formula n^3-n. The solving strategies were a continuous 'back and forth' between consideration of concrete situations (particular cubes and calculations on them) and reasoning with formulas and attempts to write them in different ways. In this phase, the teacher acted in what Vygotsky (1962) termed the *zone of proximal development*. He asked the students which ideas they were relating to divisibility. Simone mentioned multiples, Luca and Andrea decomposition. The new idea of decomposing $n^3–n$ came through a process of abduction (see Otte, 1997). At this point, the teacher suggested using the symbolic calculator to decompose the formula. Immediately after having obtained the decomposition $x(x-–1)(x+1)$, the students verbalized the solution: "Given three consecutive numbers, at least one is even and one is divisible by three."

Andrea, however, was not satisfied with this solution and looked for a different process. One of the reasons for his dissatisfaction could have been the fact that the solution was found through the teacher's intervention and thus, Andrea felt he was not controlling the situation and needed to take possession of the solution. He reflected on his drawing and we saw him make hand gestures and think intensely until he found a new solution based on the decomposition and composition of the original cube and until a parallelepiped was obtained (see Figure 24.1). The teacher asked Andrea to write out how he had reached the new solution and why he had looked for it. He wrote:

> I was not satisfied at all with the decomposition made with the symbolic calculator (I was thinking: Why have I not suddenly thought about factorizing?) [He is referring to the fact that before decomposing $n^3–n$, he had worked a lot around the figure] and I was 'looking at' [The inverted commas are in Andrea's text] the figure, partly to see that 'monster' and partly because I wished to find a geometrical proof [Andrea tries to give meaning to what he was doing. He seems disturbed by the shift from the geometric context of the problem to the algebraic one]. Rather unconsciously—may be by vent—I started to strike off the column in question. When I saw the column struck off, I realized that the two remaining columns should have been moved so that a rectangle [He means, indeed, a parallelepiped] is formed, which is high a column less (x–1), deep equally (x), and large one column more (x+1). Since the formula which gives the volume of the rectangle [parallelepiped] is b·h·p, I wrote x(x–1)(x+1), which was the same as the factorization of the calculator. To better understand my idea, see the sheet [Figure 24.1] with the steps of the operation. (our translation)

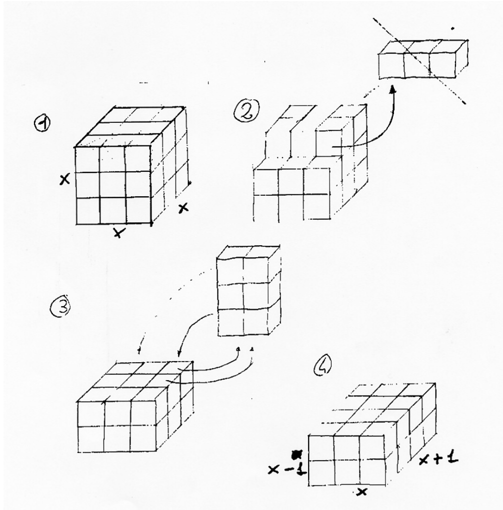


Figure 24.1 Andrea's drawing for illustrating his proving process.

The expression "to look at" suggests that the student's thinking was oriented to a deep understanding of the situation—a proof which explains. The process carried out by Andrea is mainly based on transformational reasoning. This reasoning was enhanced by three different kinds of signs used in an integrated way. We know that Peirce distinguishes between three kinds of signs: (1) icon, that is, something which designates an object on the grounds of being similar to it; (2) index, that is, something which designates an object that points to it in some way; (3) symbol, which designates an object on the grounds of some convention. Andrea uses all these kinds of signs in an integrated way. Initially, the icon (drawing) is the way of paraphrasing the problem. His hand gestures are a means of enhancing transformational reasoning. In the very words written by the student ("I would have wished to find a solution with numbers only"), we see that for him symbols hide meaning, while the drawing is a carrier of meaning. We note that the student operates on his drawing in a symbolic mode. Indeed, he manipulates the pieces of the cube as representatives of the algebraic symbols x, x–1, x+1.

The first mode of solution produced by Andrea's group may be ascribed to an axiomatic-like proof scheme (they "derive" the conclusion that the number of the remaining cubes is a multiple of 6). The second mode belongs to the transformational proof scheme (Andrea "sees" that that number is a multiple of 6). The discrepancy of schemes shown by this student is an evidence of a discrepancy between proofs which prove and proofs which explain (see Hanna, 1990). We found it interesting that in the group, Andrea's two group-mates acted in a different way. They both only worked inside the algebraic frame asking for formal aspects and avoided reference to concrete situations.

