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ORIGINAL ARTICLE

"No! He starts walking backwards!": interpreting motion graphs and the question of space, place and distance

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Abstract This article deals with the interpretation of motion Cartesian graphs by Grade 8 students. Drawing on a sociocultural theoretical framework, it pays attention to the discursive and semiotic process through which the students attempt to make sense of graphs. The students' interpretative processes are investigated through the theoretical construct of knowledge objectification and the configuration of mathematical signs, gestures, and words they resort to in order to achieve higher levels of conceptualization. Fine-grained video and discourse analyses offer an overview of the manner in which the students' interpretations evolve into more condensed versions through the effect of what is called in the article "semiotic contractions" and "iconic orchestrations."

Keywords Cartesian graphs · Interpretation · Objectification · Meaning · Semiotics · Semiotic nodes

1 Introduction

Space and time constitute two fundamental dimensions of human experience. Although their foundations can be said to be anchored in the pulsations of our biological system, they only become objects of conceptualization when they are experienced beyond the sensing body and the situated phenomenological spatial "here" and temporal "now." In other words, space and time only become conceptual objects once they turn into organizing referential elements of action and reflection.

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Cartesian graphs are semiotic constructs that offer an interesting way through which such reflections can be carried out. The use of Cartesian graphs in problems dealing with motion is a case in point. Yet, it would be misleading to think that Cartesian graphs are easy to read and interpret (diSessa, Hammer, Sherrin, & Kolpakowski 1991; Nemirovsky and Monk 2000; Radford, Miranda and Guzmán 2008; Roth and Lee 2004). Indeed, a Cartesian graph is a complex mathematical sign. As with any graph, a Cartesian one supposes a selection of elements. However, in opposition to other kind of graphs, such as medieval *mappamundi* (Woodward 1985), what a Cartesian graph depicts is not the elements themselves but specific mathematical *relationships* between them. Because they are not intended copies of the phenomena that they depict or represent, the making and interpreting of Cartesian graphs are not trivial endeavours. Cartesian graphs rest on a sophisticated manner of conveying meanings that, historically speaking, have been adjusted, refined and generalized over the course of centuries.

In this article, I deal with the interpretation of motion graphs by novice students. I am particularly interested in understanding the discursive and semiotic process through which the students attempt to make sense of motion graphs. The theoretical approach that I follow is based on a sociocultural perspective, which provides the rationale for the experimental approach, as well as its goal and methodology. The theoretical approach posits the problem of learning as a problem of objectification of cultural ideas and modes of thinking (Radford 2008a). It rests on the principle that the mathematics the students encounter in the school is a mathematics with a long history, shaped by individuals, cultures and institutions. Within this context, objectification refers to the social processes through which students become progressively conversant with cultural mathematical ideas and modes of thinking.

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Now, as considered here, objectification is not the mere exposure to a certain conceptual content and its assimilation by the student. As Vygotsky (1986) used to say against the behaviorist approaches of his time, the adult cannot inject into the child's mind his or her own concepts. Indeed, the most fundamental feature of knowledge, either in its cultural-historical or life-span evolution, is its *genetic* dimension. Knowledge is something dynamic. It has a *genesis* and a *movement* which make transmission simply impossible. The problem with pedagogical methods based on the idea of transmission of knowledge is not that they are inefficient. Rather, they are wrong.

From the objectification perspective, the learning of mathematics appears hence as a process in the course of which the students make sense of, and become conversant with, the cultural logic of mathematical objects and forms of thinking. As understood here, however, learning is not limited to gaining a degree of fluency with certain mathematical technicalities. Learning is also (and overall) the formation of the student's consciousness and subjectivity, and the student's positioning within a cultural discourse. Learning—as seen through the lens of objectification—is the dialectical result of the students' *engagement* with sensuous mediated cultural praxes and a consciousness that reflectively grows within the affordances and constraints of the praxes' historically formed epistemes and ontologies (Radford 2008b).

In the investigation of the students' interpretation of graphs, I shall focus on a small segment of what may be called 'the space of learning'—that complex space that includes the diverse elements that make objectification possible. I shall attend to the genesis and evolution of the students' interpretation of motion Cartesian graphs as a moment in which the students confront, get in touch with and try to understand the condensed, historically formed meanings of such graphs. Consonant with some contemporary views of cognition as reflective engagement with cultural practices (Ilyenkov 1977; Leont'ev 1978; Vygotsky and Luria 1994; Wartofsky 1979) and the mediating role of artifacts and the body (Arzarello 2006; Bartolini Bussi and Mariotti 2008; Edwards 2009; Radford 2009a; Robutti 2006), I will explore the role of discourse, signs, body, actions, and technological objects. The paper is divided into ten short sections. In the first six sections I discuss some brief excerpts from the students' activity, emphasizing the manner in which the students deal with some of the main concepts—including place and distance—of a motion Cartesian graph. In order to further our understanding of the students' processes of objectification, I offer in the remaining sections a detailed semiotic analysis of how the students' interpretative activity is carried out, in terms of semiotic nodes, their configuration and evolution. I conclude with some suggestions for teaching and research.

2 Interpreting graphs

In what follows, I shall discuss some passages from a videotaped sequence of Grade 8 mathematics activities. The activities were planned by our research team (which included the teacher) and focused on the interpretation, reproduction and construction of graphs.[1] In accordance with the activities' design, the students worked in groups of three on problems of increasing theoretical difficulty. In the first part, the students were presented with two graphs (Figs. 1, 2) called "Tina's Walk" and "Jean's Walk." They were asked: (1) to interpret the graphs, (2) to reproduce the walks with the help of a measuring tape and a chronometer, and, when they thought they were ready, (3) to record the walks with the help of a TI-83 + calculator and a probe motion (a CBR Calculator-Based Ranger©). Four groups of students were videotaped with one camera each. A fifth mobile camera followed the teacher around the class to capture his discussions with the eight groups in which this 24-student Grade 8 class was divided.

In the story-problem accompanying the graphs, Tina was said to be 1 m away from a fountain at the beginning of her walk. She walked on a straight line path. The graph to the left describes her walk. Jean was on the same path 4 m away from the fountain and started his walk at the same time as Tina. The graph to the right of Fig. 1 describes his walk. The following excerpt comes from Group 2 (formed by Mary, Jeff, and Daniel). As mentioned earlier, I shall focus on the way in which the students interpret the graphs, paying attention to the semiotic means to which students resort in their processes of knowledge objectification.

