CONFERENCE OF THE INTERNATIONAL GROUP FOR THE PSYCHOLOGY OF MATHEMATICS EDUCATION
PME34

RESEARCH FORUM 1

NETWORKING OF THEORIES IN MATHEMATICS EDUCATION

COORDINATORS:
ANGELIKA BIKNER-AHSBAH
TOMMY DREYFUS
IVY KIDRON
FERDINANDO ARZARELLO
LUIS RADFORD
MICHÈLE ARTIGUE
CRISTINA SABENA

COMMENTARY BY LUIS RADFORD

NETWORKING OF THEORIES IN MATHEMATICS EDUCATION

Angelika Bikner-Ahsbahs[1], Tommy Dreyfus[2], Ivy Kidron[3], Ferdinando Arzarello[4], Luis Radford[5], Michèle Artigue[6], Cristina Sabena[4]

[1]Universität Bremen (Germany), [2]Tel Aviv University (Israel), [3]Jerusalem College of Technology (Israel), [4]Università di Torino (Italy), [5]Université Laurentienne (Canada), [6]Université Paris 7 (France)

The networking of theories is a promising research practice for connecting theories systematically while valuing their identities. Strategies for the networking of theories and four case studies with different profiles are presented. Theoretical reflections on the cases show that the networking of theories can be regarded as a process that begins with differentiating among theories and develops in the direction of connecting theories towards integration. A final discussion focuses on benefits and limits of networking and a commentary describes its potential for future research.

INTRODUCTION

The idea of the networking of theories has been investigated since 2005 by a group of researchers that emerged from a working group at the CERME4 conference (Artigue et al., 2005) and can be traced back to a Research Forum about abstraction (Boero et al., 2002) that critically compared three theories of abstraction and a Research Forum about theories in general (English & Sriraman, 2005). In both cases, the germs of networking were already there: a systematic dialogue between theories improving the theoretical basis of the scientific work of the community of mathematics educators addressing the diversity of theories in mathematics education (Bikner-Ahsbahs & Prediger, 2009). Networking has become the topic of a working group at CERME5 (Arzarello et al., 2007) and CERME6 (Prediger et al., 2009) leading to several publications (see Sriraman & English, 2009; Prediger et al., 2008; Kidron, 2008).

In this Research Forum we aim to present the current state of research on networking of theories to a wider international audience, including a meta-theoretical frame, methodologies for networking, and case studies of networking that show the benefits and the limitations of this kind of research practice. Four case studies of the networking of theories are used to reflect theoretically about networking practices, about different degrees of networking, and its benefits and limitations leading to a debate about necessary ingredients to successful networking and what successful networking might mean.

THEORETICAL BACKGROUND

The networking of theories is regarded as a systematic way to connect theories and to reflect about the networking process and its outcomes leading to a "dialogue of theories". Radford (2008) describes the necessity of assuming a space for the

2010. In Pinto, M. M. F. & Kawasaki, T. F. (Eds.). *Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education*, Vol. 1, pp. 145-175. Belo Horizonte, Brazil: PME.

networking of theories that comprises at least the different theoretical traditions, a meta-language to speak about the theories and networking practices to build connections such as the networking strategies *coordinating theories* or *synthesizing theories*. Referring to Lotman (1990), he calls such a space a semiosphere, that is "an uneven multi-cultural space of meaning-making processes and understandings generated by individuals as they come to know and interact with each other" (Radford, 2008, p. 318). The semiosphere can be described by its heterogeneity concerning the different research cultures in it. It is a multi-cultural space that is dynamically changing. Theory cultures constantly produce, re-produce and develop their identities, but at the same time they establish boundaries that separate their cultures from the others and immunize their cores. However, the boundary is also the place of exchange between cultures. As Lotman (1990) stated, creative ideas are not normally born in the centre but in the periphery, at the boundaries of the cultures. Networking crosses these boundaries and therefore is a way of renewing theories in different ways. The semiosphere's main function is providing possibilities for dialogue, thus creating connections that are beneficial in different ways, such as deepening the identity of a theory, integrating different theories into a new one or just locally, or creating new kinds of research questions. According to this background, research about the networking of theories means investigating the theories within the semiosphere. In this way, sources and limits for the dialogue are uncovered through common research of different researchers representing their theory cultures. Dialogue in this sense links theories. As Radford explained, "a theory can be seen as a way of producing understandings and ways of action based on" a set of principles (P), that involves a set of methodologies (M) following a set of paradigmatic question (Q). He uses the triple (P, M, Q) to characterize a theory (Radford, 2008). Connections between different theories can be drawn between the three parts: P, M and Q.

METHODOLOGICAL CONSIDERATIONS

Different cases of networking have investigated the question under what conditions connectivity between theories is possible. For example, Gellert (2009) shows that the underlying principles have to be 'near enough' and Jungwirth (2009) shows, that the empirical load of a concept plays a crucial role if integrating is the aim. Sometimes the theoretical approaches can be connected by uncovering a common concept that connects two theories by supporting both to explain a phenomenon from two complementary perspectives leading to the further development of both theories (Arzarello, Bikner-Ahsbahs, & Sabena, 2009a). Prediger, Bikner-Ahsbahs and Arzarello (2008, p. 170) have developed a case-study-based landscape of networking strategies that are linearly ordered according to their degree of integration (Figure 1). Networking strategies are located between two ideal poles that are not regarded as networking strategies: ignoring all but one's own theory and the attempt to unify all theories. The networking approach does not regard these two poles as useful; it rather acknowledges the diversity of theories within mathematics education as a rich resource for the development of theories in the community of mathematics education.

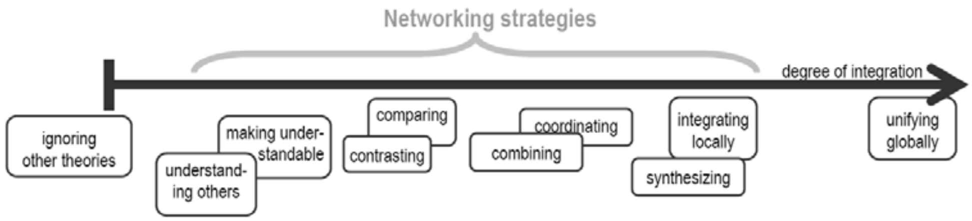

Figure 1: A landscape for connecting theoretical approaches

The first strategy pair in the landscape above describes that mutual understanding of theories is necessary when researchers start to practice networking; the second pair focuses on strategies of comparison; the third pair grasps the step that has to be gone towards other theories when linking them; and the fourth describes the balance of reducing theories by integrating at least parts of one theory into another one and building new theories that subsume others. Even if researchers want to integrate theoretical parts only locally into a new theoretical view they have to deeply understand the other theories before using the strategies of comparing, contrasting, coordinating and combining them in the course of integration.

CASES OF NETWORKING

Several case studies of networking provide the background for discussing key questions about networking in the Research Forum, including:

- What are the different goals in the networking process?
- What is the limit to what can be networked and how is this limit determined by the specificity of the theories that are being networked?

Four case studies are briefly presented below: (i) The European project ReMath has made use of networking on the basis of a powerful method: cross-experimentation. (ii) Two related epistemic action models have been developed independently of each other. One of them originates from an individual-cognitive view and considers social factors as part of the context, whereas the other one takes a more socio-constructivist view on epistemic processes. The proposers of the Research Forum are currently involved in a project that aims at networking the two models in an attempt to integrate them. (iii) Arzarello, Bikner-Ahsbahs and Sabena (2009) have analysed the same learning episode from two different perspectives. Each perspective alone was insufficient to understand why learning in this case was not successful. (iv) Sabena, et al. (2009) connect cognitive and semiotic approaches to analyse the different ways in which students use semiotic resources (inscriptions, gestures, speech), and analyse how these different approaches can be connected. In parallel, ongoing research uses networking practices in research about gesturing, connecting the semiotic perspective and the theory interest-dense situations.

The above exposition shows that research on the networking of theories means investigating the local parts of the semiosphere of mathematics education with goals like locally integrating different theories. Such efforts may

- lead to a frame for the analysis of a phenomenon from different perspectives,
- improve theories through clarifying their identities and boundaries,
- create methodologies for connecting theories,
- help researchers deepening a research question,
- lead to a renewed understanding of the role of theories in mathematics education.

While there is, even in the long term, no attempt at striving for a single overall theory of mathematics education, networking existing theories is likely to support the establishment of connections that lead to more helpful communication between researchers and hence to more results and insights that are common to a large segment of the scientific community of mathematics educators.

Case 1: The ReMath project

In the ReMath project (http://remath.cti.gr), theoretical networking was a major concern. The goal of the project was to develop an integrated vision of the potential offered by digital representations for mathematical learning, and the six teams involved considered that this goal could not be achieved without building productive connections between the respective theoretical frameworks they were relying on. The theoretical landscape offered by ReMath was indeed very diverse, involving both close and distant theoretical approaches such as the theory of didactical situations, the anthropological theory of didactics, the instrumental approach, Duval's semiotic approach, Pierce's semiotics, the theory of semiotic mediation, social semiotics, activity theory, constructionism, to mention only the main ones. Relying on the experience gained in a first European project, TELMA (see: http://telma.noe-kaleidoscope.org, Artigue, 2009), the ReMath-teams developed a specific methodology for building such connections, cross-experimentation, and this contribution to the forum focuses on this methodology.