The process conceived by Andrea resembles the 'cut and paste' process carried out by al-Khwarizmi (1838) for solving second degree equations. In the case of the equation $x^2+10x = 39$, al-Khwarizmi starts from a square of side x, sticks on the four sides four rectangles of sides 10/4 and x. He obtains a cross (see Figure 24.2) whose area is x^2+10x (which is equal to 39). Four squares of side 10/4 are added to the cross to obtain the final square whose area $(x+10/2)^2$ is equal to $39+4(10/4)^2$. By equalizing these quantities, the usual solving formula for second degree equations follows. Al-Khwarizmi was only interested in positive solutions.

We interpret this episode as a type of oblique connection between methods and forms of attending mathematically to some problems in the history of mathematics and contemporary learning. But again, we do not see the connection as a recapitulation of knowledge on the ontogenetic level. Before we discuss this point further, let us look at the third episode.

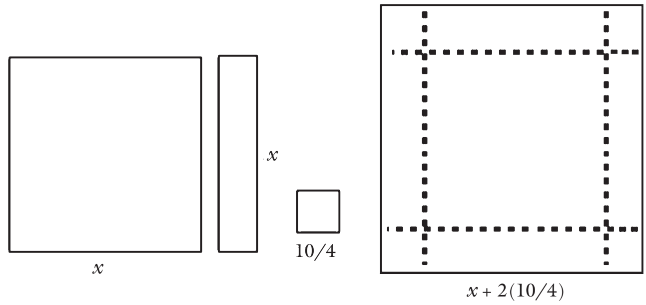


Figure 24.2 Al-Khwarizmi's process for solving second degree equations.

6.3. Third episode

As a final example, we report a case in which the construction of a mathematical object (the concept of maximum) in the classroom occurs through the embodiment of objects (via perception on the computer screen) and metaphorical language. We shall link this case to a passage by Fermat.[6]

In the mathematics laboratory, in which students use the dynamic geometric software Cabri, the following problem was assigned: "How does the area of a rectangle vary when the perimeter is constant? Take, for example, 12 as the value of the perimeter." Grade 10 students engaged in the experiment; they were rather good at using Cabri, but did not know calculus. In their classroom, exploration was regularly used as a method of approaching conjectures and proof; the work was carried out in groups. They began by drawing with Cabri a segment line *AB* of length 6 (half perimeter) and took a point *P* on it (*AP* and *PB* represent the lengths of the sides of a rectangle of perimeter 12.) After that, they draw with Cabri a rectangle of sides *AP* and *PB*. In this first phase, the students saw ("perceived") that when *P* moves on the segment *AB*, the rectangle changes. Afterwards, they perceived that the area depends on the length of *AP*. This dependence could be established in a more precise way by the use of a spreadsheet. Some students were able to produce conjectures based on the trend of the variation. At the end, the rule was shown through the command to "trace" Cabri; see Figure 24.3.

Thus the students were led to conclude that the greatest area is reached when the two sides are equal, i.e., the rectangle is a square. It was difficult for them to provide an explanation for this fact. But a group of students tried the following informal explanation:

> Look at this, teacher: if I point in the middle and after I shift a little to the left and a little to the right, the area decreases.

This is not a proof and not even an explanation, but this sentence gave the teacher an opportunity for explaining how to pass from the pure perceptive phase to a phase of using symbols as a tool for solving problems. He said:

> How can we translate into symbols […] the expression "to shift a little to the left and a little to the right from the middle point?" The answer is $3 - x$ and $3 + x$. Then the area of the rectangle is $(3 - x)\ (3+ x)$, that is $9 - x^2$.

At this point, the students easily acknowledged that the greatest area is reached when $x = 0$. The metaphor of "shifting to the left and to the right" is a bridge between perceptive situations and symbolic conceptual situations.