In the following excerpt the students discuss Tina's walk graph.

1. Mary: So… here it will be 1 meter. It would be… 1 meter in 3 seconds… (*with her left hand she points to 1 meter on the vertical axis; with the top of her pen she points to 3 seconds on the horizontal axis; see* Fig. 1, Picture 1).
2. Jeff: No, it's 2 meters in 3 seconds. Because you start… at… one meter already (Fig. 1, Picture 2). You haven't started walking yet… So from 1 to 3 (*he moves his pen along the first segment;* Picture 3), it's going to be 2 meters in 3 seconds.
3. Mary: Ah! 3 meters in 3 seconds! (*Her utterance is accompanied by the three gestures shown in* Pictures 4–6).
4. Daniel: After that, you stop… for 2 seconds.
5. Jeff: After you stop for 2 seconds.

[1] The research team included Mélanie André, Serge Demers, Alain Girouard, Isaias Miranda, Andrew Sanderson, and Sonia Gonçalves .

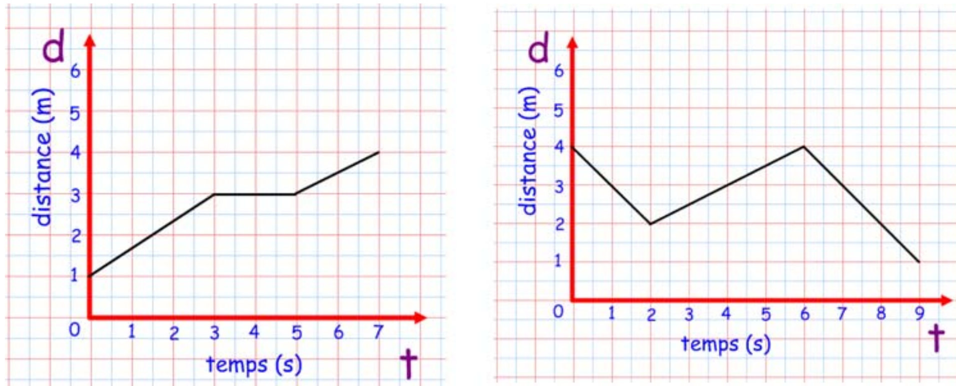

Fig. 1 The two graphs given to the students. To the *left*, Tina's walk; to the *right*, Jean's walk

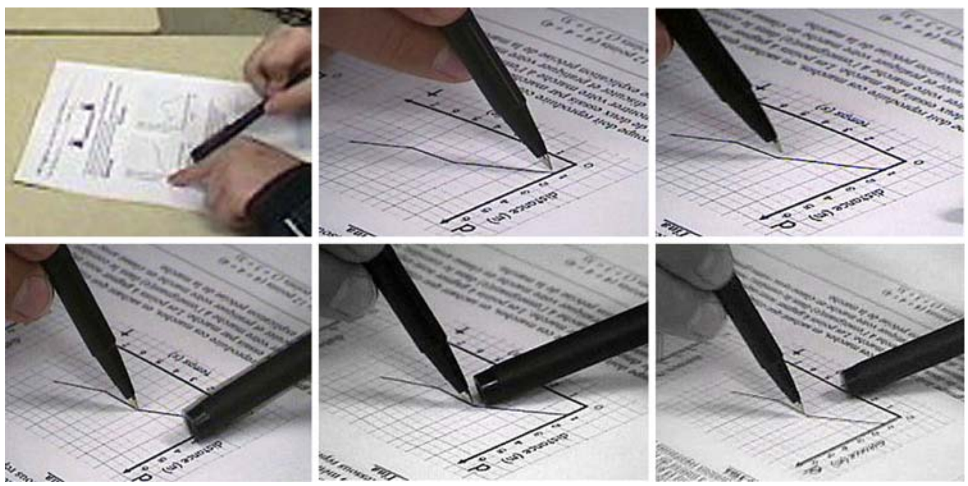

Fig. 2 Some indexical gestures made by the students in *lines 1–3*. Picture 1, 1 m in 3 seconds (*line 1*). Picture 2, You start at… 1 m already (*line 2*). Picture 3, From 1 to 3 (*line 2*). Picture 4, 3 m…. Picture 5, …in. Picture 6, 3 seconds (*line 3*)

6. Daniel: (*Referring to the second segment*) After that you go another meter in 2 seconds.

In line 1, Mary's attention rests on two signs: "1" (read on the vertical axis) and "3" (read on the horizontal axis). She interprets them as meaning Tina's travelled distance during a specific duration: "one meter in three-seconds." For her, the vertical axis indicates travelled distance and the horizontal axis elapsed time. This interpretation leads her to assert in line 3 that Tina travelled 3 m in 3 s. Jeff objects to this interpretation and offers another one, based on the idea that, in addition to conveying the measure of a distance, the vertical axis conveys also *positional information*. Thus, pointing to the beginning of the first segment (Picture 2), he says: "you start … at … one meter already." In other terms, Jeff's concept of the vertical axis is *relational*. Although the fountain is not mentioned explicitly in this part of the discussion, Jeff's interpretation takes into account Tina's position vis-à-vis the fountain at the beginning of the walk. The vertical axis means the distance measured from a particular location—a *reference point*. This is why, for him, the sign "1" on the vertical axis does not merely mean the length of an interval: it also indicates the point at which "you start" (line 2).

Jeff also brings forward another aspect that makes his interpretation of the graph different from Mary's. Jeff's interpretation includes the variables distance *and* time in a covariational manner. Thus, for Jeff, the beginning of the first segment in the Cartesian graph indicates more than distance and position: it tells us something about the temporal dimension of the phenomenon under scrutiny. The initial point of the first segment also indicates that "You haven't started walking yet…", that is to say, it brings time to the fore in the form of the action of walking in abeyance. Then, Jeff expresses the positional and covariational interpretation of the graph kinesthetically, through a pointing gesture that he makes over the first segment (Picture 3). He first mentions the signs "1" and "3" on the vertical axis of the Cartesian graph as suggested by the first segment (he says: "from 1 to 3"). Shortly thereafter, these signs are interpreted as "positional distances" (i.e. distances measured from the fountain). This interpretation allows him to infer that Tina travelled 2 m in 3 s (end of line 2). True, Mary moves her pen from point (0, 3) to point (3, 3) while saying "three meters" (Pictures 4, 5); then she moves the pen vertically from point (3, 3) to point (3, 0), while saying "three-seconds", thereby expressing her awareness of variable covariation (as her gestures clearly show). However, the nature of the covariation remains *global*—that is, covariation expresses a relationship between travelled distances and elapsed times, as suggested by line 1 and 3, and also line 7, below. Indeed, in lines 4–6, Danielle and Jeff continued to advance the interpretation of the two other segments. However, Mary was still not convinced of the meaning of the first segment. Thus, bringing the discussion back to the first segment, she asks:

7. Mary: 3 meters in 3 seconds, isn't it?
8. Jeff: No because you start at 1 [meter] (*he makes a gesture similar to the one shown in* Picture 2).
9. Mary: Ah! Yes, ok.

