Mathematical Representation at the Interface of Body and Culture

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CHAPTER 3

SIGNIFYING RELATIVE MOTION

Time, Space and the Semiotics of Cartesian Graphs

Luis Radford

Kant was perhaps the first to have realized how entrenched our knowledge of the world is in the way we experience it through space and time. Since all our acts, even the most mundane, presuppose a temporal and a spatial dimension, space and time, Kant reasoned, constitute the very conditions of knowledge: they are in us and precede all experience whence knowledge results—to use Kant's terminology, space and time are *pure intuitions*. While agreeing with Kant's emphasis on the importance of space and time in the experience we make of the world, current research on epistemology, anthropology, and the arts suggests, however, that space and time are neither apriori conceptual categories, nor the constructs of the allegedly Piagetian universal logico-mathematical structures. Space and time are rather cultural conceptual categories. The culture in which we happen to live not only provides us with the general theoretical framework in which to temporally and spatially

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experience our world but also insinuates paths to reflect about it, both at a practical and a theoretical level.

THE ELUSIVENESS OF TIME

Let us go "back in time" for a moment and have a look at a problem historically considered the first in its genre. It is a problem from the eighth century, by Alcuin of York, one of the principal figures of Charlemagne's educational reform. The problem, included in a school textbook—*Problems to Sharpen the Young*—reads as follows:

> There is a field 150 feet long. At one end is a dog, and at the other a hare. The dog chases when the hare runs. The dog travels 9 feet in a jump, while the hare travels 7 feet. How many feet will be traveled by the pursuing dog and the fleeing hare before the hare is seized? (Alcuin, 2005, p. 68)

The problem, written with a didactic purpose, lets us get a glimpse at the manner in which, at this point in the Middle Ages, space and time became object of scientific enquiry and mathematical discourse. The statement of the problem reveals the pregnant phenomenological dimension of a world still not invaded by clocks measuring time with great digital precision. Whereas space is measured by "feet"—an already abstract unit that still keeps its embodied form and evokes the spatial relationship between the motion of an individual and its surrounding—time is not explicitly mentioned in the problem. To compare the space traveled by the dog and the hare, Alcuin resorts to the idea of *jump*. In a jump, the dog travels 9 ft, while the hare travels 7 ft. How then, without explicitly employing the idea of time, can this problem be solved?

Let us turn to the solution. Alcuin says:

> The length of the field is 150 feet. Take half of 150, which is 75. The dog goes 9 feet in a jump. 75 times 9 is 675; this is the number of feet the pursuing dog runs before he seizes the hare in his grasping teeth. Because in a jump the hare goes 7 feet, multiply 75 by 7, obtaining 525. This is the number of feet the fleeing hare travels before it is caught. (Alcuin, 2005, p. 68)

The first calculation (i.e., the half of 150) corresponds to the number of jumps. The question is: are these the dog's jumps or the hare's jumps? Jump (*saltu* in the original Latin), is, like foot, an abstract idea: it is neither the dog's nor the hare's. It evokes a phenomenological action that unfolds over a certain *duration*. It is only in this oblique way that time appears in the problem. After each jump, the dog comes 2 ft closer to the hare. Thus, the dog will need 75 jumps to catch the hare. This number of jumps is multiplied by the 9 ft that the dog goes in a jump—the medieval expression of "speed"—and then by 7, that is, the number of feet that the hare goes in a jump. The resulting numbers are the space travelled by each animal.

Problems like this became popular later on. There is a problem in a fourteenth century Italian manuscript, composed by Piero dell'Abacco that reads as follows:

> A fox is 40 paces ahead of a dog, and three paces of the latter are 5 paces of the former. I ask in how many paces the dog will reach the fox. (dell'Abacco as translated by Arrighi, 1964, p. 78)

Here, distance is measured by "paces" and, like in Alcuin's problem, time is mentioned implicitly through *motion*.

The difficulties in dealing with time are well known when it comes to philosophers and epistemologists. Like Alcuin and dell'Abacco, Aristotle related time to motion. As he said, "we perceive movement and time together … time and movement always correspond with each other" (Aristotle, Physics, IV, XI, p. 62). And Grize (2005), commenting on the elusiveness of time as expressed in the works of Aristotle and Bishop Augustine, notes that within this perceptual frame of reference, "A distance is measured in relation to a length; time, however, cannot be measured by a time anymore than a temperature can be measured by a temperature unit" (p. 69). Time, hence, remained an implicit notion, embedded in the duration of motion (like the sun's analogical projected shadow on the sundials or the bodily jumps Alcuin's text talks about), until mechanical clocks, exploiting a rhythmic pendulous repetition, extricated it from its conceptual limbo and made time a precise theoretical object in its own right.

It therefore does not come as a surprise that in many Medieval and early Renaissance mathematical problems that were accompanied by drawings, time remains expressed in the perceptual motion of the moving objects (see Figure 3.1). Like Alcuin, dell'Abacco was also a teacher. We may conjecture that in discussing these problems with their students, both schematically enacted the moving objects through gestures or body movements.

Be this as it may, it might not be useless to compare the medieval and modern solutions of the previous problems. Let us consider dell'Abacco's problem. The medieval solution proceeds by a comparison of traveled distance: three dog' steps are equal to 5 fox steps or 3D = 5F. Although dell'Abacco does not mention it explicitly, he assumes that *while the fox goes one step the dog goes one step as well*. Time appears in the problem through this hidden assumption. From 3D = 5F, using a rule of three,

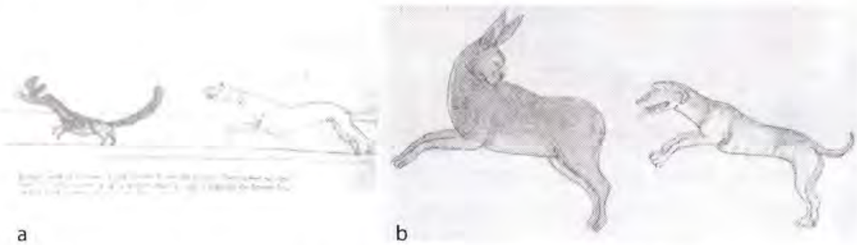

Figure 3.1. a. The drawing accompanying Alcuin's problem. b. The drawing accompanying dell'Abacco's problem.

dell'Abacco deduces that 5D = 8 1/3F. Thus, when the dog takes 5 dog steps, the fox takes 5 fox steps and the distance between the fox and the dog diminishes by 3 1/3 fox steps. Knowing that they are 40 fox steps apart, and continuing using the rule of three, dell'Abacco concludes that the dog will need to go 60 dog steps to reach the fox.