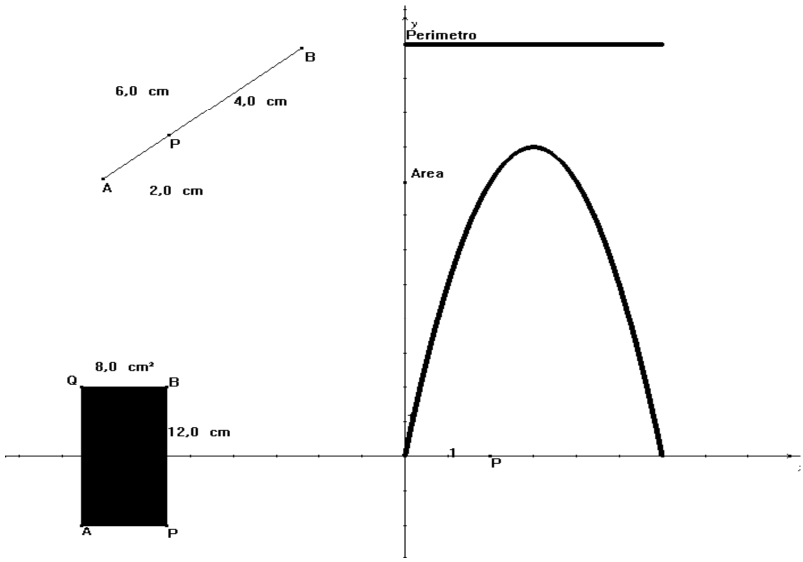
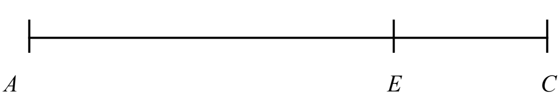


Figure 24.3 Figure made with Cabri by students for the problem of finding the greatest area.

In considering this episode from the historical perspective taken in this chapter, we note a fascinating resemblance with the following passage from *Methodus ad disquirendam maximam et minimam*, in which Fermat sets out his method for evaluating maxima and minima[7]:

> The whole theory of evaluation of maxima and minima presupposes two unknown quantities and the following rule:
>
> Let a be any unknown of the problem (which is in one, two, or three dimensions, depending on the formulation of the problem). Let us indicate the maximum or minimum by a in terms which could be of any degree. We shall now replace the original unknown a by $a + e$ and we shall express thus the maximum or minimum quantity in terms of a and e involving any degree. We shall adequate [*adégaler*], to use Diophantus' term, the two expressions of the maximum or minimum quantity and we shall take out their common terms. Now it turns out that both sides will contain terms in e or its powers. We shall divide all terms by e, or by a higher power of e, so that e will be completely removed from at least one of the terms. We suppress then all the terms in which e or one of its powers still appear, and we shall equate the others; or, if one of the expressions vanishes, we shall equate, which is the same thing, the positive and negative terms. The solution of this last equation will yield the value of a, which will lead to the maximum or minimum, by using again the original expression.
>
> Here is an example:
>
> To divide the segment AC at E so that AE x EC may be a maximum.



> We write $AC = b$; let a be one of the segments, so that the other will be $b - a$, and the product, the maximum of which is to be found, will be $ba - a^2$. Let now $a + e$ be the first segment of b; the second will be $b - a - e$, and the product of the segments, $ba - a^2$

> + $be - 2ae$ - e^2; this must be adequated with the preceding: $ba - a^2$. Suppressing common terms: $be \sim 2ae + e^2$. [Dividing all terms: $b \sim 2a + e$][8]. Suppressing e: $b = 2a$. To solve the problem we must consequently take the half of b.
>
> We can hardly expect a more general method.

Again, there is a connection here between historical conceptual developments and contemporary learning. The connection in this and in previous episodes is not a recapitulative one (i.e., it is not a connection in which the students' conceptual development is traversing the same steps as those supposedly traversed by the conceptual development of mathematics.) We see this connection as follows: the students are required to deal with problems that belong to a cultural mathematical tradition. This tradition has its own concepts, methods and ideas (e.g., what is taken to be relevant, what is considered to be an evidence, a proof, etc.) These concepts, methods and ideas have been established and refined over the course of centuries. Giulia, Andrea, and the other students meet this well-established tradition in the school. It is clear that for the students, the objects and methods of the mathematical tradition that they meet at school are not easily identifiable from the outset. They have to be *unpacked*, so to speak. Mathematical concepts and methods are general, and they cannot be fully grasped through the particulars that instantiate them. How then do the students get acquainted with the mathematical tradition? Rather than constructing the monumental knowledge that constitutes our millenarian mathematical tradition anew and from scratch, the students *make sense* of cultural mathematical concepts (Radford, 2006c). This production of sense entails noticing and understanding the cultural categories of relevance, evidence, proof, etc. But, from the ontogenetic viewpoint, it also entails a new concrete form of development that is *shaped* by the historical one. It is the historical line of development that pulls up, so to speak, the ontogenetic one and makes us "see" a kind of recapitulation. This illusion disappears as soon as we realize that the cultural context "prepares" the encounter between the ontogenetic and phylogenetic dimensions. This preparation occurs under the effects of explicit and implicit factors that underpin the various aspects of learning. One of the more prominent aspects is the mathematical problems that students are required to tackle.