The emergence of the cross-experimentation methodology: The idea of cross-experimentation emerged in TELMA as a methodological tool for better understanding our respective approaches regarding contexts, representations and theoretical frameworks in technology enhanced learning in mathematics, and making possible collaborative work in that area. It followed a first phase where such understanding was researched through a detailed description provided by each team of its approach and experience in the field, and the collective analysis and discussion of a set of selected publications. For making sense of the diversity, a first construct was elaborated, that of *didactical functionality*, which was defined for a given ICT tool as its characteristics and modalities of employment, which may favour or enhance teaching/learning processes according to a specific educational aim (Artigue & Cerulli, 2008). Didactical functionalities were thus structured around three components: a set of features / characteristics of the tool, an educational aim, and modalities of employing the tool in a teaching/learning process with reference to the chosen educational aim. This first phase was certainly useful but the mutual understanding gained through it remained rather superficial. Data collected and their analyses tended to confirm that the theoretical frameworks TELMA teams were relying on influenced the approaches they developed towards technology enhanced learning and their vision of didactical functionalities of ICT tools, but they did not give access to the underlying processes. The limited success of this first phase led to the conviction that the desired kind of mutual understanding required the development of specific methodologies, engaging the different teams in common realizations, which would become themselves objects of research. This was the origin of the cross-experimentation process. In TELMA cross-experimentation, each team was asked to build a short experiment involving an alien technology, that is to say an ICT tool designed by another team relying on different theoretical reference. The whole process was carefully monitored through a complex system of guidelines, ensuring that the experiments would be comparable, organizing their separate and comparative analyses and the collective reflection on these. It was expected that the perturbation created by the alien technology together with the guideline system would oblige the researchers involved to make more explicit the influence of their theoretical approaches on design decisions, a priori analyses and a posteriori analyses. This was indeed the case and the results obtained evidenced the interest of this methodological choice for progressing in mutual understanding as shown in (Artigue, 2009). The cross-experimentation process also led to the development of a meta-language, that of concerns, further elaborated in ReMath as explained below.

From TELMA to ReMath. The ReMath project can be seen as a development and consolidation of the TELMA enterprise with a specific focus on digital representations. Regarding theoretical networking, the goals of ReMath were more ambitious as, beyond mutual understanding, ReMath aimed at coordinating and combining theoretical approaches. The language used in the presentation of the project was indeed a language of theoretical integration, as it was planned to achieve it through a cyclic process combining the progressive elaboration of an integrated theoretical framework, the design of six dynamic digital artifacts (DDA), and their experimentation in realistic contexts. From a conceptual and methodological perspective, ReMath relied on TELMA advances and shared its vision of theoretical frameworks as functional tools. The notion of didactical functionality and the meta-language of concerns played thus a central role as well as the methodology of cross-experimentation.

At the origin of the meta-language of concerns is the hypothesis that teams working in different contexts nevertheless face rather similar educational challenges, and thus that conceptual and theoretical tools developed or appropriated for addressing them respond to similar needs. The notion of *concern* tries to capture these commonalities taken as a possible basis for networking of perspectives. TELMA research showed that concerns could obey different hierarchies, and that even when they were given the same importance by two different teams, they were not necessarily expressed and dealt with in the same way and with the same conceptual tools. From that resulted the idea that the identification of the respective attention given to these different concerns, and the precise ways they were approached could be used for elucidating the role played by theoretical frameworks, for identifying potential interesting connections and complementarities, and also potential incompatibilities and conflicts. In TELMA, a set of concerns had been attached to each of the three dimensions of didactical functionalities. The same construct was used in ReMath. Moreover teams were asked to grade from 0 to 5 the different concerns according to the importance given to them, for each dimension, and to make explicit the theoretical frameworks and constructs used for dealing with these concerns, if any, and make clear if these were used at a metaphorical or an operational level (Artigue & al., 2009).

The process of cross-experimentation was also reworked and refined in ReMath, where six different DDA were developed or enriched. Each team was asked to experiment its own DDA and an alien DDA, the experimentations being much more substantial than in TELMA. A system of guidelines was used and a specific multi-level structure of *pedagogical scenario* managed through a *pedagogical plan manager* was elaborated. Beyond that, new methodological tools were progressively created. The second year, a common research question was introduced to complement the information provided by the hierarchy of concerns. This question was the following: "*How can the representations identifiable in the DDAs be put in relationship with the achievement of specific educational goals?"* Each team was first asked to rephrase it according to its perspectives, and then to complement it with the specific questions it wanted to address in its experimentation. Each team was also asked to evaluate the rate of success of its experimentations, and the criteria underlying this evaluation. All these data were collectively analysed and compared (Mariotti, Maracci, & al., 2009). Another interesting methodological development was the cross-case analyses. Each DDA being experimented by two different teams, a specific grid was created for developing six cross-case analyses. The six analyses were then compared for identifying possible connections, developing local integrations, and also investigating the respective influence of theoretical frameworks and contexts on the similarities and differences observed. Finally, from the second year on, a minimal shared theoretical framework regarding representations was articulated and the results obtained were used for investigating the possibility of extending it. This complex methodological organization led to interesting results (Artigue & al., 2009). Among these are an extended *shared theoretical framework (STF)* regarding representations structured around nine positions, a *connected theoretical landscape*, and illustrative examples which help making sense of the connections established and the way they can be productively used while preserving the specific coherence underlying each approach. For instance, in the STF it is agreed (position 4) that "due to the diversity of semiotic systems and semiotic activities involved in mathematical activity with DDAs, an analysis in terms of semiotic registers of representations pays justice to this diversity and thus to the learning potential of DDAs, only partially." (Artigue & al., 2009, p. 60). The cross-experimentation allowed ReMath researchers to build productive combinations between Duval's semiotics approach which offers powerful constructs for analysing established semiotic registers of representations and processes involving these (expressed in terms of formation, treatment and conversion) and the theory of semiotic mediations which offers more powerful constructs for understanding the emergence of signs, the mediation role played by DDAs in this emergence, and the ways teachers can support the progressive transition from signs attached to specific activities with a DDA artefact to mathematical signs detached from the artefact and the context. The cross-experimentation also showed that a shared sensitivity to instrumental genesis and associated processes of instrumentalization and instrumentation could be compatible with rather different design decisions regarding its management depending on the main theoretical framework the researchers relied on (constructionism, instrumental approach, theory of semiotic mediations). Such an awareness helped clarify similarities and differences between the vision of design inherent to each theoretical perspective and the potential and limits for local integration at this level.

Some final comments. In such a reduced space, it is impossible to enter into more detail about these methodological constructs and their outcomes, but in our opinion, the TELMA and ReMath projects illustrate how a productive "dialogue between theories" can be established through the development of appropriate methodologies. These projects confirm that such a dialogue requires the creation of specific meta-languages, and also the organization of spaces where different research practices can become an object of common exchange and work as was pointed out in the general introduction. Up to now, most often, this exchange and work has been approached through the analysis of given data or transcripts through different theoretical lenses. At the end of ReMath, our conviction is that substantial advances on networking issues require new methodological constructions where the objects of study are in fact research praxeologies collectively organized around precise types of task. Taking research praxeologies as objects of study facilitates approaching theoretical frameworks in functional terms, better perceiving the needs they respond to, and thus why and how they can be connected and even locally integrated. In the TELMA and ReMath enterprises, some efforts have been made in that direction, and an evident methodological creativity developed, but a reflective look at these efforts shows that they remain at the level of craft work, and that a lot remains to be done in order to provide researchers with well established research praxeologies for efficiently addressing networking issues.

Case 2: Comparing, contrasting and coordinating two epistemic action models[7]

In the project "Effective knowledge construction in interest-dense situations" two theoretical frameworks are networked, Abstraction in Context (AiC; see Schwarz, Dreyfus & Hershkowitz, 2009) and Interest-Dense Situations (IDS; see Bikner-Ahsbahs, 2003, 2007). In both frameworks, epistemic action models are used to investigate processes of knowledge construction but in different ways. AiC focuses on mathematical reasoning, with social interaction as part of the context. IDS considers social interactions as basis which constitutes learning mathematics. In AiC knowledge construction is described by means of the epistemic actions of Recognizing, Building-with and Constructing (RBC). These actions pertain to the knowing of the participants and are potentially observable on the basis of how learners act, use tools, interact verbally and practically with other learners, and what and how they write. The investigation of constructing begins with the design of a task and with an a priori analysis of the task in terms of expected constructs. A posteriori, the researchers analyse the process of constructing knowledge in terms of the epistemic actions of the learners.

Interest-dense situations are epistemic situations of in-depth knowledge construction in which the students become deeply involved in a mathematical problem, progress in constructing mathematical meanings and highly value the mathematical activity. The three features of IDS are indicators for the students' situational interest. The theory of IDS describes and explains how interest-dense situations emerge in every day classrooms or groups of students, how this emergence is supported or hindered and how these situations are conducted. Within these situations, processes of constructing knowledge are described by the epistemic actions gathering mathematical meanings, connecting mathematical meanings and structure seeing. Empirical investigations show that all interest-dense situations lead to structure seeing. A basic assumption of this theory is that knowledge construction emerges out of the social interactions of students collaborating on a mathematical problem, and that the students interact according to their interpretations of the situations. Researchers can get access to the students' interpretations by re-interpreting them. This is the basis for reconstructing the social interactions and their epistemic character.