The understanding of covariation as signified in motion Cartesian graphs, I want to suggest, is based, in part, on a sophisticated understanding of the positional meaning of the Cartesian variable distance—a meaning that, in turn, is based on a clear distinction between distance and place.

3 Distance and place

Contrary to what we may think, the idea of place is far from trivial, as Aristotle confessed in Book IV of *Physics*. "The question, what is place?", Aristotle says, "presents many difficulties. We have inherited nothing from previous thinkers" (Aristotle 1984, p. 355). In fact, in its historical origins, the concept of place is an entanglement of the ideas of *body*, the *place* of the body and the *space* that the body occupies. After a lengthy and difficult discussion, place, Aristotle concludes, is something predicated on motion, for "we must understand that place would not have been inquired into, if there had not been motion with respect to place" (1984, p. 359).

However, for Aristotle, place can only be thought of in relationship to a body that qualifies it. Place, hence, is not the relative expression of a relationship between an object and another object taken as a reference point. The disentanglement in Western thought of the ideas of place, space and body was a later achievement. Such disentanglement required a tremendous ontological shift. As Burtt (2003) notes, the analysis carried out by Aristotle and medieval thinkers was aimed at answering the question of *why* bodies move. In contrast, for Renaissance thinkers like Galileo, the question was *how* they move. The old arsenal of concepts such as action, efficient cause, and natural place was no longer sufficient, and new concepts of space, place, distance, and time were required. Jeff's concept of distance rests on the conceptual idea of space as something homogeneous and isometric, and place as a relative position expressing loci in *referential terms*. Historically speaking, the attainment of such an idea was a later achievement: it appeared in a clear and explicit way in Descartes' *Principles of Philosophy*. I shall come back to this point later. For the time being let me return to the students' discussion.

4 Jean's walk: the first interpretation

After the previous episode, the students attempted a first interpretation of Jean's walk. The interpretation was made in terms of speed:

10. Jeff: I don't understand the second graph.
11. Mary: You start at 4 meters.
12. Daniel: Yeah, you go to 4 meters, and after that, you continue to go down at 1 meter per second. And after that, you go up, but when it goes up, it's 1 meter for 2 seconds; every 2 seconds you go 1 meter… yeah, this one [the third segment] he goes down, so…
13. Jeff: But how is it that [Tina's graph] goes up, and that one [Jean's graph] goes down?

The students' interpretation is based on an oblique mention of a reference point. Drawing on Jeff's utterance in line 8, in line 11 Mary says: "You start at 4 m." The question is: from where? This oblique mention reappears when the movements to be made are qualified as "down" and "up." Again, the question is: down and up in relation to what? The spatial deictics "down" and "up" suggest the pregnancy of the graph and the difficulties the students still have in flexibly coordinating *represented* distances in the graph and the key *spatial locations* (the fountain and Jean's departure point) mentioned in the story-problem. The interpretation of the segments in terms of speed is in this sense most convenient: to talk about speed, as the students do here, is indeed to put into relationship travelled distance (as opposed to relational distance) and elapsed time. Jeff agreed with this interpretation. Yet the interpretation was not enough to explain the differences in the graphs—something that requires one to resort to the relational concept of the Cartesian variable distance.

The answer to Jeff's last question (line 13) was not easily found. The students returned to Tina's graph and the way they had to walk in order to reproduce it. In the planning of their walk, the fountain became an explicit object of discourse and was thought of as being at point (0, 0) of the Cartesian graph: Jeff said: "So one meter from the fountain, so the fountain is here" and he pointed as shown in Fig. 3.

The students prepared a space beside their desks to practice Tina's walk. With the help of a marker and a measure tape, they put some signs along the walking path (see Fig. 4, Picture 1). While practicing, they adjusted their walk to the spatial signs and time results monitored with

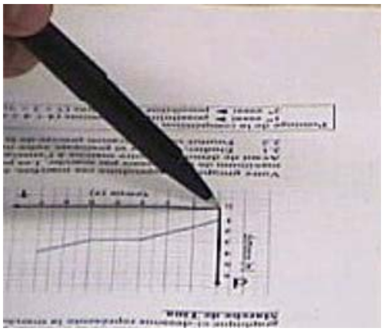

Fig. 3 Jeff indicates the place of the fountain

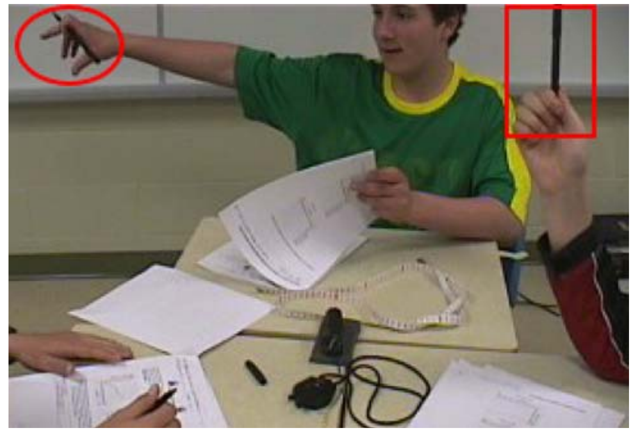

Fig. 5 Jeff says: "he [Jean] starts at 4 meters from the fountain" and points to his right. At the same time Mary makes a gesture pointing vertically upwards and says: "he starts from the top." The two gestures express two different ideas of *origo*

the chronometer. The kinesthetic context mediated by the chronometer and the measure tape helped the students to better objectify the meaning of the graph. In particular, the role of the fountain as the reference point against which distances are measured in the Cartesian graph became clearer. In accordance with the activity design, when they were ready, they called the teacher, carried out the walk and discussed the result with him (Fig. 4, Picture 2). As shown in Fig. 4, Picture 3, the students succeeded in obtaining a graph similar to the one on the activity sheet.