Mathematical problems (like those discussed in the previous three episodes) are bearers of a human intelligence deposited in them by the cognitive activity of previous generations (Radford, Bardini, Sabena, Diallo, & Simbagoye, 2005). They suggest ways of conceptually attending some aspects of the world. They offer intellectual models that suggest relevance (e.g., maximization problems within specific constraints). In the same way, the cultural artifacts that the students use to try and solve these problems (e.g., pencil and paper, the cube, written and spoken language, the language of arithmetic and algebra, software like Cabri Géomètre) are bearers of human intelligence and, like the mathematical problems, suggest lines of ontogenetic conceptual development. In getting acquainted with the mathematical tradition, the students then mobilize cultural tools that connect cultural concepts with their ontogenetic understanding in ways that do not really reveal an alleged recapitulation but rather unveil the encounter of the ontogenetic and phylogenetic developments. This encounter is produced by the effects of the school as a sociocultural institution (Bosch & Chevallard, 1999) equipped with a complex system of knowledge acquisition (teachers and other adults, a chronologically ordered curriculum, spaces of interaction, routines of socialization, experimentation, feedback and so on.)

7. THE RECOURSE TO HISTORY IN CONTEMPORARY MATHEMATICS EDUCATION

In the previous sections, we discussed some interpretations of recapitulation law emerging from the words of past mathematicians and from classroom experiences. Let us now examine a few examples of contemporary mathematics educators, confining our discussion to two specific cases. The first emphasizes (mainly although not exclusively) a theoretical interest. The second appears closer to specific contexts arising from the need to enhance teaching and learning processes in mathematics instruction. In the first case, the history of mathematics appears as a theoretical tool for understanding developmental aspects of mathematical thinking. The purpose of the second case is to facilitate, through explicit pedagogical interventions, students' learning of mathematics by attempting to relate the development of students' mathematical thinking to historical conceptual developments.

7.1. The interface between history and developmental aspects of mathematical thinking

The work of Sfard (1995) provides a clear example of contemporary views on the relation between history and the developmental aspects of mathematical thinking. She analyzed the development of algebra by blending historical and psychological perspectives. At the beginning of her article, she claimed that

> there are good reasons to expect that, when scrutinized, the phylogeny and ontogeny of mathematics will reveal more than marginal similarities. At least, this is what follows from the constructivist view according to which learning consists in the reconstruction of knowledge. (p. 15)

The similarities between the phylogenetic and ontogenetic domains, according to this account, result from "inherent properties of knowledge." For Sfard, who in the 1990s was following a Piagetian epistemological perspective, knowledge can be theoretically described in terms of genetic structural levels, and it is precisely the nature of the relationship between the different levels that accounts for the similarity of phenomena appearing in both the historical and the individual's construction of knowledge. As she noted, "difficulties experienced by an individual learner at different stages of knowledge formation may be quite close to those that once challenged generations of mathematicians" (Sfard, 1995, pp. 15–16). A large part of the text is devoted to the discussion of the development of algebraic language. Indeed, using Nesselmann's (1842) distinction between rhetorical, syncopated, and symbolic algebra, Sfard endeavored to locate those "constants" (more precisely, those "developmental invariants") that ensure the passage from rhetorical and syncopated algebra to symbolic algebra. Rhetorical algebra refers to the reliance on nonsymbolic, verbal expressions to state and solve a problem, as it appears, for instance, in Arabic, Hindu, and Italian medieval texts. Syncopated algebra is seen as a more elaborate algebra in that, although still relying heavily on verbal expressions, it introduces some symbols, the work of Diophantus being the canonical example. Viète's systematic introduction of letters epitomizes symbolic algebra. After confronting experimental classroom, results with the traditional view of the historical development of algebra, Sfard concluded that one of the development invariants underpinning the passage from rhetorical and syncopated algebra to symbolic (Vietan) algebra is the precedence of operational over structural thinking. Operational thinking, in this context, means a way of thinking about algebraic objects in terms of computational operations. Structural thinking is related to more abstract objects conceived structurally on a higher level.