The modern reader might find this fourteenth century solution a bit strange, if not unclear. A modern solution starts, in fact, by expressing the problem in terms of velocities, hence in relating traveled distance to units of time. But to think in terms of specific quantified "small" units of time (like seconds) is exactly what the medieval thinkers did not do. In tune with the modern solution, let us assume that the fox travels 5 fox steps per unit of time; then, the dog travels 5 dog steps or 25/3 fox steps per unit of time. Referring to the spatial place where the *dog* was located at the beginning of its race, the distance traveled by the dog (expressed in fox steps) is:

$$d_d = 8\frac{1}{3}t \qquad (3.1)$$

Referring to the *previous* spatial point (i.e., the *dog*'s initial place), the distance travelled by the *fox* is:

$$d_f = 40 + 5t \qquad (3.2)$$

The point at which the dog catches the fox is characterized by the equality:

$$8\frac{1}{3}t = 40 + 5t \qquad (3.3)$$

And when we solve this equation, what we get is not the distance travelled by one or the other, but *time*—the time that both animals have been in motion:

$$t = \frac{40}{3\frac{1}{3}} \text{ or } t = 12 \qquad (3.4)$$

The sense of our modern solution is quite different from Alcuin's and dell'Abacco's. We, more or less consciously, forget what $8\frac{1}{3}$ and $5t$ mean and mechanically subtract them to get $3\frac{1}{3}t$, which we then equate to 40. From there, we find t, the numerical value of this elusive concept that was not even mentioned in either dell'Abacco's statement of the problem or in its solution. Now we substitute the value of t in the first equation and get $d_d = 8\frac{1}{3} \times 12 = 100$ fox steps (or 60 dog steps).[1]

UNIFYING SYSTEMS OF KNOWLEDGE REPRESENTATION

There is also something very different in the modern solution. The motion of both the dog and the fox were referring to a *same spatial point* (the initial position of the dog in the race). In Medieval and early Renaissance problems, the motion of the moving objects involved remained without being described into a unifying system of reference (see dell'Abacco's solution to the fox problem in note 1, where calculations are made by *comparison* of "speeds" and not by *integration* of data into a same totality). From a semiotic point of view, there is a striking similarity between the mathematical problems and the paintings and drawings of the Middle Ages and early Renaissance. Their respective signs revolve around a main "subject" without being linked by a truly functional unifying system of representation. For instance, objects surround the main subject (e.g., the saint) in a juxtaposed manner. Each object contributes to the whole meaning of the drawing or painting by addition of its particular meaning (see Figure 3.2a).

The order of the signs in paintings and mathematical texts became profoundly transformed by the concurrent invention of the technique of perspective and algebraic symbolism (Radford, 2006). The new cultural forms of knowledge representation continued to privilege a certain subject, but now there was a relational link ensuring the relationship between a chosen central object and other objects (see Figure 3.2b, c).

The emergence of a Cartesian system of coordinates and its central point (0, 0) was one of the most sophisticated ways in which to express the complex set of relations between the objects described in the situation at hand. This was one of the crucial developmental steps in the mathematical


A
B
C
fox's initial position
dog's initial position
F
40
D
$d_d = 8\frac{1}{3}t$
$d_f = 40 + 5t$

Fig. 3.2

study of motion. The creation of such a system required, in particular, a mathematical reconceptualization of space and time. It is hence not surprising that Cartesian graphs rest on a sophisticated manner of signifying that, from an ontogenetic viewpoint, is far from transparent to novice students.

LEARNING AND THE HISTORICAL DIMENSION OF KNOWLEDGE

What is it then that we ask the students to accomplish when we expect from them to describe problems about motion through algebraic formulas and Cartesian graphs? As intimated by my previous remarks, a graph is a complex mathematical sign. It serves to depict, in specific ways, certain states of affairs. Instead of being merely a reproduction of these, a graph supposes a selection of elements: what it depicts is *relationships* between them. This is why the making of a graph of an elementary phenomenon, such as the motion of an object, is like putting a piece of the world on paper (or electronic medium). But because Cartesian graphs are not copies of the phenomena that they depict or represent, making and interpreting them is not a trivial endeavor. A Cartesian graph rests on a sophisticated syntax and a complex manner of conveying meanings. It is the understanding and creative use of this complex historically constituted cultural form of signification that we expect the students to accomplish when dealing with graphs, and, as we know, it does not go without justified difficulties.

The investigation of the difficulties surrounding students' understanding of graphs has been an active research area in mathematics education.[2] In this chapter, I contribute to the research on graphs by looking at students' processes of graph understanding. I am interested in particular in researching the way in which students attempt to make sense of graphs related to problems of relative motion—an area little investigated thus far. To so do, drawing on a Vygotsky's historical-cultural school of thought, I consider here mathematical thinking as a cultural and historically constituted form of reflection and action, embedded in social praxes and mediated by language, signs and artifacts (Radford, 2006). A Cartesian graph is an artifact for dealing with and thinking of cultural realities in a mathematical manner. But, as mentioned previously, this artifact is not transparent: it bears the imprint and sediments of the cognitive activity of previous generations which have become compressed into very dense meanings that students have to "unpack," so to speak, through their personal meanings and deeds.