5 Jean's walk: the second interpretation

After carrying out Tina's walk, the students returned to their desks and continued their discussion about Jean's graph. Again, Daniel was the first to suggest a way to interpret the graph:

14. Daniel: Jean has to walk 2 meters in 2 seconds; after that he has to back up 50 centimeters in a second for 4 seconds.
15. Mary: (*Interrupting and referring to the second segment*) [He must] walk backwards…
16. Jeff: No! he starts walking backwards!
17. Daniel: Humm? No!
18. Mary: He starts from the top! (Fig. 5)
19. Jeff: (*Talking at the same time as Mary*) he starts at 4 meters from the fountain! (Fig. 5).

In line 14, Daniel offers an interpretation that differs from the one he gave in line 12. As we may recall, in line 12, the interpretation of the walk was made in terms of the *perceptual features* of the graph. Indeed, perceptually the graph "goes down" and "up"; thus, if one is supposed to match Jean's walk, "you continue to go down" (first segment) and then "you go up" (second segment; see line 12). In short, on line 12 the graph is interpreted as a kind of *map* of the walk. Like geographic maps, the graph is seen as providing its reader with "up" and "down" locations, affording interpretative narratives such as "you go up", "you go down", etc. In contrast, in line 14, the interpretation is made in terms of the students' recent kinesthetic experience. Both interpretations presuppose, even if only implicitly, what linguist Karl Bühler (1979) called *origo*, that is, an origin or *reference point* that serves to organize actions and interpretations. (For a detailed discussion of the idea of *origo*, see Radford 2002).

Now, what is this implicit reference point that Daniel and Mary have in mind in lines 14, 15, 17 and 18? Is it the same as the one to which Jeff resorts? No. In line 16, Jeff opposes Daniel's and Mary's idea. And even though he uses the term "backwards," he does so in a different sense: for Mary and Daniel the *origo* is the point at which Jean started walking. For Jeff, the *origo* is the fountain. In line 17, Daniel sharply rejects Jeff's idea. In line 18, in tune with Daniel, Mary shows through a clear gesture her

Fig. 4 In *Picture 1*, the students make some spatial signs. In *Picture 2*, they discuss their results with the teacher. *Picture 3* shows the calculator screen shot of the students' reproduction of Tina's walk as measured by the CBR

disagreement with Jeff: pointing up with her pen, she says: "He starts from the top." At the same time, Jeff says: "he starts at 4 meter from the fountain," and indicates a hypothetical place where Jean is supposed to start his walk. The spatial relationship between Jean's departure point and the fountain is emphasized verbally and through a gesture (see Fig. 5, 6). It is implicit in this utterance (but this will become explicit a few seconds later) that Jeff's body's place plays the role of the fountain's locus.

6 Descartes' idea of place

In Principles of Philosophy, Descartes says:

> the words place and space signify nothing really different from body which is said to be in place, but merely designate its … situation among other bodies. For it is necessary, in order to determine this situation, to regard certain other bodies which we consider as immovable; and, according as we look to different bodies, we may see that the same thing at the same time does and does not change place. For example, when a vessel is being carried out to sea, a person sitting at the stern may be said to remain always in one place, if we look to the parts of the vessel, since with respect to these he preserves the same situation; and on the other hand, if regard be had to the neighboring shores, the same person will seem to be perpetually changing place, seeing he is constantly receding from one shore and approaching another. (Descartes 1644, Part II, XIII)

The Cartesian idea of place makes it possible to think of the latter as *position* within a descriptive framework organized around an arbitrary reference point—what I have been calling here, after Bühler, the *origo*. Place as position (that which tells us the "situation" of the body) appears hence as a relative attribute (Slowik 1999). In our problem-story the choice has been made. The *origo* is the fountain. Usually, though, the students do not find it easily. There is a strong tendency to adopt as *origo* the initial point of movement and see the graph as a kind of map (see Radford, Demers, Guzmán, & Cerulli 2003; Radford, Cerulli, Demers, & Guzmán 2004).

The important conceptualization of place accomplished by Descartes, where place is understood as the relative position of things defined by their relationship to an arbitrary reference point, made it possible to describe problems about motion in a formal relational way. The peculiarity and complexity of Descartes' idea has puzzled philosophers and epistemologists like Ernst Cassirer. In his book, *An Essay on Man*, Cassirer distinguishes between practical and abstract ideas of space. The former, he says, "is a space of action; and the action is centered around immediate practical needs and interests" (Cassirer 1974, pp. 44–45). Things are referred to each other as practical needs require, without reaching a global or systematic organization, much in the way we talk about things in a house or other quotidian surroundings. It would sound very awkward, and indeed impractical, as I have experienced myself, to beg someone to fetch the newspaper by its spatial coordinates within the house. It is simpler to say something like "beside the coffee table." The mode of signifying in practical spaces is *local*, as opposed to the global and systemic way of signifying in abstract spaces (Radford 2009b). Furthermore, as Cassirer notes, in practical spaces the way we signify things and the places wherein they are located is "fraught with concrete personal or social feelings, with emotional elements" (p. 45).

Much like Lévy-Bruhl (1922), Cassirer was impressed by the reports of anthropologists who found that people in some non-Western communities displayed an unusual

Fig. 6 Jeff makes some pointing gestures to indicate key places that would serve as the remarkable loci of Jean's walk

sensitivity to the finest details of their environment and to every change in the position of the common objects of their surroundings. He was astonished to learn that these people, while hunting or sailing, could find, even under the most difficult circumstances, their way home. Yet,

> If you ask him to give you a general description, a delineation of the course of the river he is not able to do so. If you wish him to draw a map of the river and its various turns he seems not even to understand your question. (Cassirer 1974, p. 46)

For Cassirer, the previous differences mark two distinct conceptions of space and the way we refer to places therein. Abstract space rests on a global and systematic description of things and their places. As mentioned previously, crucial to the emergence of the concept of abstract space was Descartes' idea of relative place and a homogeneous and isometric space. As the German philosopher Martin Heidegger remarked, the Cartesian ideas of space and place offer a *symbolic objectivity* to the objects of enquiry. These ideas not only break with the Aristotelian tradition but overall make possible something completely new: "the apprehension of nature through calculative measurement" (Heidegger, quoted in Elden 2006, p. 135). Without such a conceptual shift, Cartesians graphs of the sort that we find in school textbooks would be impossible.