As we can see, the use of history in Sfard's approach is an attempt to corroborate parallelisms between ontogenetic and phylogenetic developments. As she said, "history will be used here only to the extent which is necessary to substantiate the claims about historical and psychological parallels" (Sfard, 1995, p. 17). Although she stressed the importance for teachers to be aware of the historical development of mathematics, the intention is not that of creating an historically inspired classroom activity. This is the goal of another perspective in contemporary mathematics education, discussed in section 7.2. For the time being, we only want to discuss a sociocultural approach that shares Sfard's use of history for epistemological reasons but, in contrast, emphasizes the crucial link between cognition and the practical human activity in which cognition is embedded. This approach (see Radford, 1997a; Radford, Boero, & Vasco, 2000) is inspired by key ideas of the Vygotskian and cultural perspectives alluded to in section 3 of this chapter and is driven by a conception of knowledge that differs from Piagetian genetic structuralism, inasmuch as it perceives knowledge and individuals' intellectual means to produce it as intimately and contextually related to their cultural setting. Knowledge, in fact, is conceived as the product of a *mediated cognitive reflexive praxis* (see Radford, 2000b). The mediated character of knowledge refers to the role played by artifacts, tools, sign systems and other means to achieve and objectify the cognitive praxis. The reflexive nature of knowledge is to be understood in Ilyenkov's sense, that is, as the distinctive component that makes cognition an intellectual reflection of the external world in the form of the individuals' activities (Ilyenkov, 1977, p. 252). Knowledge as the result of a cognitive praxis (*praxis cogitans*) emphasizes the fact that what we know and the way we come to know it is framed by ontological stances and by cultural meaning-making processes that shape a certain kind of rationality out of which specific kinds of mathematical questions and problems are posed.

Theoretically, however, this does not mean that the study of knowledge is determined by social, economic and political factors because these factors are also historically produced. Certainly, the link between culture and cognition is more subtle than the distinction between the "internal" and "external" realms employed in many historiographic approaches that see the external as only a mere stimulus for conceptual changes and developments. Methodologically, this means that the study of the historical development of mathematics cannot be reduced to the sociology of knowledge. This also means that such a study cannot be done through the analysis of texts only. The "archive" (to borrow Foucault's (1969) expression), as a historical repository of previous experiences and conceptualizations, bears the sediments of social, economic, and symbolic human activities. Therefore, understanding the rationality within which a mathematical text was produced requires the relocation of the text within its own context. The goal of this kind of epistemological reflection is not to find a parallel between phylogenetic and ontogenetic developments. In the sociocultural approach that we advocate, mathematical texts from other cultures are investigated while taking into account the cultures in which they were embedded. This allows the researcher to scrutinize the way mathematical concepts, notations, and meanings were produced.

Through an *oblique contrast* with the notations and concepts taught in contemporary curricula, we seek to gain insights about the intellectual requirements that learning mathematics demands of our students. We also seek to broaden the scope of our interpretations of classroom activities. In designing classroom activities, we aim at eventually adapting conceptualizations embedded in history to facilitate students' understanding of mathematics. Our work on Babylonian algebra and the teaching of second-degree equations (Radford, & Guérette, 2000) is an example of the latter. Our classroom research on the strategies that students use to deal with the algebraic generalization of patterns and the way they conceive relations between the concrete and the abstract (see Radford, 1999a, 2000c)—research based on our investigation of pre- and Euclidean forms to convey generality (Radford, 1995a)—is an example of oblique contrast between past developments and contemporary students' conceptualizing processes.

Our classroom research on the introduction of algebraic symbolism also benefited from our epistemological inquiries based on editions of original texts from medieval and Renaissance Italian mathematics (Radford, 1995b, 1997b). Space constraints do not allow us to go further, but this anthropological approach to the epistemology of mathematics offers a new view of the rise of symbolic algebra in the 16th century. The difference from traditional views stressing the passage from syncopated to abstract algebra in terms of abstractive processes is that, in our account, changes in development are related to changes in societal practices and the way in which mathematical conceptualizations are subsumed in them (Radford, 2006a). Briefly, what we find in our analysis is that there were two main mathematical practices in the early Renaissance, that used by merchants and abacus mathematicians and that used by humanists and court mathematicians. While the latter were busy with the restoration of Greek texts, the former were applying Arabic algebraic techniques to practical as well as nonpractical problems (e.g., problems about numbers). Symbolic algebra was a time-consuming effort made by Italian humanist and engineer mathematicians, such as the priest Francesco Maurolico, who eradicated all commercial content in his *Demonstratio Algebrae*, which was completed October 7, 1569, and edited by Napoli in the 19th century (Napoli, 1876). Another example is the engineer Rafael Bombelli, who, after having learned that the first books of Diophantus' *Arithmetic* were on the shelves of a Roman library, studied them and ended up eliminating the commercial problems in his *Algebra*. Bombelli provided a final version of it that conformed much more to the humanist understanding of Greek mathematics. In France, a similar effort was made by the humanists Jacques Peletier and Guillaume Gosselin (although in this case, the promotion of French as a scientific language was an important drive; Cifoletti, 1992). The underlying reason for the effort to introduce a specific symbolism in algebra was not due to the limitations of vernacular language. Mathematicians working within the possibilities offered by rhetorical algebra produced many difficult problems involving several unknowns, as can be seen in Fibonacci's *Il Flos* (Picutti, 1983). These problems could not be simplified by the introduction of letters because what was being symbolized in the emergence of symbolic algebra did not include all of the unknowns mentioned in a problem but only one of them. (See, for instance Bombelli's symbolism or the neogeometrical example in Piero della Francesca's *Trattato d'Abaco*, edited by Arrighi, 1970.) It was only later that some mathematicians in Germany began using letters for several unknowns (see Radford, 1997b). In our approach, the emergence of algebraic symbolism appears to be related to the efforts made by humanists and court-related mathematicians to render the merchant's algebra noble and court-worthy (details in Radford, 2000b). This was accomplished by the lawyer and mathematician François Viète, at the French court, who followed the prestigious Greek traditions typified by Diophantus' *Arithmetic* rather than the multitude of 15th- and 16th-century abacus treatises.