While both models use epistemic actions, a comparison shows important differences.

Recognizing and gathering: Recognizing happens to an individual if she realizes that a previous mathematical construct is inherent in the given mathematical situation. Gathering mathematical meaning is a heuristic strategy to gain information that might show a direction of how to solve the task. Gathering provides material to think with, it can be done by a group or by a solitary learner, while recognizing is individual.

Building-with and connecting: Building-with consists of combining existing knowledge elements in order to meet a goal such as solving a problem or justifying a statement. It may include establishing connections. Connecting is a strategy in which gathered material has to be connected to solve a task. Connections are worked out among a limited number of things.

Constructing and structure seeing: In AiC, constructing is the main step of abstraction. It consists of assembling and integrating previous knowledge constructs to produce a new one. While constructing refers to the construction of new knowledge only, structure seeing only says that the students are all of a sudden able to perceive a mathematical structure in the material given. This step can be understood as the result of structuring activities made of gathering and connecting actions leading to saturation of information. Structures need not necessarily be new to the learner.

Methodical considerations for networking

Comparing and contrasting the two approaches uncovers essential differences. However, each approach can benefit from the other. AiC is a theoretical tool to investigate processes of in-depth knowledge construction and conditions that favour such processes are particularly important for AiC researchers. AiC postulates that the first stage of the genesis of an abstraction is the need for a new construct. This need can arise from interest. Therefore we expect that reconstructing the emergence of IDS might shed light on how the need arises and how it leads to constructing knowledge in the sense of AiC. On the other hand, IDS are social processes that at the same time are shaped by individuals and influence individuals' coming to know mathematics. AiC provides the tools for a micro analysis of individual processes of coming to know within a social situation. The following methodical steps for networking permit to highlight the insights offered by each of the two models to the other one.

The networking process in this project has four steps. The first step was the design of tasks. Both teams developed tasks that were piloted, exchanged, revised according to the needs of both groups. Already this first step uncovered differences in the theory based approaches. Based on an increased mutual understanding of each other's underlying assumptions, the research teams modified conceptual frames and methodological approaches of the activities. The teams also developed a method of adapting the design by means of the use of adaptive interviewer intervention. The second step was collecting and transcribing video data about the processes in which pairs of students solved the tasks in the presence of an interviewer. Differences in the transcription method mirror differences in the principles and methodologies. The IDS-group analyses social interaction through an interpretative approach. Therefore the transcription key used by this team grasps the illocutionary (meaning making at a level beneath the content) level of discourse by paralinguistic signs whereas the AiC-team is satisfied with the usual punctuation marks in the transcribed utterances. The third step is a cross-analysis, in which every team transcribes their data, and analyses them as well as the data of the other group. Meetings between the teams support matching the results and the methodologies. The fourth step consists of reflections and establishing complementary insights into processes of constructing mathematical knowledge from the individual and the social view. In addition, the search for conditions that favour processes of in-depth construction of knowledge demanded connecting the theoretical concepts of need and interest. An example illustrates how new insights arose by the attempt to overcome methodological differences.

An example of linking need and interest (Kidron et al., 2010). A pair of students worked on an activity asking them to interpret a continued fraction. The students were asked to find the first seven elements of the sequence, reflect on how they computed them, and extend the computations to 20 elements, writing them as simple fractions as well as decimal fractions. They were also asked to find the decimal representation of $\frac{1+\sqrt{5}}{2}$ (the limit of the infinite process), make a hypothesis on the sequence from the first part, and justify it. The process of working on the task was videotaped and transcribed. This transcript was analysed by both teams. An exchange of the analysis led to the networking focusing on the students' needs and interests.

The AiC-team made an a priori analysis of the limit constructs they expected (for example the closeness construct: the sequence comes closer and closer to the limit number) and focussed mainly on the constructing parts of the transcript. In their a posteriori analysis, the AiC team identified students' constructs of the intended knowledge elements (and, in fact, some unintended ones), some of which were partially or not at all constructed. The AiC researchers expected to identify the students' need for the new constructs during the process of knowledge construction. However, they found it difficult to identify a need for a specific new construct.

Analysing the process of socially constructing knowledge the IDS-team produced a diagram representing the different phases of the process and thus got an overview about the interest-dense situation that occurred during the process. The main phase comprised a long interaction sequence of structure seeing that seemed to be in-depth and crucial for structure seeing, but the IDS-team could not exactly distinguish the quality of this part from the others. Stimulated by the a priori analysis of the AiC-team with its list of constructs about the limit, the IDS-team reconstructed the flow of ideas within this phase as a process of becoming more precise.

Through exchange, both teams could identify a source for making progress in constructing mathematical meanings which was neither interest nor the need for a new construct, but might be connected to both: a *general epistemic need* (GEN) in which the need for a new construct is implicitly nested. This epistemic need can be identified in the students' attempts to strengthen mathematical utterances, to be more precise, to look for mathematical patterns, to grasp the specific mathematical idea in terms of descriptions, etc. It empowers the students' flow of mathematical ideas. But this epistemic need also can be regarded as a need that is nested in situational interests of the two students.

With a view to networking, we note that the IDS-analysis focuses and reconstructs the whole situation sequentially on the basis of segments that show intense social interactions, whereas the AiC analysis focuses on segments that appear relevant to the constructing process. However, the IDS analysis led the AiC team to closely consider the phases in which structure seeing occurred according to IDS. This led to an additional focus of the AiC analysis on excerpts which were ignored by them at first. These excerpts did contribute to the constructing process. This only became clear to the AiC researchers after the social interaction based on the IDS analysis provided an overview by means of the phase structure diagram. Only then, the AiC researchers, who had encountered difficulties to identify students' need for the new constructs, realized that the students, in the course of gathering information, arrived at some “seeds” of constructing. These seeds of constructing together with a general epistemic need for more precision were sufficient to lead to some constructing actions. Networking assisted AiC and IDS researchers to discover the concept of *epistemic need* It is not a need for a specific, well defined new construct but it is a need to progress, to reinforce a vague image into a more definite one. Interestingly, networking assisted AiC and IDS researchers to discover this concept of general epistemic need which can be traced back to the roots of AiC in the work of Davydov. Indeed, the view of abstraction underlying AiC is based on Davydov’s (1972/1990) ideas, according to which the process of abstraction starts from an undifferentiated and possibly vague initial notion, which need not be internally and externally consistent. The development of abstraction proceeds by establishing an internal structure by means of links and results in a differentiated, structured, consistent entity. This way, networking contributed to deepen insight into the identity of AiC.

Case 3: Locally integrating two theoretical perspectives

In this third case of networking, the networking strategy has been worked out through analyses of empirical data that had been presented at the CERME6 conference (Arzarello, Bikner-Ahsbahs & Sabena, 2009). The same teacher-student-interaction has been analysed from two theoretical perspectives: the *interest-dense situation* and the *semiotic bundle* analysis.

The semiotic bundle perspective is based on two basic assumptions:

- the teaching-learning process inherently involves resources of different kinds, in a deep integrated way: words (orally or in written form); extra-linguistic modes of expression (gestures, gazes, …); different types of inscriptions (drawings, sketches, graphs, ...); instruments (from the pencil to the most sophisticated ICT devices), and so on (for some examples see Arzarello, 2006);
- such resources may play the role of signs (according to Peirce's definition: As sign or semiotic resource, we consider anything that "stands to somebody for something in some respect or capacity", Peirce, 1931-1958, vol. 2, paragraph 228) and therefore can be considered as semiotic resources.

An interesting phenomenon identified in the teacher-students interaction is the *semiotic game* (Arzarello et al., 2009), which happens when the teacher tunes with the students' semiotic resources and uses them to guide the evolution of mathematical meanings. A typical example is when the teacher repeats a student's gesture, and correlates it with a new term or with the correct explication using natural language and mathematical symbolism. Semiotic games may therefore constitute an important strategy in the process of appropriation of the culturally shared meaning of signs.

The Interest-Dense Situation (see: the previous section) perspective (Bikner-Ahsbahs, 2003, 2007) focuses on those situations of in-depth knowledge constructions in which the students of a maths class show situational interest in the mathematical topic or activity, that is they become involved in the activity and experience meaningfulness about it. To achieve some mathematical knowledge the students activate epistemic actions, i.e. actions that are executed in order to come to know more. Through social interactions the class or the group of learners collectively coordinate the epistemic process. In this way collective epistemic actions are constituted by social interactions. In contrast to non interest-dense situations, all interest-dense situations lead to the epistemic action of *structure seeing*, i.e. perceiving a mathematical pattern or rule referring to an unlimited number of examples.

Through analysing utterances of social interactions on three levels of speech act (locutionary, illocutionary, perlocutionary levels, see for example Davis, 1980) this approach investigates conditions that foster or hinder the emergence of interest-dense situations and their in-depth knowledge construction. Meaning on the locutionary level is concerned with *what is said*, meaning on the illocutionary level can be described as *what is said through saying something the way it is said* and the perlocutionary level describes the intentions and factual effects of utterances. The non-locutionary comprises the illocutionary and the perlocutionary level of speech acts.

The episode being crucial for the networking process. Two students explore the exponential function $y=a^x$ using Cabri (figure 2). *a* is a parameter whose value can be changed in a sliding bar. The line in figure 2 is actually a secant line; but the secant points are so near that the line appears on the screen as tangent to the graph. This issue has been discussed in the classroom in a previous lesson. As figure 2 appeared on the computer screen the teacher asked: what happens to the exponential function for very big x?