This process of "unpacking" is the socially and culturally subjective situated encounter of a unique and specific student with a historical conceptual object—something that I have previously termed *objectification*

(Radford, 2003). The construct of *objectification* refers to an active, creative, imaginative, and interpretative social process of gradually becoming aware of something. Within this context, understanding the making and meaning of a graph, the way it conveys information, the potentialities it carries for enriching and acting upon our world, rests on processes of objectification mediated by one's voice, others' voices and historical voices (Boero, Pedemonte, & Robotti, 1997). Objectification is indeed a multi-voiced encounter between an "I," an "Other," and (historical and new) "Knowledge."

Now, one of the distinctive traits of human cognition is its *multimodal* nature. What this means is that thinking is profoundly dependent upon the cultural artifacts that we use and our own body. As Gallese and Lakoff (2005) expressed the idea, "the sensory-motor system not only provides structure to conceptual content, but also characterizes the semantic content of concepts in terms of the way that we function with our bodies in the world" (p. 456). In other words, signs (language included), artifacts, and our body along with its various senses are vehicles for thought. Within this context, in the objectification of mathematical knowledge, recourse is made to body (e.g. kinesthetic actions, gestures), signs (e.g., mathematical symbols, graphs, written and spoken words), and artifacts of different sorts (rulers, calculators and so on). In the practical investigation of students' understanding of graphs, I therefore pay attention to the students' gestural, kinesthetic, symbolic and discursive activity as they attempt to make sense of a graph.

MAKING SENSE OF GRAPHS—A CLASSROOM EXAMPLE

In this chapter, I discuss the attempt made by one group of tenth-grade students in their effort to understand a graph representing the relative motion of two bodies. The data, which comes from a five-year longitudinal research program, was collected during classroom lessons that are part of the regular school mathematics program in a French-language school in Ontario. In these lessons, designed by the teacher and our research team, the students spend substantial periods of time working together in small groups of 3 or 4. At some points, the teacher (who interacts continuously with the different groups during the small group-work phase) conducts general discussions allowing the students to expose, compare and improve their different solutions.

The data that will be discussed here comes from a lesson featuring a graphic calculator TI 83+ and a probe—a Calculator-Based Ranger or CBR (a wave sending-receiving mechanism that measures the distance between itself and a target). The students were already familiar with the calculator graph environment and the CBR. In previous activities, they had dealt with a fixed CBR and one moving object. In the activity that I discuss here, the students were provided with a graph and a story. The graph showed the relationship between the elapsed time (horizontal axis) and the distance between two moving children (vertical axis) as measured by the CBR (see Figure 3.3). The students had to suggest interpretations for the graph and, in the second part of the lesson (not reported here), to test it using the CBR.

Here is the story:

> Two students, Pierre and Marthe, are one meter away from each other. They start walking in a straight line. Marthe walks behind Pierre and carries a calculator plugged into a CBR. We know that their walk lasted 7 seconds. The graph obtained from the calculator and the CBR is reproduced below.

(See Figure 3.3 for the illustration and the graph.) The disposition of the axes in the Cartesian graph reflects the modern concept of space and time as continuous variables represented by oriented lines. We chose three main "events" to be interpreted—the segments AB, BC and CD. They were different not only in their successive positions in the graph but also in their orientation. Within the Cartesian semiotic system, "events" signify in a *relational* and *unifying* manner. Of course, as already mentioned, this historically constituted manner of signifying is far from trivial. In what follows, we see that in order to be able to interpret these events, the students will have to unpack space and time from the phenomenological expression embedded in motion.

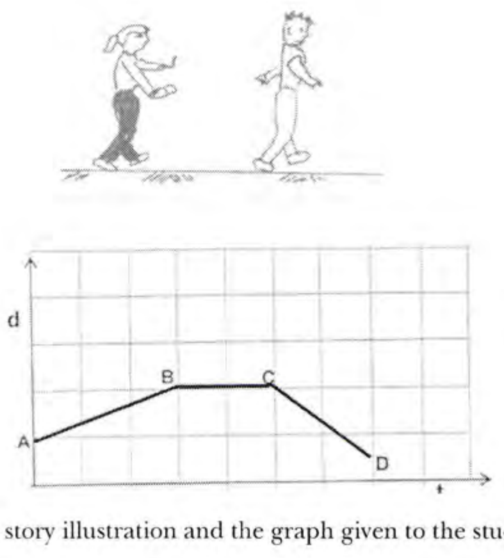

Figure 3.3. The story illustration and the graph given to the students.

STUDENTS' FIRST INTERPRETATION

I focus on one 3-student group and present some excerpts of the students' processes of interpretation, with commentaries on the progressive manner in which objectification was accomplished. The students were Maribel (M), Marie-Jeanne (MJ) and Carla (C). After discussing the problem for a few seconds, in Line 1 (L1), Maribel offers an interpretation of segment AB:

1. M: Then, there (*she moves the pen along the segment AB*) he moves for what … 3 seconds?
2. MJ: Yeah! …
3. M: Then here (*referring to segment CD*) he …
4. MJ: (*Moving the pen over segment CD, from D to C*) He goes backwards for 2 seconds … (*See Figure 3.4a*)
5. M: (*Summarizing the discussion,* she *moves the pen on the desk and says*) He moves away from Marthe for 3 seconds, and then he stops (*she stops the pen on the desk at a point that would correspond to the point B in the graph; see Figure 3.4b*), so (*referring to segment BC, she moves the pen further along the desk; see Figure 3.4c*) he might have like dropped something for 2 seconds, and (*moving the pen back this time; Figure 3.4d*) he returns towards Marthe …
6. MJ: Well … does it (*referring with a pointing gesture to the distance axis*) have to have like … specific things?
7. C: His speed increases a bit …
8. MJ: … What I mean, it's, like, the distance, does it have to be specific?
9. M: No.
10. C: … If the speed increases … it would be a curve, right?
11. MJ: (*Referring to the speed*) It's constant.
12. C: So … Pierre moves away from Marthe at a constant speed for 3 seconds …
13. MJ: (*Continuing C's utterance*) takes a 2 second pause and returns …
14. M: (*Continuing MJ's utterance*) towards Marthe…
15. MJ: (*After a short reflection*) Well, if she walks with him, so, it [the graph] doesn't really make sense!