7 Jean's walk: the third interpretation

Most (if not all) mathematical ideas—precisely because they have been forged and refined through centuries of cognitive activity—are far from trivial for the students. It is not surprising that the relational meaning of the variable distance, as conveyed by Cartesian graphs, remains opaque, to varying degrees, for the students. The mathematical activity that our research team designed is in fact an educational artifact to help the students objectify the historical meanings embedded in the target mathematical ideas. We carefully chose two graphs with different shapes, each susceptible to appeal to the target ideas in different ways and with different intensities—remaining nonetheless within the confines of the school curriculum.

It would be misleading, though, to think that a mathematical problem alone (or a sequence of them) can make the target concept appear. The logical necessity with which modern mathematical concepts seem to be intrinsically endowed is the illusion of a coherence that is provoked by the retrospective manner in which we look at past events. The apparent logical necessity of mathematical concepts is just a reconstructive hypothesis (Radford 1997). Developmentally speaking, there are several possible aftermaths to the same mathematical situation. Framed by cultural-epistemological conditions that I cannot discuss here, Descartes, in a brilliant and skillful move, offered a very particular way of thinking about space and place. But this move does not exclude the possibility that there might have been other possible ways that we cannot even imagine now. Each creative act—as Descartes'—is at the same time the renouncement of other potential creative acts. To invite the students to interpret Tina's and Jean's walk in Cartesian terms is already an invitation to embark on a particular developmental cognitive path. The questions we ask and the problems we pose in the classroom are not innocent. They convey cultural epistemological and ontological assumptions. The students encounter and objectify the Cartesian concepts of space and place as cultural evolution has crafted them through centuries of refinements and abstractions. They objectify these concepts through interaction with the activity, the teacher, and peers.

To come back now to our students, it is indeed in this way that Jeff (line 16), after interacting with the activity, the teacher and his peers, was in a position to answer his own question (line 13). Although from the beginning of the mathematics activity he was advocating a relational interpretation of the vertical axis, he did not apply such an interpretation to answer his question in line 13. The kinesthetic and technologically mediated experience of matching Tina's walk served as the bridge to extend the relational interpretation to new conceptual lands.

Through a series of gestures that indicated the material space of the walk and suggested key places that would serve as the remarkable loci of Jean's walk, Jeff continued his interpretation in this way:

20. Jeff: It [the walk] starts from there (Fig. 6, *Picture 1*), then you back up to here (*Picture 2*), and then you go there (*Picture 3*), then you back up again (*Picture 4*).
21. Mary: Yes, because it [the graph] goes down towards the fountain… So Jean must back up 2 meters in 2 seconds… Then, [he] must go forward… And then, it's… it backs up again… [it] backs up 3 meters in…. 3 seconds.

In line 20 Jeff resorts to the positional or relational sense of distance and, through a pointing gesture that indicates the imagined position of bodies and things, suggests that Jean was 4 m away *from the fountain* and walked *backwards*—that is, towards the fountain. There is a series of coupled pointing gestures and deictic words ("there," "here") that schematically describes Jean's imagined successive positions. In the beginning of line 21, Mary translates Jeff's gestural and verbal described motion in terms of the graph. Then her interpretation is made in terms of the key forms of motion as seen from the fountain, which she now explicitly mentions.

Line 20 presents a clear example of objectification. Without perhaps being necessarily a rule, in our experimental research we have found that objectification is accomplished through an intense semiotic activity. More importantly, in this intense activity, signs play *different* and *complementary* roles. In an effort to better understand the students' processes of objectification, my collaborators and I found it useful to introduce the concept of *semiotic node*. A semiotic node is a part of the students' general semiotic activity, where action, gesture and word work together to achieve knowledge objectification (Radford et al. 2003). In line 20, we see how Jeff crucially resorts to pointing gestures and words. The pointing gestures sketch Jean's possible trajectory. Words (more precisely, the spatial deictics "there" and "here", the personal deictics "you", and action verbs "to go", "to back up") contribute with specific meanings out of which a complex meaning is achieved. The pointing moving gestures offer an *analogical* sort of meaning while deictics and verbs qualify them in precise ways. The complementary role of signs in a semiotic node can be easily grasped if we imagine Jeff gesturing without talking (or vice versa).

Of course, these remarks do not amount to saying that speech alone (or written language, or algebraic language) cannot express the meaning of the graphs under study. Speech, written and symbolic language certainly can do it. But for many students, this is only possible after objectification has reached a certain level of depth. Objectification, as a social process in which we notice and become aware of a historically condensed meaning, rarely occurs instantaneously. The famous "Aha!" of traditional cognitive psychology is, indeed, the apex of a long process in the course of which we notice things and link them in a meaningful manner.

8 Jean's walk: the final interpretation

The bell rang and put an end to the students' discussion. The next day, the students returned to the math class and continued working on the activity. As in the previous case (Tina's walk), before the students could physically carry out the walk with the CBR and the calculator, they had to discuss and write a detailed interpretation of the graph to match. This pedagogical choice was motivated by our experience with other classes in which we have observed that the technological environment can block the students' reflective attitude. The students are often tempted to try different walks over and over until they find a similarity between the CBR-Calculator graph and the graph to match. Mathematical thinking is certainly a sensuous form of theoretical and practical action (Radford 2009a), but that does not mean that it can be reduced to a kind of empirical attitude based on trial and error.

Encouraged hence to reflect on the meaning of Jean's walk, the students started discussing the graph, drawing on their accomplishments from the day before. Daniel was designated to carry out Jean's walk. This time it was Mary who started the interpretation. She drew on her activity sheet the numbers 0, 1, 2, 3, and 4, at more or less equal distances. Putting her pen on number 4, she said:

22. Mary: So, Daniel, you have to start at 4.
23. Daniel: I back up 2 meters… I go to 2 meters in 2 seconds.
24. Mary: Ok…You start… at 4 (*she puts her pen around number 4*). Ok, you back up how much?
25. Daniel: 2 meters in 2 seconds.
26. Mary: So until 2 meters (*she draws an arrow as shown in* Fig. 7, Picture 1) in 2 seconds (*she writes the elapsed time above the arrow*). Ok.
27. Daniel: Then I go back to 4 meters in 4 seconds.
28. Mary: (*She slowly repeats Daniel's utterance*) Then you go back to 4 meters (*while speaking, she draws another arrow that goes from 2 to 4, as shown in Picture 2*) in how much?
29. Daniel: 4 seconds.
30. Jeff: In 4 seconds. So he advances 2 meters, then he … backs up.
31. Mary: Then… (*She leaves her pen still at the place indicated by number 4 in the drawing and waits for Daniel's answer*).
32. Daniel: I back up 3 meters (*Mary begins to draw an arrow that starts from 4 and goes to her left; she pauses),* in 3 seconds… Then, I go to 1 meter *(Mary continues drawing the arrow and ends it at number 1; see* Picture 3), ah! 9 meters.
33. Jeff: (*Correcting Daniel*) One meter!
34. Daniel: Yeah, one.
35. Mary: So, you should be at 1 meter when you're done?
36. Jeff: (*Answering the question*) At 1 meter when you're done.
37. Daniel: Yeah.
38. Mary: Then, you back up from 4 to 1 in how much?
39. Daniel: From 4 to 1 in 3 seconds … So 1 meter per second (*Mary writes "3 sec" above the last arrow; see Picture 4*).
40. Mary: 3 seconds. So it's how much? One, one-second per meter?
41. Daniel: Yeah
42. Mary: (*Satisfied with the result, she joyfully says*) Ah!!

As shown in Picture 4, Mary made five small marks (vertical lines). These marks are signs of some remarkable

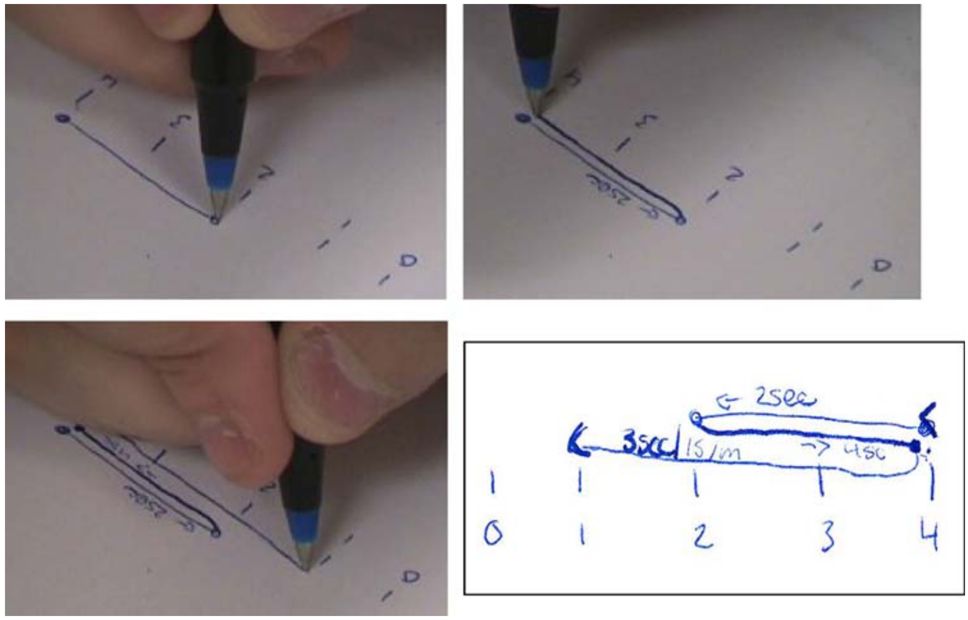

Fig. 7 Some aspects of the students' semiotic activity in the final interpretation of Jean's graph. Picture 1, M: "So until 2 meters…". Picture 2, M: "you go back to 4 meters". Picture 3, D: "… I go to 1 meter". Picture 4, M's drawing

places to be visited by Daniel in his walk to match Jean's graph. What distinguishes these places is their position (their farness or closeness) vis-à-vis a reference point. In the drawing, the reference point is identified by the mark associated with the sign "0." The places are "named" according to their distance from the reference point. This is what Mary means when she says: "you have to start *at* 4" (line 22). Line 23 shows in an unambiguous manner the distinction that the students are making between place and distance. Daniel says: "I back up 2 meters… I go to 2 meters in 2 seconds." The first "2 meters" mean *distance*; the second "2 meters" mean *place* ("I go *to* 2"). This distinction between place and distance is also made by Mary, but she does not do it verbally. Instead she has recourse to a sophisticated linkage of gestures, words and signs. This linkage constitutes another semiotic node.

In the next section I shall attempt to disentangle the semiotic node's elements and their objectifying organization.

9 An anatomical portrait of the semiotic node's configuration

9.1 Arrows as gestures

In line 22, Mary starts drawing the first arrow from place 4 to place 2. The arrow, I want to suggest, is a *gesture* mediated by the pen. But the pen's role is not merely to help Mary and her group-mates to imagine Daniel's walk. The arrow is more than a moving pointing gesture, for through the pen's ink, the gesture acquires a *permanent* perceptual status that makes it available to continuous scrutiny. For one thing, the arrows can be seen as occurring in a certain order. They depict Daniel's imagined global trajectory. But this trajectory would be incomplete if the remarkable places of Daniel's walk were not made explicit. The remarkable places must be brought to the realm of attention. Moving these places into the foreground is the role played by words and signs, such as the aural sign "two" and the written sign "2." Places like 1.3 or 1.9 do not need to be named. Not that they do not exist. They do, but, they are not relevant in the interpretation of the graph. Hence they remain ignored.

However, there is something that the *arrows-as-gestures* cannot signify: *time*. This is the price that the inked-gesture has to pay in order to gain perceptual permanence. Time was certainly involved in Mary's gestural movement from place 4 to place 2 as well as in the other movements. It appeared *embedded* in Mary's gestural motion as a sensorimotor act. But it vanished as soon as the gesture was completed. As mentioned in the introduction, space and time only become objects of conceptualization when they are experienced beyond the sensing body and the situated phenomenological spatial "here" and temporal "now." Mary hence writes the duration of the first part of the walk on the arrow. She writes: "2 sec" (see end of line 26 and Picture 4). Within the semiotic system of the drawing, the spatial and temporal description of the walk is *not* made in the form of a co-occurrence of spatial–temporal variables. The co-occurrence of spatial–temporal variables is in fact what a Cartesian graph does. It is the Cartesian expression of covariation that makes a Cartesian graph so particular and powerful. Within the written semiotic system of the drawing, however, co-occurrence is replaced by a *juxtaposition* of information: the arrow tells us the places from which we have to move and the duration of each trajectory.