Based on the observation of the graph, a student (Gabriel) stresses "but always for *a* very big this straight line (pointing at the screen), when they meet each other, there it is again…that is it approximates the, the function very well, because…"

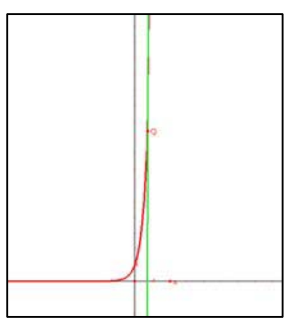

Figure 2: The observed computer screen.

On the locutionary level this utterance means *for a very big basis a, when the graph and the vertical straight line meet, hence the line approximates the function very well*. Considering the broken language on the non-locutionary level, Gabriel seems to be thinking and speaking in parallel just beginning to give reasons for his view as the teacher interrupts him: "what straight line, sorry" Factoring the mode of speech into the non-locutionary level, there is a negotiation between the teacher and the student about whose train of thought is followed. G extrapolates what he observes on the computer screen *for very big x* and the teacher interactively works out a proof of contradiction concerning the statement about approximation: if the graph of the exponential function would meet a straight line (for a fixed x) it would have to cross the graph. During the process, Gabriel reduces his intensive engagement ending up answering the teacher's questions by one word sentences such as *yes, no, infinity*. The IDS-analysis allows understanding how the emergence of the interest-dense situations (and Gabriel's situational interest) is interactively interrupted but it cannot explain the deeper reason why this happens.

The semiotic bundle perspective integrates the analysis of speech with the analysis of accompanying gestures and other semiotic resources. Gabriel's first utterance (see above) is in fact accompanied by a gesture (figure 3) showing the behaviour of the exponential function. A few seconds later in the same interaction, the teacher repeats Gabriel's words, and accompanies them with a similar but not identical gesture (figure 3).

The teacher is in fact showing with his hands both the vertical line (right hand) and the exponential graph (left hand, and in particular the forefinger). In this way the teacher is performing a semiotic game with the aim of guiding the student to think again (and possibly change his mind) about the behaviour of the exponential function with respect to the vertical line, for very big x.

Differently from other cases, this time the semiotic game does not appear to work, i.e. the student does not appear to profit from it, and will continue to refer to the function as becoming *almost a vertical line*. One limit of the semiotic bundle perspective put to the fore by this episode is that the theory cannot explain the deeper reason why the semiotic game does not work.

Figure 3: Gabriel's gesture and the teacher's gesture

An empirically based integration:

Based on the theoretical account and the empirical analysis, we considered the two theories as *complementary*: they shed light on different aspects of the teacher-students interaction. However, by using the two theoretical lenses separately it appeared that something important was missing in each case. The strength of the interest-dense situations perspective is the possibility to predict their emergence according to the type of social interactions that hinder or foster it. In fact it includes the analysis of the locutionary and non-locutionary levels of speech and shows negotiations underneath the content. This approach is able to describe how the epistemic process proceeds and provides deeper insights into the social interaction process that foster or hinder the emergence of interest-dense situations, including structure seeing. However, in the analysed example the student and the teacher are not able to merge their argumentations although there is a lot of negotiation about whose train of thought will be followed. Neither the teacher nor the student is able to engage with the other's perspective. The analysis showed a gap that cannot be overcome, but was unable to give the tool to find out the deeper reason why this is so. By looking at a wide range of signs (in Peirce's sense), the semiotic bundle analysis identified the semiotic game between teacher and student, and allowed the game to be properly described. However, the theory is not able to fully explain the reason why the student does not gain much from such a semiotic game. In most other cases we had observed that the students succeeded to learn through semiotic games (e.g. see Arzarello et al., 2009). One difference that can be identified within the theoretical frame is that this time the semiotic game applies the gesture-speech resources in reverse way with respect to semiotic games analysed as "successful". In this case, in fact, the teacher tunes with students' speech and uses gesture to foster meaning development; in other cases, it was the other way round: tuning with gestures and fostering meanings through words. We could conjecture that the characteristics of gestures as semiotic resource are not apt to this kind of didactical support, and indeed this can be a research problem to investigate. But within the semiotic bundle theory we are not able to say why this semiotic game did not work. The discussion so far leads us to argue that the simple juxtaposition of the two perspectives is not enough to deeply understand what's going wrong in the analysed episode.

To go a step further, we started from the example to combine and locally integrate the two theories. The *combination* provides a tool to investigate how each sign of the semiotic bundle may contribute to the locutionary or non-locutionary aspects of the interaction. For instance, a gesture can support locutionary as well as non-locutionary features that play important roles in the interaction. In the episode, gestures show the behaviour of the graph in iconic way, but also that the student is trying to agree with the teacher's perspective. The hands in fact are used in the same configuration as the teacher previously did; in the entire episode this is the only case in which it happens. In all the other cases, the student's gestures have very different configurations. Concerning the words, a similar situation is constituted; at the locutionary level the student's words affiliate to the teacher's perspective. But at the non-locutionary levels the teacher and the student do not fully agree with each other using words. The combination of the two approaches is diagrammatically summed up in Figure 4.

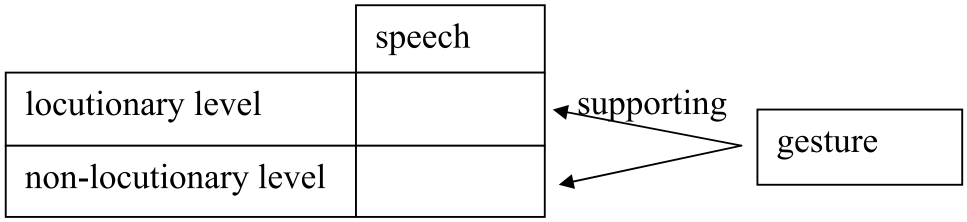

Figure 4: Two-level-analysis of semiotic resources

With the aim to answer the question what exactly did not work in the student-teacher interaction of the episode, we propose an integration of the two combined theories adding an *epistemological* dimension to the analysis above; that means to carefully consider the epistemological points of view of the teacher and of the students. By *epistemological points of view* we mean the background of the piece of knowledge that a subject thinks might give sense to a specific situation. The epistemological point of view is not always explicit: it appears not only from the locutionary dimension of the semiotic resources used by a subject but also from the non-locutionary ones. Moreover, it can be partially revealed by the epistemic actions produced by the subject. Of course, the epistemological point of view with respect to a situation can vary with the subjects. For example, that of a student can be different from that of the teacher. This difference might not be apparent although the dynamics of a didactic situation in the classroom might be deeply influenced by it, especially when the teacher is not aware of it or does not take into account the epistemological points of view of his students. This is exactly what happened in the episode analysed above. We observe a semiotic game articulated in a tuning in words and a dissonance in gestures: the teacher is repeating G's words, but he is performing completely different gestures. The dissonance in gesture is a signal that the teacher and the student are showing different points of view: the teacher relies on a formal theory; the student relies on his perception imagining what happens in a very intuitive way. The analysis of the semiotic game *including* the epistemological dimension allows us therefore to say that there is an *epistemological gap* between views of the teacher and student, and to hypothesise that this gap prevents the teacher from suitably coaching the student's knowledge evolution and the student from profiting from the interaction with the teacher. Therefore an interest-dense situation did not emerge.

Concluding comments: Presenting this empirical case of the networking of theories, we showed that by means of a local integration two theoretical approaches can be enriched. This was possible because the theories provided two complementary observation tools: one at the level of discourse analysis describes social interactions and their epistemic processes; the other at the level of gesture analysis describes learning from a semiotic perspective. The starting point of the theoretical integration was based on the empirical data analysis whose meaning was not clarified by either of the two theories. This impasse was overcome by suitably combining the two approaches: adding an epistemological dimension made possible to locally integrate the two theories, so uncovering blind spots in both.

Case 4: Integrating two semiotic approaches to graphical thinking tasks

This case study emerged from the collaboration of two research teams, who had independently assigned related tasks to their students and interpreted the students' work, each team according to their own theoretical framework. In project SB (semiotic bundle, in Italy) students were assigned D-tasks (D stands for derivative): Given the graph of a function *f*, propose a graph for the derivative *f'* of *f*. In project VS (virtual space, in New Zealand), students were assigned AD-tasks (AD stands for antiderivative): Given the graph of a function *f*, propose a graph for an antiderivative *F* of *f*. The tasks were contextualized in different ways, but we ignore this fact here (see Sabena et al., 2009, for more detail).

These two tasks are similar since both belong to calculus, both are set in a graphical framework and both can be answered by graphical reasoning, based on relationships between graphical properties of *F*, *f* and *f'*, such as the relationship between the concavity of *F*, the increase of *f* and the sign of *f'*. (We use terms such as 'increase' to stand for increase or decrease.) In spite of these similarities, however, the two tasks have crucial differences. As is usually the case in mathematics, the D-task is less difficult than its inverse, the AD-task. Some graphical relationships are simpler to identify, like "if *f* has an extremum at $x=a$, then $f'(a)=0$"; others are more complex, like "if f has an extremum at $x=b$, then *F* has a point of inflection at $x=b$". This can be explained as follows: The D-task can be solved by (Di) drawing segments of the tangent at different points of the given graph of *f*; (Dii) evaluating the slopes of these tangent segments; and (Diii) graphing the values of these slopes in the same or a new coordinate system to obtain the graph of *f'*. These are pointwise actions, requiring only local considerations on the graphs. The AD-task, however, requires non-local, integrating (pun intended) considerations; it can be solved by (ADi) interpreting the value *f* at a given point as a slope; (ADii) visualizing this slope for 'all' points in sequence; and (ADiii) combining the slopes so visualized into a single graph, the graph of *F*. In addition to the non-locality, the AD task has an additional difficulty: while Di can be carried out physically, there exists no appropriate medium in which to carry out ADi physically so as to be useful in ADii.