We now discuss a second position towards the use of history in contemporary mathematics education, that which aims at enhancing, through explicit pedagogical interventions, the students' learning of mathematics.

7.2. Students' mathematical thinking and historically based pedagogical actions

Boero and collaborators (see Boero, Pedemonte, & Robotti, 1997; Boero, Pedemonte, Robotti, & Chiappini, 1998) made use of the history of mathematics to investigate the nature of theoretical knowledge and the conditions by which it emerges. Their historico-epistemological analysis aims at looking for elements considered typical of mathematical thinking, such as organization, coherence and systematic character. They have investigated the role played by definitions and proofs, as well as by the type of theoretical discourse. Their framework draws on Bakhtin's theory of discourse, mainly the theoretical construct of "voice" (Bakhtin, 1968; Wertsch, 1991), and on Vygostky's distinction between scientific and everyday concepts (Vygotsky, 1962). The historico-epistemological inquiry is subsequently invested in the design and implementation of teaching settings based on a careful selection of primary sources, the main objective of which is to allow the students to echo the voices of past mathematicians. Through the echoing process, the students bring in their individual subjective and cultural backgrounds to build a "voices and echoes game," which proves to be fruitful for the acquisition of theoretical knowledge. The voices from the past are not listened to passively but actively appropriated through an effort of interpretation. Usually the students' echoes may take various forms. Boero and his team have provided a categorization of some of the ways in which the students enter the dialogical game. For instance, a "mechanical echo" consists in precise paraphrasing of a verbal voice, whereas an "assimilation echo" refers to the transfer of the content and method conveyed by a voice to other problem situations. A "resonance" is a student's appropriation of a voice as a way of reconsidering and representing his or her experience.

Among the concrete instances of theoretical knowledge examined by the authors are the theories of the falling bodies of Galileo and Newton, Mendel's probabilistic model of the transmission of hereditary traits, and theories of mathematical proof and algebraic language, all of which are characterized by aspects of a counterintuitive nature.

Another example of the contemporary use of history in the classroom is the research of Sierpinska and collaborators (e.g., Sierpinska, Trgalová, Hillel, & Dreyfus, 1999). One of the goals of this research is to provide an alternative, based on the use of the Cabri-Géomètre software, to the traditional axiomatic approach to the teaching of linear algebra in undergraduate courses. A problem examined in this research, which underlies important aspects of the learning of basic linear algebra, is that of understanding key differences in the representations of mathematical objects. In this line of thought, Sierpinska has emphasized the distinction between "numerical" and "geometrical" space.

The difference between geometrical and numerical space is clear in the history of linear algebra. Sierpinska, Defence, Khatcherian, and Saldanha (1997) identified three modes of reasoning, which they labeled "synthetic-geometric," "analytic-arithmetic," and "analytic-structural." As they noted (a more detailed report is in Bartolini Bussi, & Sierpinska, 2000), the concepts of linear algebra do not all have the same meaning and, in the classroom, they are not equally accessible to beginning students. The design of the teaching activities as well as the understanding of students' answers took into account the modes of reasoning as determined in the historico-epistemological analysis. (An extended account of the teaching activities can be found in Sierpinska, Trgalová, Hillel, & Dreyfus, 1999 and Sierpinska, Dreyfus, & Hillel, 1999.)