In L1, Maribel attends to the first event by *moving* the pencil from A to B while mentioning *verbally* its *duration* ("*he moves for what … 3 seconds?*"). In L4, MJ interprets the third event, segment CD, as Pierre going back for 2 seconds, moving her pen over segment CD in the direction from D to C (Figure 3.4a). In L5, the ideas are synthesized in a way that the segments AB, BC and CD represent Pierre moving away, stopping and coming back. The synthesis is organized in terms of *Pierre's motion* and its *duration*. Distance has not been mentioned. In L6, MJ asks if they have to consider particular values for the distance. However, the focus is put on a vague

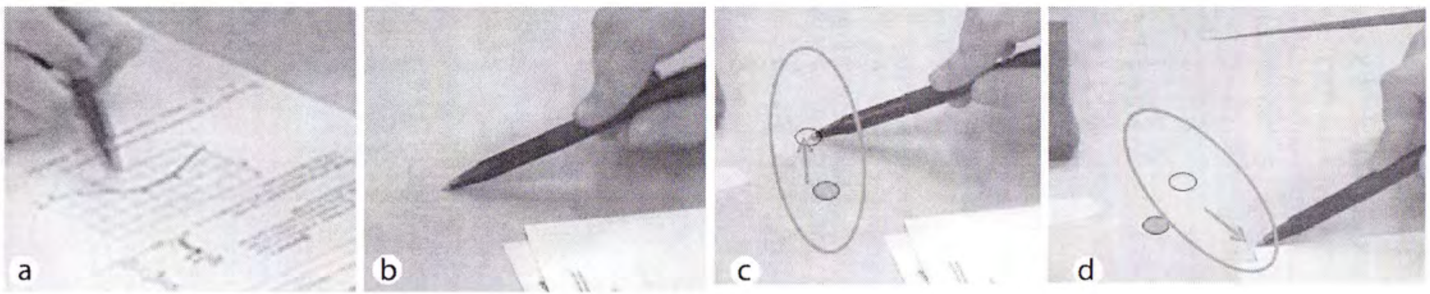

Figure 3.4. Some gestures made by MJ (a) and M (b–d) during the first interpretation of the graph.

qualitative idea of speed and, in L9, the idea of distance is quickly dismissed. The students' approach resembles Alcuin's and dell'Abacco's in one point: the emphasis on the phenomenological aspect of motion. But it differs in other important aspects. Duration is here unproblematically quantified in terms of seconds; furthermore, in contrast to the historical texts discussed previously, the quantification of speed is not brought into the students' discourse.

Although the students' current interpretation is not yet resonant with the expected mathematical interpretation, we can see that the students' ideas have been forged through a complex coordination of perceptual, kinesthetic, symbolic, and verbal elements. The students' dynamic pointing gestures and actions with the pen are not merely redundant mechanisms of communication, but key embodied means of knowledge objectification.

A closer look at the gestural and verbal interaction reveals some aspects of the students' unfolding interpretation. In particular, in her synthesis in L5, Maribel's gestures and actions allow us to see that the interpretation is entangled in a *juxtaposition of spaces*. On her desk, while referring to segment AB and saying "He moves away from Marthe," she moves the pen as if enacting Pierre's walk (Figure 3.4b). This motion occurs in what we may term the *phenomenological space of imagined motion*. While she says, "he stops," she *continues moving the pen* in a direction that now evokes the horizontal segment BC (Figure 3.4c). Here, Maribel's motion is not enacting Pierre's walk, but the *passing of time* (as they explain, in this part, Pierre is considered to be still). This gesture hence occurs in the *Cartesian space of representation*, where a horizontal segment represents the passing of time. And right after this, she goes back to the *phenomenological space of imagined motion*, where her gesture continues evoking Pierre's walk: Now Pierre is imagined as if being at the point that in his walk would correspond, in the students' interpretation, to point C. Instead of following the inclination of segment CD in the Cartesian graph, Maribel moves her pen *back*, towards what would be Marthe's position, saying, "and he returns towards her" (Figure 3.4d). It is at the end of this episode that Marie-Jeanne reminds her group-mates that Marthe is moving too, so that, according to the current interpretation, the graph "doesn't really make sense!" For, if Marthe is moving, Pierre will no longer find her when he walks back towards her initial position!

A SECOND INTERPRETATION

Twenty seconds later, Maribel offers a refined interpretation that tries to address the issue raised by Marie-Jeanne:

16. M: Well technically, he walks faster than Marthe ... right?
17. MJ: She walks with him, so it could be that [...] She is walking with him, so he can walk faster than her (*she moves the pen on segment AB; see Figure 3.5*). [He] stops (*pointing to points B and C*) ...
18. M: No, there (*referring to the points B and C*) they are at the same distance ...
19. C: (*After a silent pause, she says with disappointment*) Aaaaah!

The graph interpretation has changed: In the previous episode, the segments were seen as predicating something about *Pierre*. Marthe was not really part of the story told by the graph. In L16, Maribel introduces the two-variable comparative expression "X walks faster than Y." In L17, Marie-Jeanne reformulates Maribel's idea in her own words, while producing a more sophisticated interpretation. Indeed, L17 contains three ideas: (a) Marthe walks with Pierre; (b) Pierre walks faster than her, and (c) Pierre stops.