The Semiotic Bundle (SB) for analyzing students solving the D-task: The Semiotic Bundle (SB) frame that was already presented in Case 3, has been introduced by Arzarello (2006) out of the necessity of taking into account the variety of semiotic resources activated by the students and by the teacher in their mathematical activities. A semiotic bundle is a system made of the different semiotic resources and of their mutual relationships that are produced by one or more interacting subjects. As such, it encompasses the classical semiotic registers as particular cases, but includes a richer variety of semiotic resources (compared to standard semiotic registers), and in particular gestures. Following Peirce, 'semiotic resources' or 'signs' are meant as anything that "stands to somebody for something in some respect or capacity" (Peirce, 1931/1958, vol. 2, paragraph 228). In short, a sign is a triad (R,O,I) composed by a *representamen* (that represents), an *object* (that is represented), and an *interpretant* (related to the interpretation process).

Using the SB, Arzarello and his group have analysed secondary school students solving D-tasks and emphasised the role of gestures therein (Arzarello et al., 2009c; Sabena, 2007). Specifically, gestures have sometimes a mediating role in the semiotic activity Di on R1, in order to interpret it and to produce a new sign (R2,O2,I2), namely the graph of *f'*. However, Arzarello and Sabena show how in some unsuccessful cases, students confused the graph of the given function *f* with the graph of the derivative function that they have to produce, with the gestures being of little help (Arzarello et al., 2009b; Sabena et al., 2009). Misleading interpretations are especially observed in positive decreasing graphs, where the function is positive and the derivative is negative (as the boxed part in figure 5).

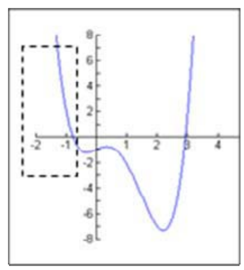

Figure 5

In these cases, both the gestures and the term "slope" show an interference with the Figure 5 everyday meaning of "slope of a path", that is partially in contrast with the mathematical meaning. In fact, a positive decreasing graph reaching its minimum has an increasing slope (negative and approaching zero), whereas the slope of a similar downhill path is said to be (and perceived as) decreasing. If we frame the previous analysis using Peirce's definition (R,O,I), we can say that the graph R is associated (through interpretants I1 and I2) both to the picture of a real path (O1), and to the graph of a mathematical function (O2). Such signs in which the representamen is linked or may be linked by the interpretant to two (or more) objects have been called *blending signs* (Sabena, 2010), since they blend the references of two different domains. Blending signs are probably related to the iconic aspects of the mathematical representations, and are often used successfully by expert mathematicians. We may find many examples in the classroom as well, both by students and by teachers. The difference between success and failure in using blending signs appears related to a certain consciousness and control over signs (i.e. over the three components R, O, I) by the subject.

The Virtual Space (VS) for analysing students solving the AD-task: Yoon, Thomas & Dreyfus (2009, 2010) have closely observed two teachers solving AD-tasks. Their solution process can be sketched as follows: The teachers initially used speech and deictic gestures to interpret the graph of *f*, while tracing along it with their fingers and simultaneously using speech to verbalise their interpretation of the corresponding portions of the graph. Their use of these semiotic resources led to some confusion, inhibiting the construction of an image of the graph of *F*. The researchers interpret this as due to the segmented, linear, and analytic nature of speech and hence its limited ability to convey the holistic and dynamic features needed in ADiii. The teachers soon turned to using iconic gestures, showing the slopes of *F*, as read off the graph of *f*, by means of the back of their hand placed in an imagined coordinate system within their gesture space. This device not only created a medium within which to physically carry out ADi, but also allowed the teachers to move the hand along the imagined coordinate system while changing its slope, thus achieving ADiii by combining the varying values of the slope of *F* into a holistic graph of F, which could then be inscribed onto a piece of paper.

In order to analyse the performance of two teachers on the AD-task, the researchers proposed the notion of a virtual space (VS) populated by virtual objects. A VS is a space that is created within the confines of a subject's gesture space by means of a set of gestures and the mathematical meanings associated with them; more technically, it is defined as a grounded conceptual blend of Real Space and a mathematical space that is projected onto the subject's physical gesture space (Yoon et al., 2010). The VS is populated by virtual objects, which are representations of mathematical objects constructed by a subject in virtual space through the use of gestures. The VS becomes meaningful for the subject via the virtual objects and their properties. A VS can be used by more than one person to construct and communicate mathematical understandings via gestures: If two or more people view a gesture being performed, they may construct similar virtual objects if they interpret the gesture using similar and similarly positioned virtual spaces. A crucial feature of VS is that it is temporary and hence lends itself to experimentation and error correction.

Networking: Both research studies, though carried out independently, are situated in a semiotic framework. This, together with the similarity of the tasks, allows us to carry out the following common analysis, which for both tasks starts from the interpretation of a given sign (R1,O1,I1) of *f*.

In the D-task, successful students carry out the semiotic activity Di on R1, interpret it in Dii, and in Diii build up a new sign (R2,O2,I2), namely the graph of *f'*. The Dii phase is accomplished by considering (and possibly tracing) a mediating sign, i.e. the tangent line. Such a mediating sign has an important role with respect of the subjects' control in their semiotic activity from (R1,O1,I1) to (R2,O2,I2).

A priori, the AD-task is more difficult, since no direct activity with the sign (R1,O1,I1) is apparently possible for accomplishing the task. Hence, the production of the sign (R2,O2,I2) for *F* may be inhibited. However, the VS makes it possible to overcome this inhibition. The virtual objects then enter into the SB managed by the subjects. This implies a further interpretative activity namely to manage the coordination between the VS and the sheet where (R1,O1,I1) lives and thus to control the relationships between the two signs (R2,O2,I2) and (R1,O1,I1). In a parallel manner, Di can be seen as managing the coordination between the collection of relationships between graphical properties of *f* and *f'* (mentioned in the introduction) and the sheet where (R1,O1,I1) lives and thus to control the relationships between the two signs (R2,O2,I2) and (R1,O1,I1).

The common frame that was used here to interpret the solution behaviour of successful students on both tasks corresponds to what Vygotsky (1978) called a second order auxiliary sign: "Because this auxiliary stimulus possesses the specific function of reverse action, it transfers the psychological operation to higher and qualitatively new forms and permits humans, by aid of extrinsic stimuli, *to control their behaviour from the outside*" (p. 40). The auxiliary stimulus for the D-task is part of the collection of properties; the auxiliary stimulus for the AD task exists in the VS. Introducing this auxiliary sign, allows for what has been called duplication (Duval, 1983; Fischbein, 1987), namely a further interpretation of the first sign. Both tasks can be solved through a duplication: Reading in *f* the value of its slope, i.e. of the function *f'* (for D); or reading the value of *f* as a slope, the slope of *F* (for AD). This way of looking at the solution process allows us to distinguish between successful and unsuccessful students. The duplication leads to a structure of the kind (R,O,I1-I2), which is a blending sign (Sabena, 2010). Students who are able to use semiotic control to distinguish between the two interpretants of the blending sign, tend to be successful; students who are unable to control the activity with the new sign, presumably because they have only one interpretant, are led to a mismatch such as *f*=*f'* and are likely to fail solving the task.

Concluding Comments: This networking case study was facilitated by the fact that from the outset, the SB perspective and the notion of VS both had a semiotic basis. This made it possible to view VS from an SB perspective and carry out a local integration of VS into SB, providing an integrative comparison of the successful and the unsuccessful student solutions in the two research studies, based on the notions of duplication, blended signs, and control. These are more specific notions that can focus different didactical phenomena in a rather precise way. Integrating them has allowed clarifying the similarities and differences between the two tasks, the dynamics of solution processes, and has produced a picture

- of the didactical situations that have been proposed,
- of the nature of the two tasks we have given,
- and of the possible ways that a teacher can design her/his interventions in the proposed didactical situation to support the students in solving the tasks.

THEORETICAL REFLECTIONS

The four cases described in the previous section offer different profiles of networking concerning their starting points und their aims from which the methodology is extracted. In Case 2, the IDS- and the AiC-researchers started from two different theories and focused on the relationships among them. The aim of networking was to put the two frames on the table in order to see which insights of each frame will enrich the other. In Case 3, the SB- and the IDS-approach start from an empirical phenomenon with the aim of developing their understanding of it better by means of connecting the two different perspectives. Each theoretical tool turns out to be insufficient to properly analyse the data.

Case 1 represents a new kind of profile of networking quite different from the two other cases: It offers a new profile while starting from a given set of theories but aiming at designing a theoretically integrated development of digital artefacts. This is done by using concrete empirical research to develop conceptual and methodological tools for coordinating and combining theoretical approaches.