Recently, C. Tzanakis and his collaborators have been conducting a cross-curricular project that involves teachers of mathematics, language, and history (see Tzanakis, 2006). The main innovation of this project is the introduction of original texts in the normal teaching of Euclidean Geometry in the first year of the Greek Lyceum (16-year-old students). They selected some parts of Euclid's *Elements* and Proclus' *Commentary on the 1st Book of the Elements* to create, among students and teachers, the atmosphere of a debate over the concept of proof and the issue of critical thinking. Selected propositions taken from Euclid's *Elements* and some of Proclus' commentary on ancient critiques of these propositions were examined from different points of view: the linguistic, the historical and the mathematical. Students and teachers engaged in the interpretation and discussion of Euclid's theoretical choices and Proclus' views on ancient critiques of Euclid's proofs (Tzanakis, 2006).

In addition to these studies that interpret the didactical phenomena in the light of educational research, there are studies stemming directly from teachers' experience in classroom. The proceedings of European Summer Universities (Lalande, Jaboeuf, & Nouazé, 1995; Lagarto, Vieira, & Veloso, 1996; Radelet-de-Grave, & Brichard, 2001; Furinghetti, Kaijser, & Tzanakis, 2006) offer good examples of teaching sequences entirely based on history. Usually, the authors are teachers with a deep familiarity with the history of mathematics and a good teaching experience. They also are at ease with the use of primary sources. Generally speaking, they may not be considered the mathematics teacher that we find usually in school, nor is their work common in school environments. On the contrary, there is evidence that they are exceptions in the school panorama. Siu (2006), indeed, has listed 15 reasons given by teachers for not using history in the classroom, in spite of decades of encouragement provided by curriculum developers and researchers. Also, the TIMSS 1999 Video Study seems to show that history of mathematics does not constitute an important part of teaching in the seven visited countries (see Smestad, 2006).

8. SYNTHESIS AND CONCLUSION

The history of mathematics can be used for different reasons in the teaching of mathematics and in the training of prospective teachers. In this chapter, we dealt with one of the many uses of the history of mathematics in mathematics education, namely, the investigation of historical conceptual developments to deepen our understanding of mathematical thinking and our capacity to enhance the students' learning of mathematics. In the first part of the chapter, we focused on the psychological recapitulation. In particular, we discussed how psychological recapitulation was imported from biological recapitulation and gave rise to a discourse that framed much of the discussions about child development from the beginning of the 20th century. Psychological recapitulation was adopted by some eminent mathematicians who, in one form or another, supported the idea that in developing their mathematical thinking, children would traverse similar steps as those found in the history of mathematics. Within this conception, during their development, children would supposedly face some problems, difficulties, or obstacles similar to those encountered by past mathematicians. Recapitulationism, we argued, served as the means by which some mathematicians wanted to counter a teaching orientation that was based on the commitment to rigor and logical structures which had arisen in the flow of the research on the foundations of mathematics at the turn of the 20th century.

Nonetheless, one of the problems with the recapitulationist approach is that conceptual developments are seen as chronologically self-explanatory and psychological evolution is taken for granted. Furthermore, knowledge is conceived as having little (if any) bond to its context and history is reduced to a linear sequence of events judged from the vantage point of the modern observer.[9] In all likelihood, the extremely low number of studies that attempt to check the validity of recapitulation law is evidence of the impossibility of reproducing the conditions in which ideas developed in the past. As Dorier and Rogers noted, "'naive recapitulationism' has persisted in many forms and now we accept that the relation between ontogenesis and phylogenesis is universally recognized to be much more complex than was originally believed" (Dorier, & Rogers, 2000, p. 168).

This statement corresponds well to recent nonpositivist epistemological and anthropological trends (see, e.g., Cole, 1996; Gould, 1979; Shweder, 1991). Indeed, in emphasizing the relation between knowledge and social practices, these trends have raised some criticisms to the acultural stance conveyed by the general and universal character of the recapitulation law, thereby opening new ways to reconceptualize the relationship between historical conceptual developments and the teaching of mathematics (Radford, 1997a).

In the course of our discussion, we mentioned two different and critical stances toward the relation between ontogenesis and phylogenesis as elaborated by Piaget and Garcia on the one hand and by Vygotsky and his collaborators on the other. The way Piagetian and Vygotskian epistemologies have inspired current work on contemporary mathematics education was made clear in the brief presentations of specific traits in the works of Boero, Radford, Sfard, and Sierpinska. We also presented a different interpretation of recapitulation based on the idea of conceptual connections between ontogenetic and phylogenetic developments. These connections—induced by the complex learning system of the school—appear as parts of the students' process of *objectification* (Radford, 2002) and *making sense* of a historically and culturally constituted knowledge deposited and mobilized by the school (Radford, 2006c).