Although improved, the interpretation, as the students realize, is not free of contradictions. These contradictions result in part from incautiously endowing the segments with meanings coming from the *phenomenological space of imagined motion* and the *Cartesian space of representation*. The meanings overlap, resulting in a global incoherent interpretation. Even if, at the discursive level, Marthe is said to be walking (L17), segment AB is still understood as referring to Pierre's motion (MJ says at the end of the movement of his pen in Figure 3.4c, "[He] stops"). However, segment BC is interpreted not in terms of *motion* but of *distance* (L18): BC is interpreted as indicating that the distance between Pierre *and* Marthe remains the same during this period of time. So, while segment AB is about Pierre's motion, segment BC predicates something about both children's distance. The interpretation of the events does not yet fit into a unifying systemic logic of

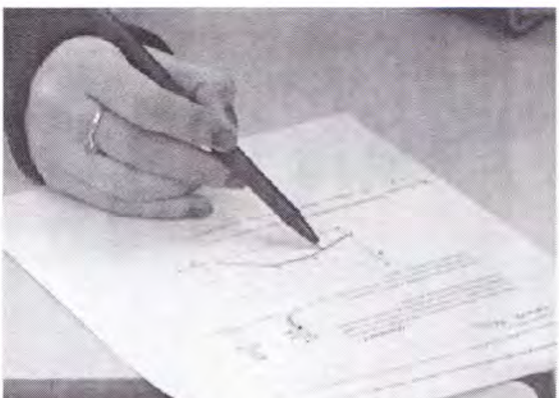

Figure 3.5. MJ moves the pen from A to B, meaning Pierre's motion (L17).

knowledge representation. The oddity of the interpretation leads to a tension that is voiced by Carla in Line 17 with an agonizing "Aaaaah!" The partial objectification bears an untenable incongruity.

THE TEACHER

The students continued discussing and arrived at a new interpretation: In the story-problem, the students are told that Pierre and Marthe are 1 meter apart from each other. Thus, the new interpretation: Pierre and Marthe maintained a distance of 1 meter apart throughout. The students could not agree on whether or not this interpretation was better than, or even compatible with, Maribel's interpretation (L16). Having reached an impasse, the students decided to call the teacher (T). When he arrived, Marie-Jeanne explained her idea, followed by Maribel's opposition; it is this opposition that is expressed in L20:

20. M: No, like this (*moving the pen along segment AB*) would explain why like, he goes faster, so it could be that he walks faster than her ...
21. T: Then if one is walking faster than the other, will the distance between them always be the same?
22. M: No, (*while moving the pen along AB, she says*) so he moves away from the CBR and then.... What happens here (*pointing to segment BC*), like?
23. MJ: He takes a brake.
24. T: So, is the CBR also moving?
25. M: Yes.

In L21, the teacher rephrases the first part of Maribel's utterance (L20) in a hypothetical form to conclude that, under the assumption

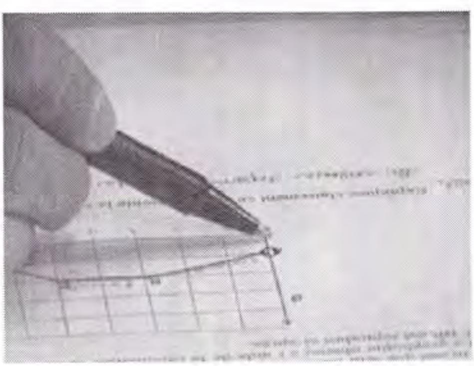

Figure 3.6. The teacher moves the pen back and forth between the intersection of the axes and point A.

that Pierre goes faster, the distance cannot be constant. Although inconclusive from a logical point of view, the teacher's strategy helps move the students' discourse to a new conceptual level. Maribel's L22 utterance shows, indeed, that the focus is no longer on relative speed but on an emergent idea of *relative distance*. In L22, the moving gesture along segment AB is the *same* as Marie-Jeanne's in Figure 3.5, but its *content* is *different*: although the gesture still enacts Pierre's motion in the *phenomenological space of imagined motion*, it signifies Pierre *moving away* from the CBR. However, as shown in L22, the students still have difficulties providing a coherent global interpretation of the graph. How to interpret BC within the new relative motion context? Drawing on Maribel's utterance (L22), in L24 the teacher suggests a link between Marthe and the CBR, but the idea does not pay off as expected. He then tries something different.

THE MEANING OF SEGMENT 0A

26. T: OK. A question that might help you.... A here (*he circles point A*). What does A represent on the graph? (*He moves the pen several times between the intersection of the axes and A; see Picture 6*)

27. MJ: Marthe.

22. M: No, (*while moving the pen along AB, she says*) so he moves away from the CBR and then.... What happens here (*pointing to segment BC*), like?

28. T: This here (*pointing to the intersection of the axes*), is it zero? (*he writes 0 at the intersection of the axes*)? We'll only talk about the distance. OK? (*He moves the pen again over 0A as in Line 26*).

29. MJ: 1 meter.

30. T: (*Rephrasing MJ's answer*) it (*i.e., the segment 0A*) represents 1 meter, right? ... 1 meter in relation to what?

31. M: The CBR ...

32. T: OK. So, does it represent the distance between the two people?

33. M: So this (*moving the pen along the segments*) would be Pierre's movement and the CBR is 0.

34. MJ: (*Interrupting*) First he moves more ...

Capitalizing on the emerging idea of relative distance, the teacher's strategy now becomes one of calling the students' attention to the *relational meaning* of a particular segment—the segment OA. This segment has passed unnoticed so far. It refers, in relational terms, to the beginning of the story-problem—the distance between Pierre and Marthe. In this sense, it appears as a good point from which to launch a Cartesian interpretation of the graph in general and segment AB in particular.

The teachers captures the students' attention in three related ways:

- writing (by writing 0 and circling point A);
- gesturing (by moving the pen back and forth between 0 and A);
- and verbally (L26).

Since the students see the graph as a kind of map of the position of Pierre and Marthe, they insist on locating them somewhere in the Cartesian graph. Thus, in L27, point A is associated with Marthe. However, it is precisely not this phenomenological reading that the teacher aims at, but a relational one. So, in L28, he formulates the question in a more accurate way and takes advantage of the answer to further emphasize (L30) the idea of the relative meaning of the distance. Line 33 includes the awareness that the CBR has to be taken into account, while L34 is the beginning of an attempt at incorporating the new significations into a more comprehensive account of the meaning of the graph.