In Case 4 (SB and VS), we are offered still another kind of profile. It starts with two empirical case studies using tasks that demand working into opposite direction. The aim is to get an integrated view of both results, the phenomena of success and failure. Contrary to Case 2 (AiC and IDS) or Case 3 (SB and IDS) the common exchange and work between the theoretical frames are not approached through the analysis of common data and transcripts through different theoretical lenses. Although the two frames use semiotic resources, it is done in different manners and the exchange between the frames is done only after each team has analysed its proper relevant data within the appropriate frame in which the data was created. It is also different from Case 1, for which the object of study are research praxeologies collectively organized around precise types of tasks. Nevertheless, the networking of the SB and the VS frames offers an interesting kind of local integration of neighbouring approaches. As we will see in the following, the different profiles of networking offer different possibilities of dialogue between the theoretical frames and words like local integration might have different meanings according to the different profiles.

From differentiation to integration

These case studies represent different kinds of networking but nevertheless have principles in common that will show what a framework for the networking of theories might look like. There are two steps in the course of networking, the first step is concerned with mutual understanding of theoretical cultures that means understanding the theory's identity and periphery in terms of a specific question, problem or idea by the interaction with other theories, research teams, cultures,... The second step is concerned with the goal of overcoming differences and difficulties by creating connections in the direction of integration. Regarded from the semiosphere the goals of the two steps in terms of the networking of theories can be described as a pathway from differentiation to integration.

Differentiation

In all the studies mutual understanding and finding similarities and differences were the main common goals of the first step. When networking only two approaches, this dialogue may be conducted by the researchers; no explicit framework is needed for this dialogue and for reflecting afterwards about the networking process. This is impossible when more approaches are involved like in the case for ReMath (see also

Kidron et al., 2008). However, looking deeper into the case studies, we can always find a framework, sometimes an implicit one. The framework of the networking of two approaches can just be a complementary analysis driven by common research questions defined by the phenomenon investigated. For example, investigating the construction of knowledge by two epistemic action models has the goal to get a more inclusive insight about epistemic processes. In the case of ReMath, the many theoretical backgrounds demanded to explicitly shape a framework for all the teams directing attention on specific categories or dimension: the didactical functionalities of the ICT-tools.

The differentiation step often implicitly leads to what we might call difficulties in networking. The fact that the basic assumptions of different theories are different might be expressed already in the first phases of the process of networking. Indeed, difficulties were observed in the networking of AiC and IDS already in the first stage of designing the activities and defining the role of the interviewer in conducting the activities. Experiencing such perturbations and attempting a mutual understanding, the researchers observed that the fact that in the AiC perspective there is an a priori analysis of the knowledge constructs, which are expected to be constructed, influences the design of the activities. On the other hand, IDS has no such an a priori analysis. In addition, a different view of "learning" itself in IDS as social interaction and a very specific view of the role of the teacher in learning situations led to a different design of the activities and a different definition of the role of the interviewer. Even the ways of transcribing were different.

The ReMath-researchers encountered similar difficulties. Already in previous TELMA cross-experimentation, each team was asked to build a short experiment involving an alien technology, that is to say an ICT tool designed by another team relying on different theoretical reference. The ReMath researchers wrote:

> It was expected that the perturbation created by the alien technology together with the guideline system would oblige the researchers involved to make more explicit the influence of their theoretical approaches on design decisions, a priori analyses and a posteriori analyses. (see the section: Case 1: The ReMath project)

In both cases of networking, the case of AiC and IDS and the case of ReMath, examining the first difficulties has, at the same time, led to benefits. The difficulties, the perturbations pointing to the differentiation clarify what kind of data is requested by each theoretical perspective according to its underlying assumptions. This clarification is a necessary stage towards an intermediary step before the integration.

The differentiation step helps the researchers towards an awareness of the underlying assumptions of their own theories. In the intermediary step, as a consequence of the perturbations, the researchers reflect on the boundaries of both their own theory and the others. This is an indispensable stage towards a productive dialogue between theories. The "negative" face of the perturbations is transformed into a more positive mise en scene: not only the specificity of each theory is clearer but some germs of the possible benefits are already here.

According to the profile of the networking of IDS and SB, a perturbation was even the starting point: The results of analyses of the same data set from two perspectives seemed, at first glance, to be contradictory. This specific set of data, which was created in the SB frame included a part in which the role of the teacher was important, and which, hence was suitable for IDS analysis. For the SB-team, the analysis within their lenses proved to be insufficient when analysed from the IDS-view and vice versa. The semiotic game was not successful as in previous SB analyses. With IDS with its focus on teacher-students interaction, the germs of the possible benefits were already there without provoking perturbations that appear in the other cases of networking that we presented.

Perturbation through other theories helps researchers to better understand their own theories in terms of principles, methodologies, paradigmatic questions, and their boundaries. The strategy after mutual understanding is comparing and contrasting to explore differences and commonalities. However, this part is not always clearly separated from the second step; it rather prepares the shift to the second step of integration that cannot always be reached. Comparing and contrasting often leads to translating, conversing, transferring ideas, concepts, methodical principles. The networking of the epistemic actions models shows that the AiC-team's a priori analysis of the expected mathematical constructs conducted the IDS-team's analysis towards the flow of mathematical ideas that deepened insight into the episode. In the networking process of the SB- and IDS-approaches, initially, a similar aspect emerged: the question why epistemologically the specific situation was not successful, understood from both perspectives differently. In the ReMath projects this part was a step of analysis when taking the approaches as objects of analysis.

Integration

Integration is a conscious step in the networking of theories after experiencing perturbations and after the attempt at mutual understanding, knowing about the differences and commonalities. It is driven by the question: What can each approach learn from the others? In the course of answering this question central concepts can be conversed or transferred and connections are built by the strategy of coordinating and combining. The underlying contract is to solve a specific common problem that all the teams share.

As mentioned before, the difficulties point at the benefits; for example in the ReMath-project, encountering the perturbations at the design stage helped the researchers towards the conclusion that the analysis of given data or transcripts through different theoretical lenses was not sufficient in their effort of networking. At the end of their project, the researchers were convinced that substantial advances on networking issues requires new methodological constructions where the objects of study are research praxeologies collectively organized around precise types of task.

Taking research praxeologies as objects of study facilitates approaching theoretical frameworks in functional terms, better perceiving the needs they respond to, and thus why and how they can be connected and even locally integrated.

Trying to understand the source of difficulties in the first phase of networking was also a crucial stage for the AiC and IDS researchers before performing their cross-analysis. In their cross-analyses, the researchers' search for conditions that favour processes of in-depth construction of knowledge led them to connect the theoretical concepts of need and interest. Comparing their different analyses using a common object of study, a common focus on the concept of need for a new construct enabled the researchers to find a *general epistemic need* which appeared only after analysing the entire session from both points of view: IDS (with social interaction and a suitably large grain size) as well as AiC (with its a-priori analysis and its fine grained a posteriori analysis). This general epistemic need was shared by the students but had different effects on them according to their different situational interests. Implicitly nested in this general epistemic need was the need for a new construct that might be weak or strong; in this case it was weak, in other cases it was strong, hence, explicit.

In the case of the networking of the SB- and IDS-approach, both methodologies were integrated into a common methodology creating the concept of the epistemological view that showed a deep gap between the one of the teacher and the one of the student. This gap could not be overcome by the semiotic game experienced.

Comparing different analyses with a common object of study permits to better profit from the networking effort. This was demonstrated for the networking of two different approaches like AiC and IDS and is crucial for a larger number of approaches (the ReMath-project; see also Kidron et al., 2008). The networking of the SB- and VS-approach is different from the other cases. The theoretical perspectives have a common semiotic basis, the differences refer to the concepts and the methodologies, but connections are already there. The SB includes gesturing and gestures shape the VS, the task posed to the students are connected since one group is supposed to build the graph of the derivative and the other the graph for an antiderivative. However, this study is interesting because we may study how two neighbouring approaches whose theories are close can benefit from each other leading to strong local integration. First of all, there is no step of explicitly perturbing the approaches. We may want to attribute this fact to the common semiotic basis of the two frames but the real reason for it was that the kind of data, which was requested by each theoretical perspective according to its underlying assumptions was already defined by the choice of the appropriate tasks. Moreover, in the networking process, the researchers were aware of the crucial differences in the nature of the two tasks. The first task requests a local, pointwise view while the second task requests a more global view. Similarly, the semiotic resources like gestures accompanied by speech, are different in the two tasks. The reason is that in addition to the non-locality, the AD task has an additional difficulty: While the first task can be carried out physically, there exists no appropriate medium in which to carry out the second task physically. This awareness of the differentiation between the tasks and the appropriate frame used to analyse each task permitted this specific case of networking, in which the researchers conceptually integrated the VS-approach into the SB-view and conducted a parallel analysis inventing the same language based on Peirces' concept of signs networked with Vygotsky's second order auxiliary sign and Duval's concept of duplication.

It is interesting to note that the enriching experience in this case of networking was a consequence of the fact that the two tasks were different in their nature, each one inviting a different kind of semiotic resource. In spite of the fact that the theoretical frames were neighbouring approaches, the requested data for each frame was different. All these conditions permit this specific case of local integration and its strength for further research.