To sum up, through words, symbols and gestures the students attend to different facets of the graph: words and symbols allow the students to capture elements such as places, distances, duration, while arrows-as-gestures allow them to deal with one aspect of time: the succession of events. All of these signs are *tied* together in a specific way—a way that shapes the configuration of the semiotic node.

9.2 Iconic orchestration

Let me continue the anatomy of the semiotic node produced by the students at this point in their discussion. In line 27, Daniel quickly mentions the spatial and temporal parameters of the second part of the walk: "Then I go back to 4 meters in 4 seconds." In line 28, Mary *repeats* Daniel's utterance. However, we should avoid seeing Mary's utterance as a mere *copy* of Daniel's. Mary's utterance is rather a personal attempt to understand the problem at hand. It is thus better to conceptualize Mary's discursive action as an *iconic orchestration*.

By iconic orchestration I mean here the personal expression of somebody's utterance reformulated with our own gestures, actions, words, tones and intentions. It is iconic in the sense that my formulation *resembles* the previous utterance. It is an *orchestration* in the sense that it is more than a copy: it allows my consciousness to reach a realm of understanding that is new for me. Iconic orchestration is a powerful mechanism of objectification—one that is frequently used by the students in the classroom.

Mary's semiotic activity shows perfectly the complexity of iconic orchestration. In addition to words, Mary incorporates gestures.[2] While she says "you go back to 4 meters," she makes the arrow-gesture that goes from place 2 to place 4. She loses track of time and asks Daniel "in how much?" Jeff, in line 30, specifies further Daniel's answer. He carefully distinguishes between time, place and distance. In line 35, Mary asks if the walk ends at place 1. Again, since her gestural focus is on place and distance, she asks for the duration. Daniel answers: "3 seconds" and concludes "So 1 meter per second," a space-time relationship that, after iconic orchestration, Mary transforms into "1 second per meter," showing that for her, time is not the variable that serves to express space; rather, it is the other way around.

[2] By utterance I do not mean something necessarily verbal. As Nemirovsky and Ferrara (2009) cogently argue, utterances can also incorporate actions, gestures and other sensorial elements.

Underneath the complex linkage of words, gestures, and symbols that gives a semiotic node its specific configuration is hence an undistinguishable fabric of voices and perspectives that makes the semiotic node and its configuration truly social. Through the prism of the semiotic node we find the students accessing a cultural signification. Without losing their subjective identity, the students' voices and actions become fused to, and undistinguishable from, those of their group-mates.

10 Semiotic contraction: the evolution of semiotic nodes

Right after this passage, Mary rehearsed the attained interpretation, this time in a more schematic form. Instead of using the inked-gestures, she made a series of gestures on her desk with her arms miming Daniel's walk:

43. Mary: So you go, you back up 2 in 2 seconds (*her utterance is accompanied by the hands' movement shown in* Fig. 8, Pictures 1 and 2). You move again to 4 (*she moves the hands from the bottom to the top of the desk as show in* Picture 3), it would be, you move 2 meters in 4 seconds this time, then you back up 3 meters in (Picture 4) 3 seconds.

Mary carries out here a semiotic contraction of the previous semiotic node—that is to say, a tightening of the complex linkage of arrow-gestures, symbols and utterances through which she was objectifying the meaning of the graph in the previous episode (Fig. 7). The lengthy linkage of arrow-gestures, symbols and utterances from line 24–42 lasted about one minute. The contracted version (Fig. 8) lasted about 11 s only. A lot of details were skipped. The initial place of the walk was now signified by the position of the hands on the desk (Picture 1); her hands moved from the loosely signified position of place 4 to place 2, imagined around the bottom of the desk, with Mary's body's position acting as the fountain or "place 0." The hands' movement was verbally qualified by travelled distance and elapsed time ("You back up 2 in 2 seconds"). The same gestural-verbal connection happened in the description of the next two segments of the walk. Mary did not need any longer to ask her group-mates for the duration of the various segments of the walk.

Fifteen seconds later, she goes still further in the semiotic contraction and, referring to the remarkable places to be visited, says, in an utterance that lasts 1.82 seconds:

44. Mary: It's 4 (Fig. 9, Picture 1), 2 (Picture 2), 4 (Picture 3), 1 (Picture 4) …

Through the effect of another semiotic contraction, words and gestures became coordinated in a more precise

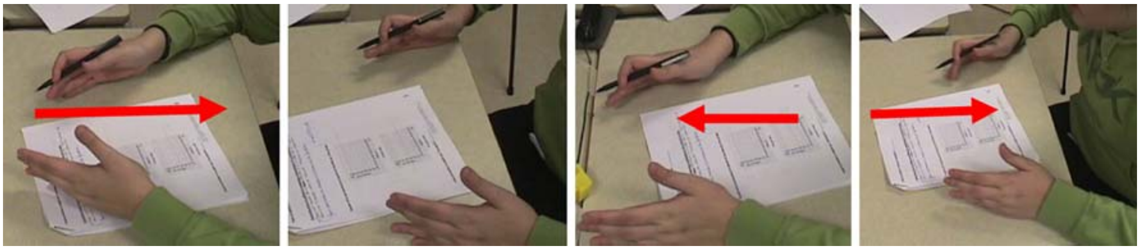

Fig. 8 *Pictures 1–4* from left to right. Mary makes a series of gestures to synthesize her interpretation of the graph

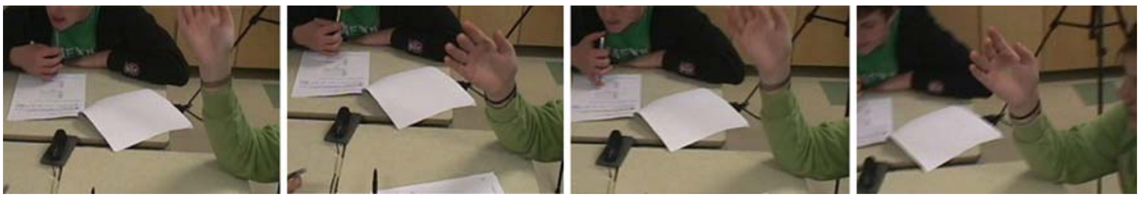

Fig. 9 Another semiotic contraction leading to a sharper link between gestures and words to signify the places to be visited in order to match Jean's walk

and direct way. Here what is brought to the fore is the question of *place*. The spatial gestures are reminiscences of the inked-gestures as words are the reminiscence of the vertical marks. The question of time was the object of another set of gestures made immediately after those shown in Fig. 9. When Mary finished making the previous gestures, Jeff said:

45. Jeff: (*Rephrasing Mary*) 4, 2, 4, 1. Yeah, but… the seconds… you would do
46. Mary: (*Interrupting Jeff and talking and gesturing quickly*) it's 2 (Fig. 10, Picture 1), 4 (Picture 2), 3 (Picture 3).