The teacher left the students saying, "I do not say anything more!" and went to talk to another group. The students thus entered into a new phase of knowledge objectification. They continued discussing in an intense way. Here is an excerpt:

35. C: He moves away from her, he stops then comes closer.
36. M: But she follows him.... So, he goes faster than she does, after, they keep the same distance apart.

In L35, Carla still advocates an interpretation of the graph that suggests a fragile understanding of relative motions. In the first part, she makes explicit reference to Marthe ("He moves away from her"), but in the second and third part of the utterance, Marthe remains implicit. The *phenomenological space of imagined motion* and the *Cartesian space of representation* are not linked suitably yet. As a result, an ambiguity remains.

In L36, Maribel offers an explanation that seems to overcome the ambiguity. Even though the segment AB is expressed in terms of rapidity, the previously reached awareness of the effect of rapidity in the increment of distance makes the interpretation of BC coherent. Maribel recapitulates the students' efforts and says before the group starts writing an interpretation:

> Maybe he [Pierre] was at 1 meter (*pointing to A*) and then he went faster; so now he is at a distance of 2 meters (*moving the pen in a vertical direction from BC to a point on the time axis, see Figure 3.7a, b*); and then they were constant and then (*referring to CD*) they slowed down. Would that make sense?

For the first time in their process of objectification, the students provide an interpretation that stresses the relational meanings of the Cartesian space of representation. The interpretation still needs to be refined. For instance, in the interpretation of CD, Maribel did not specify in which manner they slowed down. Was it Pierre who slowed down? Was the reduction of distance the effect of Marthe increasing her speed? Was it something else? Nonetheless, the students were able, to a certain extent, to put into correspondence the relational meaning and the phenomenological space of imagined motion. Key in this accomplishment was Maribel's vertical gesture (Figure 3.7), which makes clear the explicit insertion of the idea of distance in the students' discourse. To understand the students' process of objectification, this gesture needs to be put into correspondence with the teacher's gesture shown in Figure 3.6.

Literary critic Mikhail Bakhtin (1986) once remarked that "Each word contains voices that are sometimes infinitely distant, unnamed, almost impersonal (voices of lexical shadings, of styles, and so forth), almost undetectable, and voices resounding nearby and simultaneously" (p. 124). Figures 3.6 and 3.7 suggest that the same is true of gestures: Maribel's gesture contains the conceptual intention of the teacher's gesture. Naturally, Maribel's gesture is not just a copy of the teacher's; it has been endowed with personal tones and displayed in a different part of the graph. Nonetheless, it bears an almost undetectable voice that has served as inspiration for seeing something new.

In writing their answer, the students, however, realized that something important was missing: the interpretation needed to include Marthe in an explicit way. Naturally, writing requires one to make explicit, and thereby objectify, relationships that may remain implicit at the level of speech and gestures. Maribel's activity sheet contains the following answer:

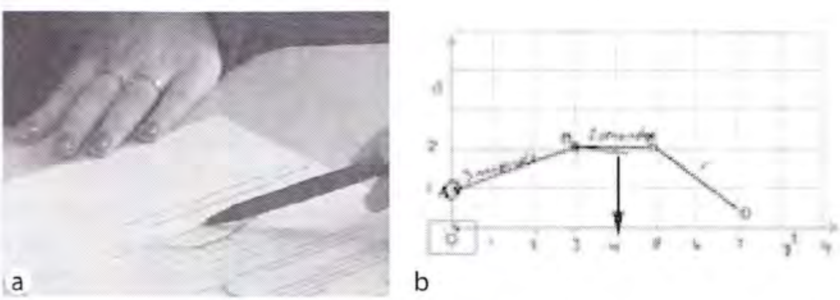

Figure 3.7. a. Maribel makes a vertical gesture that goes from BC to the time axis and is indicated by an arrow in (b). This gesture is a generalization of the teacher's gesture (Figure 3.6).

> Pierre moves away from Marthe by walking faster for 3 seconds. He is now 2 meters away from her. They walk at the same speed for two seconds. Pierre slows down for two seconds so he gets closer to Marthe. (Maribel)

The text is now about the relative distance of Pierre and Marthe (compare this text to Maribel's interpretation in the previous section). It is organized in accordance with the sequentiality of a *common* time—an element whose importance was pointed out in the second section and in Figure 3.2. The first sentence tells us how to imagine what happened during the first three seconds. The emphasis is on the distance: As a result of walking faster, Pierre is "now" 2 meters away from Marthe. The interpretation of segment BC follows: Pierre and Marthe are said to be walking for 2 seconds, at the same speed. Here, the students do not feel the need to tell us that the distance between them remains the same. What was hard to figure out was indeed that here they were walking at the same speed. Thus, this is what needs to be said. In addition, as in the first and last sentence, speed is left without being quantified. In the third part, the idea of the distance is brought to the fore again: Pierre slows down and so he gets closer to Marthe.

Here we can see that the link between the *phenomenological space of imagined motion* and the Cartesian space of representation was improved. The static segments of the Cartesian graph were endowed with a dynamic interpretation where relational aspects of relative distance became partially linked to the phenomenological space of imagined motion. The central concept in the production of the students' narrative was motion. Moving in a certain way (faster, at the same speed, slowing down) explained what happened with the distance between Pierre and Marthe. We may say that the students' interpretation remains primarily phenomenological, rather than relational. In other words, the relational meanings conveyed by the Cartesian graph are still not the primary motor of interpretation.

SHARING IDEAS

This primacy of the phenomenological over the relational became clearer when, in the next part of the activity, the students were asked to calculate the speeds involved in the graph. For segment AB, two solutions were obtained: Maribel suggested 1/3 and Carla 2/3. The second speed was based on the idea that Marthe was at the origin of the graph. Following a classroom practice encouraged by the teacher—where students are invited to visit other groups to submit, compare and discuss their ideas—Marc

(Mc), a student from a group at the opposite side of the classroom, came to discuss with Carla's group.