COMMENTARY (by Luis Radford**)**

The systematic investigation of the limits and possibilities, challenges and problems that result from the efforts of "connecting" theories in mathematics education has gained a substantial impetus in the past few years and has become a new research field of its own. As I mentioned in a commentary paper written for the ZDM issue where connecting theories were the central theme (Radford, 2008), such efforts may reflect direct or indirect actions to cope with some of the needs that were brought to the fore by the new educational, political, and economical structures of the European Union and its institutionalizing forms of academic research. Be that as it may, the systematic encounter of different research traditions came with a multitude of unforeseen problems that are of great interest—both theoretical and practical—in the elucidation of what theories are and how they evolve. For one, encountering different research traditions requires the participants to make clear the ideas, principles, and assumptions of their own theoretical approaches and, in doing so, to acquire a certain form of self-consciousness. As Mikhail Bakhtin remarked, "I am conscious of myself and become myself only while revealing myself for another, through another, and with the help of another. The most important acts constituting self-consciousness are determined by a relationship toward another consciousness (toward a *thou*)" (Bakhtin, 1984, p. 287). The encounter with theoretical approaches also offers participants the opportunity to recognize theoretical similarities and differences and inquire to what extent two or more approaches are opposed, congruous, compatible, complementary, combinable, and so on. However, the encounter of theoretical approaches does not proceed without difficulties. Each theory is made of principles, methodologies, and paradigmatic questions that are expressed in theoretical terms that do not necessarily bear the same meaning in other theories. In other words, each theory has its own language. More than 10 years ago, I received a message from a good friend who commented on an article in which I made my first attempt at articulating the question of mathematics representations from a Vygotskian perspective. My friend made a list of the key concepts in my paper (e.g., signs, meaning, social interaction, artifactual action) and sent me a short verdict: "We [socioconstructivists] have all of them too; so, what's the difference? What's new?" What surprised me the most was that our discussion became quickly entangled in a nightmare of terminological misunderstandings. We were using the same terms but with different meanings. To get out of our terminological mess was not an easy task. To be frank, we did not succeed! As I later realized, the semantic value of a theoretical term (e.g. *social interaction*) in a theory results from its position in the main web of dynamic interconnections that characterize the theory as a whole. There is hence something positional or hierarchical about theoretical constructs that makes them impossible to be extracted from the whole, contrary to, e.g. the unproblematic way we extract a weed from the grass. This is why it is important to attempt to clarify first the principles and methodological tools of a theory. But it would be a mistake to think that our theories in mathematics education should be explicitly formulated *in toto*. With the exception of formal theories, where everything is specified beforehand and everything is explicitly stated, at least within a certain conception of what explicitness is about, theories work, function, and move under a great deal of implicit assumptions. Theories in mathematics education reflect and refract implicit specific national-cultural "world views". They are unavoidably immersed in those symbolic systems of cultural significations that Cornelius Castoriadis (1975), Ernst Cassirer (1957) and others have pinpointed in their investigation of the symbolic structures of society—structures from where (implicitly or explicitly) our theories draw their views of what constitutes a good student, a good teacher, and, of course, a good researcher.

Now, the fact that two theories can *be* different is not a reason to imagine that a dialogue between them cannot be fruitful. An interesting question is to determine that which makes them different. Another interesting question is to determine the extent to which, given those differences, contrasts can be achieved or connections made in order to come up with new theoretical understandings and new forms of practical action. Some theories may position themselves closer to each other in the (meta)space of theories in mathematics education and constitute a kind of *cluster* of this (meta)space—i.e., the semiosphere of cultural exchanges. It might be the case sometimes that their proximity affords connections at the methodological level, and sometimes at the level of the conceptual principles. At any rate, the connection of theoretical approaches is based on the philosophy of acknowledging both differences between theories and the possibilities of building links of different kinds among them. The landscape of possibilities that was suggested by Prediger, Bikner-Ahsbahs and Arzarello (2008, p. 170) includes several strategies and the four cases presented in this Research Forum illustrate some of them. Let me review these cases briefly.

The four cases can be read at different levels. One of them—the one I will follow here—is the methodological level. The four cases provide us with clear efforts for creating semiospheric methodologies pertaining to the general problem of *connecting* theoretical approaches. The ReMath project elaborated a *cross-experimentation*, i.e., a collaborative practice that would help the researchers of two different teams in clarifying what is needed to achieve mutual understanding and theoretical integration. The collaborative practice involved two teams that constructed a technological tool, where each became part of an experimentation carried out by the other team. The DIDIREM team was informed by the major French theories of *didactique des mathématiques*, while the Italian team was informed by theories of activity and semiotic mediation. By merely glancing at the approaches, you can feel "areas of tension" or difference. Thus, while in the French theory of didactic situations the student is an epistemic and abstract entity, in the activity theory the student is a real, concrete, sensing, and breathing individual. The teams had hence the same word "student" in their lexical repertoire, yet the meaning was not the same. You can also feel a "tension" in the epistemic role that is attributed in these respective theories to language. Despite all the differences, the teams collaborated and achieved a better understanding of each other. To facilitate dialogue and joint work, the teams came up with a list of common general *concerns*. Yet, those concerns did not play the same role during their corresponding experimentation or were not dealt with through the same conceptual tools. This is an example, I think, of the general cultural worldview in which each theory finds itself subsumed. The *concerns* refract sometimes specific commitments and orientations of the corresponding theoretical approach; sometimes they are operationalised by principles of a more general cultural nature. These differences do not necessarily impede connections and complementarities to be obtained.

The second case involved the AiC and IDS theories and the goal was to compare, contrast and coordinate them. These theories have from the outset clear theoretical similarities as well as differences. As mentioned in previous sections, these theories study processes of knowledge construction but in different ways. AiC focuses on knowledge construction from a personal viewpoint while IDS puts a larger emphasis on the social construction of knowledge. In this case, the methodology was not a cross-experimentation: it revolved around the creation of same tasks. The teams proceeded to collect data and make transcriptions and, finally, they reflected upon the establishment of complementary insights into processes of constructing mathematical knowledge from the individual and the social views. As the teams reported, differences appeared from the outset during the design of tasks, and reappeared in the methodology used to deal with the transcriptions. In the end, an interesting coordination of theoretical viewpoints led to the addition of a new construct that was not part of the theories, namely a *general epistemic need*. This new construct was able to fill an explanatory gap and account for the manner in which students were making progress in constructing mathematical meanings.

In the third case, the establishment of a semiospheric methodology revolved around a same piece of data, analysed through two different approaches: Semiotic Bundle and the Interest-Dense Situations (IDS). Although different in several respects, the dialogue resulted in an interesting combination of tools emphasizing gestures and modalities of discourse. And as in the previous case, the collaboration resulted in the recognition of a new construct that was not part of the theories initially—*an epistemological gap*.

In the fourth case, the semiospheric methodology focused not on the same corpus of data, but on different data related to the same subject matter (graphs) investigated through the use of gestures. A team working within the SB approach looked at its own data (students understanding derivatives) and another team working within the Virtual Space (VS) approach looked at its own data (high school teachers understanding antiderivatives). The research teams exchanged their analyses and, as a result, gained insights that had previously remained out of their focus of attention.

These four cases present us with four different semiospheric methodologies that were designed with the intention to foster dialogue and mutual understanding, as well as to provoke a more or less extended form of integration. These methodologies are interesting exemplars and, of course, are open to refinements. But there is something else that we can learn from those cases. They show how *peripheral entities* of a cluster of theories ended up gaining a more central role. Take for instance the *general epistemic need* that was not an organic part of the involved theories (i.e., the AiC and the IDS theories). This marginal entity made its entrance through interaction. Exactly the same can be said of the *epistemological gap* construct brought to the fore by the IDS and SB theories. It seems then that when two (or more) theories position towards each other to enter into a semiospheric dialogue, a halo of new conceptual possibilities is formed. Potential entities appear. But they remain in the periphery of the cluster that the theories constitute. They remain "revolving around", as the etymological sense of *periphery* intimates. An effort of objectification is required to bring the peripheral entities into attention. And, in this objectifying movement, in order to accomplish the crossing of the peripheral threshold, we need someone else. For in the end, it turns out, as Bakhtin was suggesting, that "every internal experience occurs on the border, it comes across another, and in this tension-filled encounter lies its entire essence." (Bakhtin, 1984, p. 287, adapted from Todorov, 1984, p. 96).

References

Artigue M. (Ed.) (2009). Connecting approaches about technology enhanced learning in mathematics: the TELMA experience. *International Journal of Computers for Mathematical Learning,* Special issue, 14.3.

Artigue M. et al. (2009). *Integrated Theoretical Framework. Version C.* Deliverable 18. //cti.remath.gr.

Artigue M., & Cerulli M. (2008). Connection theoretical frameworks : the TELMA perspective. In O. Figueras & al. (Eds), *Proceedings of the Joint Meeting of PME 32 and PME-NA XXX,* vol. 2 (pp. 81-88). Morelia, Michoacan (Mexico): Universidad Michoacana de San Nicolas de Hidalg.