In line 45 Jeff makes an iconic orchestration of Mary's gestural and verbal utterance. In line 46, Mary brings to the fore the question of time as duration.

The text that the students wrote was the following:

> Jean starts his walk at 4 meters from the fountain. Jean must back up 2 meters in 2 seconds and then he must advance 2 meters in 4 seconds, and then back up 3 meters in 3 seconds.

11 Synthesis and concluding remarks

In a highly Kantian vein, Cassirer reminds us that "Space and time are the framework in which all reality is concerned. We cannot conceive any real thing except under the conditions of space and time" (Cassirer 1974, p. 42). Yet mathematical conceptualizations of space and time as conveyed by motion Cartesian graphs (and other complex signs, like algebraic formulas; see Radford 2009b), are far from trivial for novice students. Despite its apparent simplicity, a Cartesian graph entails a particular way of signifying that is at variance with more mundane or phenomenological ways of thinking and talking about things around us. For one thing, space and time have to be understood in *relational* terms. Both space and time have to be measured numerically against an arbitrary starting point. The spatial–temporal "coordinates" of points in a Cartesian space can only be expressed under these conceptual conditions. Our short historical incursion into the works of Aristotle and Descartes gave us some hints of the difficulties that had to be overcome in order to end up with a relational and systematic concept of origin, place and space.[3]

The analysis carried out in this article, whereby the students interpreted simple graphs, was based on the concept of objectification. I paid particular attention to the semiotic means to which the students resorted in order to connect with, and make sense of, the historically and culturally constituted meanings as found in contemporary Cartesian graphs. The analysis showed two different interpretations: one was based on the relational situation of the loci that define the walk as seen from the fountain (a relational interpretation). The other was based on the shape of the graph and its interpretation as a map whose centre is the walker's initial position (a map interpretation). The kinesthetic experience the students underwent in enacting the first graph (Tina's walk), and the intensive use of measuring tools (measure tape and chronometer) and the

[3] It was beyond the scope of this paper to investigate the question of time. Suffice it to say that the concept of time followed a similar historical developmental path as that of space (see, e.g., Radford 2008c, 2009b).

Fig. 10 Mary makes a sequence of gestures linked to words to signify the duration of each segment of Jean's walk

social space of interaction ensured by the activity's pedagogical design, allowed the students to overcome the first obstacles and engage in a process of objectification from which a meaningful interpretation of Jean's walk started to emerge.

The interpretation evolved as the students progressively disentangled the conceptual differences between space, place and origin. This progressive disentanglement is a central feature of objectification processes. They account for the manner in which, through action and reflection, the students become gradually aware or conscious of conceptual distinctions and links embedded in complex mathematical meanings.

In their attempt at reaching higher levels of mathematical meaning, the students moved into more and more profound levels of consciousness. There is indeed a dialectical movement between accessing higher layers of generality and meaning of mathematical objects and forming and reaching deeper levels of consciousness. While one movement goes "upwards," so to speak, the other goes "downwards."

To form and access those deeper levels of consciousness and conceptualization, the students resorted to gestures, symbols, and speech. These signs belong to different semiotic systems. This is why Arzarello (2006) says that they form a "semiotic bundle." Now, because of their difference in origin, these signs entail different modes of signifying (Radford et al. 2007). They are not fully exchangeable or translatable, much as it is impossible to translate a poem into a string quartet or vice versa. What these signs bring is a certain complementarity in meaning formation. Following Vygotsky's idea of the child's first writings as gestures,[4] and in an attempt to capture as much as possible the meaning of the kinaesthetic action underpinning the drawing of the arrow, I suggested to conceive of the students' arrows as inked gestures. The problem was then to understand the respective role of these signs vis-à-vis the other signs. Aural words and written symbols have a way of signifying that cuts, so to speak, the realm of meaning along different axes from those cut by arrows-as-gestures.

To understand the role of signs, I drew on the concept of semiotic node. A semiotic node is a theoretical way to account for the complementary role of signs in the process of objectification. We observed how through words and written signs the students brought to their consciousness a clear relationship between precise locations and travelled distances, while arrows-as-gestures helped them to cope with one aspect of time—succession or the *chronological* dimension of events. The bringing of time into the realm of consciousness was in fact a slow process that somehow lagged relatively behind the consciousness of space. "Temporal relationships," Piaget remarked, "are organized in the things before they are organized in our own consciousness" (Piaget 1973, p. 28). Time was brought forward in the sensorimotor action of drawing the arrow, but became an object of discourse as a form of temporal organization only, clearly marked by the adverb "then," in the students' discourse and text of their final written interpretation. "Time proceeds from the organization of movements and this is why it is dominated from the outset by spatial coordination." (Piaget 1973, p. 30).

The anatomy and evolution of semiotic nodes offered us a window into the students' conceptual path. The mechanism of iconic orchestration clearly showed the social dimension of knowledge objectification.[5] Voices merged in a splendid way, making knowledge attainment a truly joint endeavor. As the mathematical relationships embedded in the graphs were objectified, the students proceeded to contract semiotic nodes and their sign-configurations. Interpretations were made in shorter spans of time. The contraction of signs entailed an abstraction of meanings and deeper levels of consciousness. As semiotic contractions proceed, the students did not feel the need to take into account details that became integrated into the surviving signs. For example, the students' first sentence of the final written text, namely "Jean starts his walk at 4 meters from the fountain," integrates the intensive gestural activity show in Pictures 5 and 6, among others.[6]

[4] "[T]he written sign is very frequently simply a fixed gesture" (Vygotsky 1997, p. 133).

[5] I have been using the term orchestration in the manner of Bakhtin (1981).

[6] For other examples of the role of iconicity and contraction in objectifying processes, see Radford (2008d).

The investigation and pedagogical use of the different ways in which students and teachers resort to signs and artifacts in processes of teaching and learning still deserves more investigation. The awareness of some general paths followed by the students' objectification and their relationship with the activities we offer to them in the classroom may help us to design more encompassing contexts where students can engage in meaningful ways with mathematical concepts.

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