Maribel explains her calculations to Marc:

37.	M:	This (*referring to the speed associated with the first event—segment AB*) is one third.... Do you understand?
38.	C:	(*Opposing Maribel's ideal*) No!
39.	M:	(*Pointing to the origin of the Cartesian graph; see Figure 3.8a*) Marthe, isn't she here?
40.	Mc:	No, Marthe is (*trying to point somewhere, but his finger never lands on the graph*).... Okay. (*He abandons the idea of locating Marthe in the Cartesian graph and starts a different line of thought*). They both begin at 1 meter (*pointing with the back of the pen to segment 0A; see Figure 3.8b*) ...
41.		No, she starts here (*pointing to the Cartesian origin*) ...
42.		'Kay. There is 1 meter here (*pointing to OA; Figure 3.8c*). This is the distance between both people (*he draws an arrow between Pierre and Marthe to signify the initial distance between them in the drawing accompanying the story-problem; Figure 3.8d*).
43.	M & C:	(*at the same time*) Yeah.
44.	Mc:	Here (*he draws an arrow in from of Pierre*) he moves forward faster or (*he draws an arrow behind Marthe*) she moves more slowly. So then, it makes a difference in the distance (*Figure 3.8e*).
45.	M:	Yes.
46.	Mc:	The line increases (*moving the pen along segment AB; the arrow in Picture 6 indicates the sense of the gesture*) because you have more than one [unit of] distance. Ok? Here (*pointing to segment BC*), it's a straight line ... they are moving at the same speed ...
47.	C:	Yeah.
48.	Mc:	Here (referring to segment CD), she moves forward faster or he slows down so that the distance is smaller.
49.	C:	Why does the distance go down again?
50.	Mc:	Because they are closer. So the distance between the guy and the CBR ... is smaller.

Marc displays an ability to move between the phenomenological and relational spaces. He shows a clear understanding of how these two spaces relate to each other. The length of the Cartesian segment OA is translated into the phenomenological space (L40 and L42). In L46, the increase of distance in the phenomenological space is related to the inclination of segment AB.

This discussion can be seen as occurring in a zone of proximal development created by the students within the spirit of the circulation of ideas

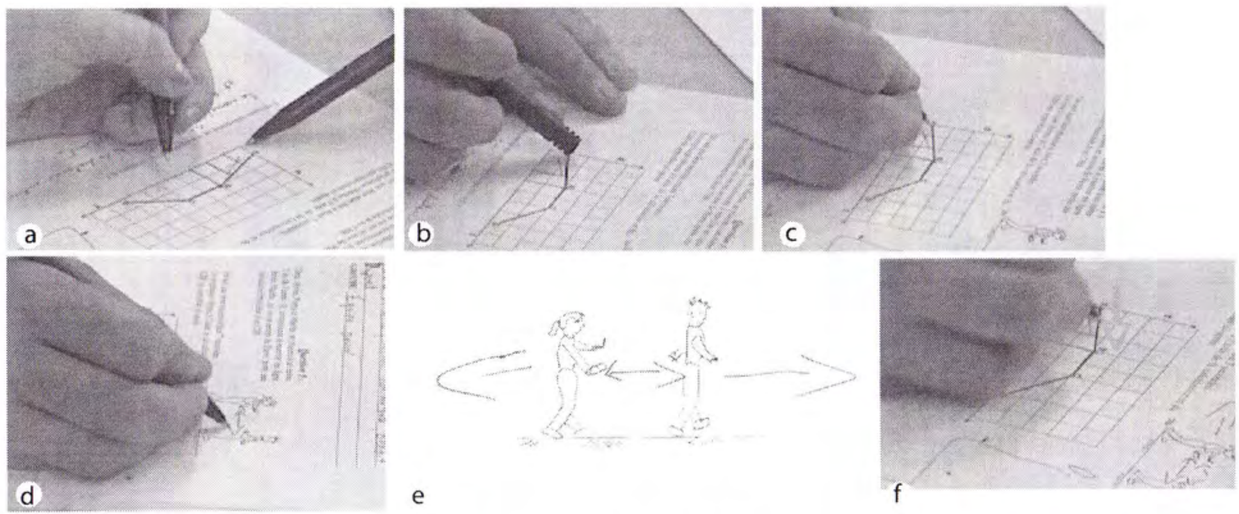

Figure 3.8. Marc's gestural activity during the discussion of the graph. Marc links the phenomenological space of imagined motion and the Cartesian graph in a clear way.

in the classroom encouraged by the teacher. The zone of proximal development allowed Marc to better understand that the graph is not really about locating Marthe somewhere in it: the graph is rather about relative distances. The zone of proximal development helped Carla's group to enhance their understanding of the graph and the complex historically formed cultural logic behind the Cartesian graph.

SYNTHESIS AND CONCLUDING REMARKS

In the first part of this chapter, I discuss two historical problems about the meeting point of two objects moving at different speeds. As pointed out, the solution involved calculations made by *comparison* of "speeds" and not by the *integration* of data into a same totality. Our brief historical excursion allowed us to remark that the concept of time remained rather implicit in the formulation and the solution of these types of problems.[3] It may be true, as Koyré (1966) notes in his studies on pre-modern scientific thought, that, in problems about motion, it was more difficult to think in terms of time than in terms of space. Time appeared imbricated in the concept of motion and could only be extracted from it at great pains.[4] Galileo's own account is eloquent. Commenting on his experiment on an inclined plane, he says:

> As to the measure of time, we had a large pail filled with water and fastened from above, which had a slender tube affixed to its bottom through which a narrow thread of water ran; this was received in a little beaker during the entire time that the ball descended along the channel [carved on the inclined plane] or parts of it. The little amounts of water collected in this way were weighed from time to time on a delicate balance, the differences and ratios of the weights giving us the differences and ratios of the times, and with such precision that, as I have said, these operations repeated time and again never differed by any notable amount. (Galileo, 1638, p. 170)

The new needs brought about by the cultural and economical contexts of the Renaissance led to a reconceptualization of space and time and the emergence of new unifying systems of artistic and scientific knowledge representation (Figure 3.2). One of the most sophisticated examples of such systems is the Cartesian plane, whose laborious constitution required centuries of progressive refinements. In motion problems, the Cartesian plane entails the description of events in terms of *common* spatial and temporal points of reference. The spatial-temporal location of *all* objects is described in relation to these distinguished referential points.