Artigue, M., Alexopoulou, E., lshwaikh, J., Cazes, C., Chaachoua, H., Chiappini,G., Kahn, K., Kynigos, C., Lagrange, J.-B., Latsi, M., Maffei, L., Maracci, M., Mariotti, M. A., Markopoulos, C., Morgan, C., Moustaki, F., Nicaud, J. F., Pedemonte, B., Psycharis, G., Robotti, E., Sabena, C., Trgalova, J., Vandebrouck, F., & Winters, N., (2009). *Representing Mathematics with Digital Media (ReMath)*. Research Report for the Information Technology Programme. Retrieved September 28, 2009 from http://remath.cti.gr/content_by_cat.asp?contentid=857&catid=34&how=&keywords,

Artigue, M., Bartolini-Bussi, M., Dreyfus, T., Gray, E., & Prediger, S. (2005). Different theoretical perspectives and approaches in research in mathematics education. In M. Bosch (Ed.), *Proceedings of the Fourth Congress of the European Society for Research in Mathematics Education* (pp. 1237-1402). San Feliu de Guíxols, Spain. [On CD]

Arzarello, F. (2006). Semiosis as a multimodal process, *Revista latino americana de investigación en matemática educativa, vol. Especial*, 267–299.

Arzarello, F., Bikner-Ahsbahs, A. & Sabena, C. (2009). Complementary networking: Enriching understanding. *Proceedings of the 6th Congress of the European Society for Research in Mathematics Education (CERME6)*, Université Claude Bernard, Lyon, France. http://educmath.inrp.fr/Educmath/recherches/actes-en-ligne/1wg9.pdf. (in press)

Arzarello, F., Bosch, M., Lenfant, A., & Prediger, S : (2007). Different theoretical perspectives in research from teaching problems to research problems. In D. Pitta-Pantazi, G. Philipou, & A. Gagatsis (Eds.). *Proceedings of the 5th Congress of the European Society for Research in Mathematics Education* (pp. 1617-1818). Cypris : Larnaca [On CD].

Arzarello, F., Paola, D. & Sabena, C. (2009b). Logical and semiotic levels in argumentation. In F.-L. Lin, F.-J. Hsieh, G. Hanna & M. de Villiers (Eds.), *Proceedings of Icmi Study 19 conference: Proof and Proving in Mathematics Education* (vol. 1, pp. 41-46). Taipei, Taiwan: Department of Mathematics, National Taiwan Normal University. ISBN 978-986-01-8210-1.

Arzarello, F., Paola, D. Robutti, O., & Sabena, C. (2009c). Gestures as semiotic resources in the mathematics classroom. *Educational Studies in Mathematics*, 70(2), 97-109.

Bakhtin, M (1984). *Problems of Dostoevesky's poetics*. Minneapolis: University of Minnesota Press.

Bikner-Ahsbahs, A. & Prediger, S. (2009). Networking theories – an approach for exploiting the diversity of theoretical approaches. In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education* (pp. 589-592). Heidelberg, New York: Springer - Advances in mathematics education series.

Bikner-Ahsbahs, A. (2003). A social extension of a psychological interest theory. In N. A. Peitman, B. J. Dougherty & J. T. Zilliox (Eds.), *Proceedings of the 2003 Joint Meeting of PME and PME-NA,* Vol. 2 (pp. 97-104). Honululu, HI: University of Hawai'i. College of Education.

Bikner-Ahsbahs, A. (2007). Semiotic sequence analysis – Constructing epistemic types. In Jarmila Novotná, Hana Moraovsá, Magdalena Krátká, Nad'a Stehliková (Eds.): Mathematics in the centre. *Proceedings of the 30th Conference of the International group for the Psychology of Mathematics Education*, Vol. 2 (pp. 161-168). Prague (Czech Republic): Charles University, Faculty of Education.

Boero, P., Dreyfus, T., Gravemeijer, K., Gray, E., Hershkowitz, R., Schwarz, B., Sierpinska, A. & Tall, D. (2002). Abstraction: Theories about the Emergence of Knowledge Structures. In A. Cockburn & E. Nardi (Eds.). *Proceedings of the 26th International Conference on the Psychology of Mathematics Education*, Vol. 1 (pp. 111-138). Norwich, UK: East Anglia University: PME.

Cassirer, E. (1957). *The philosophy of symbolic forms*, Vol. 3 (Vol.3): Oxford University Press.

Castoriadis, C. (1975). *L'institution imaginaire de la société*. Paris: Seuil.

Davis, St. (1980). Perlocutions. In J. R. Searle, F. Kiefer & M. Bierwisch (Eds.), *Speech act theory and pragmatics* (pp. 37-55). Dordrecht, London: D. Reidel Publishing Company.

Davydov, V. V. (1972/1990). *Soviet studies in mathematics education: Vol. 2. Types of generalization in instruction: Logical and psychological problems in the structuring of school curricula* (J. Kilpatrick, Ed., & J. Teller, Trans.). Reston, VA: NCTM.

Duval R. (1983). L'obstacle du deboublement des objets mathematiques. *Educational Studies in Mathematics, 14*, 385-414.

English, L. & Sriraman, B. (2005). Theories of mathematics education. Research Forum. In Chick, H. L., & Vincent, J. L. (Eds.). *Proceedings of the International Group for the Psychology of Mathematics Education,* Vol. 1 (pp.170-204). Melbourne, Australia: PME.

Fischbein, E. (1987). *Intuition in Science and Mathematics, an Educational Approach*. Dordrecht, Holland: Reidel.

Gellert, U. (2009). Modalities of local integration of theories in mathematics education. In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education* (pp. 623-646). Heidelberg, New York: Springer - Advances in mathematics education series.

Jungwirth, H. (2009). On networking strategies and theories' compatability: Learning from an effective combination in a research project. . In B. Sriraman & L. English (Eds.), *Theories of Mathematics Education* (pp. 593-622). Heidelberg, New York: Springer - Advances in mathematics education series.

Kidron, I. (2008). Abstraction and consolidation of the limit procept by means of instrumented schemes: the complementary role of three different frameworks. *Educational Studies in Mathematics* 69(3), 197-216.

Kidron, I., Bikner-Ahsbahs, Cramer, J. A. Dreyfus, T. & Gilboa, N. (2010). Construction of knowledge: need and interest. *These proceedings.*

Kidron, I., Lenfant, A., Artigue, M., Bikner-Ahsbahs, A., & Dreyfus, T. (2008). Toward networking three theoretical approaches: the case of social interactions. *Zentralblatt für Didaktik der Mathematik - The International Journal on Mathematics Education 40* (2), 247-264.

Lotman, Y. (1990). *Universe of the Mind*. Bloomington, Indianapolis: Indiany University Press.

Mariotti, M.A., Maracci M. & al. (2009). *Design-Based Research: Process And Results*. Deliverable 13. //cti.remath.gr .

Peirce, C. S. (1931/1958). In C. Hartshorne, P. Weiss, & A. Burks (Eds.), *Collected papers* (vol. I–VIII). Cambridge, MA: Harvard University Press.

Prediger, S., Bikner-Ahsbahs, A. & Arzarello, F. (2008). Networking strategies and methods for connection theoretical approaches: first steps towards a conceptual framework. *ZDM-International Journal on Mathematics Education*, 40(2), 165-178.

Prediger, S., Bosch, M., Kidron, I, Monaghan, J., & Sensevy, G. (2009). Different theoretical perspectives and approaches in mathematics education research - strategies and difficulties when connecting theories. In *Proceedings of the Sixth Congress of the European Society for Research in Mathematics Education (CERME 6)*, Lyon (France): Université Claude Bernard (in press).

Radford, L. (2008). Connecting theories in mathematics education: challenges and possibilities. *Zentralblatt für Didaktik der Mathematik - The International Journal on Mathematics Education 40* (2), 317-327.

Sabena, C. (2007). *Body and signs: A multimodal semiotic approach to teaching-learning processes in early Calculus*. Tesi di Dottorato, Università degli Studi di Torino.

Sabena, C. (2010). Are we talking about graphs or tracks? Potentials and limits of 'blending signs'. *These proceedings.*

Sabena, C., Yoon, C., Arzarello, F., Dreyfus, T., Paola, D., & Thomas, M. O. J. (2009). Relationship and control within semiotic bundles. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5, (pp. 33-40). Thessaloniki, Greece: PME.

Schwarz, B. B., Dreyfus, T., & Hershkowitz, R. (2009). *The nested epistemic actions model for abstraction in context.* In B. B. Schwarz, T. Dreyfus & R. Hershkowitz (Eds.), *Transformation of knowledge through classroom interaction* (pp. 11-41). London, UK: Routledge.

Sriraman, B. & English, L. (Eds.) (2009). Theories of mathematics education. Seeking new frontiers. Series: *Andvances in mathematics education*. Heidelberg, New York: Springer.

Todorov, T. (1984). *Mikhail Bakhtin: The dialogical principle*. Minneapolis: University of Minnesota Press.

Vygotsky, L. S. (1978). *Mind in society: The development of higher psychological processes* (M. Cole, V. John-Steiner, S. Scribner, & E. Souberman, Eds.). Cambridge, MA: Harvard University Press. (Original work published 1930)

Yoon, C., Thomas, M. O. J., & Dreyfus, T. (2009). Gestures and virtual space. In M. Tzekaki, M. Kaldrimidou & H. Sakonidis (Eds.), *Proceedings of the 33rd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 5 (pp. 409-416). Thessaloniki, Greece: PME.

Yoon, C., Thomas, M. O. J., & Dreyfus, T. (2010). *Gestures and Virtual Space: A Framework for Graphical Antiderivatives.* Technical report available from the authors, The University of Auckland.

[7] Case 2 belongs to a research project supported by the German-Israeli Foundation for Scientific Research and Development grant 946-357.4/2006.