Regarding recommendations for future research, it can be suggested, in light of the previous discussion, that a pedagogical use of the history of mathematics committed to enhancing students' conceptual achievements requires a critical reflection about the conceptions of ontogenesis and phylogenesis and, of course, about knowledge itself. But to be fruitful in practical terms, such a critical reflection must be clear about its classroom implications (see Demattè, 2006). The episodes described in section 6 suggest that to make the use of history effective, teachers need to be able to create suitable learning environments. This requires that teachers gain an appropriate understanding of the differences between ontogenetic and phylogenetic developments and maintain a critical stance toward recapitulation views. As the sophisticated methodology of Boero's approach suggests, this requires that teachers be sufficiently comfortable in handling cognitive and historical aspects. Let us make three suggestions concerning actions for research.

1. On a theoretical level, discussions about recapitulation and its different meanings should be promoted among historians, epistemologists, psychologists, anthropologists, and mathematics educators.
2. On a practical level, models of contrast and conceptualizations between ontogenetic and phylogenetic developments should also be considered further. Models of contrast may help us to better grasp specific traits of mathematical thinking, its relation to cultural settings and the mathematical concepts thereby produced. This can lead to a better understanding of the kinds of practical pedagogical interventions that can be envisioned.
3. Theoretical reconceptualizations of recapitulation and contrasts and comparisons between ontogenetic and phylogenetic domains should be explicit as to how they can frame the engineering of material and teaching sequences.

We consider these related research topics to be interactively fed by theoretical inquiries, historical studies and also classroom observations.

The course of the three aforementioned actions for future research will ultimately depend on the very conception of mathematical knowledge to be adopted. At this point, two main contrasting trends seem to be emerging. In the first trend, what makes for the specificity of mathematical knowledge is its systemic, objective and logical nature (see Fujimura, 1998.) In the second trend, which is much more anthropologically driven, knowledge is conceived as a kind of culturally framed activity enabling individuals to inquire about their world and themselves. Here "systematicity" and "logicality" are seen as circumscribed characteristics of knowledge that can differ from culture to culture (see Radford, 1999b). Between them, of course, many theoretical possibilities can be envisaged. What they may have in common, though, is the growing awareness that "a concrete understanding of reality cannot be attained without a historical approach to it" (Ilyenkov, 1982, p. 212).

NOTES

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   This chapter is based on our previous work (Furinghetti & Radford, 2002).
2. Nikolai Alexandrovich Izvolsky was born in 1870 in Tula, Western Russia. He worked as a teacher at the 2nd Moscow Military School and from 1922, he was a professor at the 2nd Moscow State University (now Moscow State Pedagogical University). He wrote papers on mathematics education and some textbooks in arithmetic, algebra, and geometry. Izvolsky died in Yaroslavi in 1938. The authors are grateful to Professor Ildar Safuanov from the Pedagogical Institute of Naberezhnye Chelny for the information he kindly provided concerning the life of Izvolsky.
3. A complete study of the genetic method as envisioned by Toeplitz can be found in Schubring (1978).
4. This passage is taken from (Toeplitz, 1963). The original German version appeared in *Jahresbericht der Deutschen Mathematischen Vereinigung*, XXXVI, 1927, 88–100 with the title "Das Problem der Universitaetvorlesungen Ueber Infinitesimal-rechnung und ihrer Abgrenzung gegenueber der Infinitesimalrechnung an hoeheren Schulen."
5. For a wider account of the experiment see (Cartiglia, Furinghetti, & Paola, 2004).
6. For a full account of the experiment see (Paola, 2004). It is remarkable that the experiment was set in a computer science class, without any reference to history.

7. I report Fermat's passages as reprinted in Fauvel, (1990, pp. 28–30.)
8. The sentence in square brackets is not in (Fauvel, 1990): it is my translation of the sentence "*et, omnibus per E divisis, B adaequabitur Abis + E*" (Fermat, 1891, tome.I, p.134). The same sentence is translated as "*Divisant tous les termes: b ~ 2a + e*" in the French version (Fermat, 1891–1922, tome III issued in 1896, p. 122).
9. Schubring (2006) writes, "The historiography of mathematics has hitherto concentrated on the peaks', on the 'heroes' of mathematics, and it has practiced a resultatist view, searching for forerunners of present mathematics" (p. 339).

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