The Cartesian plane allows one to grasp *visually* the evolution of a phenomenon. For Alcuin the problem was not to determine the remaining distance between the dog and the hare after each jump, but the point at which the former catches the latter. He might have found it very curious that one could be interested in calculating the remaining distance for each value of time. Indeed, the interest in following with minute detail the evolution of phenomena became important when attention started to be given to problems of *variation* in the eighteenth and nineteenth centuries.

Now, the systemic and formal structure of a Cartesian plane affords the representation of other more general phenomena—like equipotential curves (see Roth, 2003) or relative motions. In the case of relative motions, what is signified by the distance axis is the relative distance between the moving objects. An important level of indeterminacy is introduced: It is not possible to tell where the objects are in the phenomenological world, for what is known is only the distance between them. This historically constituted form of representing knowledge is far from evident for the novice students. The Cartesian graph bears the sediments of previous generations of cognitive activity and understanding its mode of signifying is, for the students, the outcome of a lengthy process of unpacking knowledge that is termed here *objectification*. Objectification, in fact, is a social process related to the manner in which students become progressively aware, through personal deeds and interpretations, of the cultural logic of mathematical entities—in this case, the complex mathematical meanings that lie at the base of the ways in which Cartesian graphs are used to describe some phenomena and convey meanings.

The data I present here suggest that one of the most important difficulties in understanding the graph was: (a) overcoming an interpretation based on a phenomenological reading of the segments and their descriptions in relation to a fixed spatial point, and (b) the attainment of an interpretation that puts emphasis on relative relations. The question is not to "forget" the phenomenological realm. It is rather to link, in a suitable way, the phenomenological space of imagined motion with the Cartesian space. The logic of interpreting a Cartesian representation of relative motion became progressively apparent for the students through an intense activity mediated by multiple voices, gestures, and mathematical signs. Crucial to this endeavor was the teacher's intervention and the group's discussion with Marc. The teacher was indeed able to call the students' attention to the relationship between segment 0A and the initial distance between Pierre and Marthe, thereby creating some conditions for the evolution of meanings both at the discursive and gestural levels. The teacher's coordination of words with the sequence of similar gestures and signs in the Cartesian graph (Figure 3.6) helped the students understand the meaning of segment 0A in the context of the problem. Segment 0A entered the universe of discourse and gesture, and its length started being considered as the distance between Pierre and Marthe at the beginning of their walk. Without teaching the meaning directly, the teacher's interactional analysis of the meaning of segment 0A was understood and generalized by the students in a creative way (Figure 3.7). After the teacher's intervention, the students' gestures became more and more refined as did their words: whereas their first gestures were about Pierre's motion, their last gestures were related to distances in a meaningful relational way.

The evolution of meanings was deepened during the discussion with Marc, who also insisted on the meaning of 0A, but went further, offering a way to relate the other segments of the Cartesian graph with Pierre and Marthe's walk in a manner that was consonant with what the students had accomplished by themselves. Borrowing a term from Bakhtin (1981), I want to call the transformative objectifying process *heteroglossic*, in that heteroglossia, as I intend the term here, refers to a locus where differing views and forces first collide, but under the auspices of one or more voices (the teacher's or those of other students), are then momentarily resolved at a new cultural-conceptual level, nonetheless awaiting new forms of divergence and resistance.

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NOTES

1. Dell'Abacco's solution is as follows: "Do in this way: if 3 is worth 5, how much is 5 worth? Multiply 5 by 5, which is 25, and divide by 3, you will have 8 1/3. Now you may say: for each 5 of those (paces) of the dog, you have 8 1/3 (paces) of the fox; so the dog approaches the fox 3 1/3 (paces). In how many paces will he (the dog) reach her (the fox) by (covering) 40 paces? Then say: if 5 is worth 3 1/3 for 40, how many will I have? Multiply 5 by 40, which is 200, and divide by 3 1/3. Bring (i.e., reduce) to thirds, thus multiply 3 by 200, which makes 600, and divide by 3 (and) 1/3 , that is 10/3 and then divide 600 in 10, it gives 60. And the dog will do 60 paces before it reaches the fox. And it is done. And the proof is that in 60 paces the fox goes 60, and the dog in 60 paces is worth 100 [i.e., 60 steps of the dog are worth 100 steps of the fox], because three of his (dog's paces) are worth 5 (of the fox); therefore 60 paces (of the dog) are worth a good 100 (of the fox). It is done." (Arrighi, 1964, p. 78). I am grateful to Jens Høyrup, Fulvia Furinghetti and Giorgio Santi for translating dell'Abbaco's problem into English and for their precious help in the analysis of the solution.
2. Since the pioneering work of Clement (1989) and Disessa, Hammer, Sherrin, and Kolpakowski (1991), informed by cognitive science and constructivism, recent work includes Arzarello and Robutti (2004), Arzarello (2006), Nemirovsky (2003), and Roth and Lee (2004), inspired by embodied psychology.
3. Similar problems and solutions can also be found in many other manuscripts, for example, in the thirteenth century Fibonacci's Liber Abacci (Sigler, 2002).
4. The fourteenth century mathematicians at Merton College in Oxford did not deal with two moving bodies, but rather with theoretical investigations of uniform and non-uniform speed (see Clagett, 1959).

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