Research in Mathematics Education
Vol. 12, No. 1, March 2010, 1–19

Routledge
Taylor & Francis Group

Algebraic thinking from a cultural semiotic perspective

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In this article, I introduce a typology of forms of algebraic thinking. In the first part, I argue that the *form* and *generality* of algebraic thinking are characterised by the mathematical problem at hand and the embodied and other semiotic resources that are mobilised to tackle the problem in *analytic* ways. My claim is based not only on semiotic considerations but also on new theories of cognition that stress the fundamental role of the context, the body and the senses in the way in which we come to know. In the second part, I present some concrete examples from a longitudinal classroom research study through which the typology of forms of algebraic thinking is illustrated.

Keywords: algebraic thinking; semiotics; embodiment

Dedication: I dedicate this article to the memory of Giorgio T. Bagni, 16th June 1958 to 10th June 2009

Introduction

To deal with algebraic thinking is not a simple matter. It supposes that you have some sort of theory about thinking, or at least a clear idea of what you mean by thinking in general. Before you continue reading, please pause for a moment and try to answer this question: what do you take 'thinking' to mean?

As psychologists, philosophers, anthropologists and others are willing to acknowledge, there is no simple and direct answer to this question. As odd as it may seem, thinking is something that we do continuously. Thinking is as ubiquitous as breathing. Yet, we still do not know how we think! Commenting on the elusiveness of thinking, Dan Rappaport said: "The knowledge that thinking has conquered for humanity is vast, yet our knowledge of thinking is scant. It might seem that thinking eludes its own searching eye." (Rappaport 1951; quoted in Benson 1994, 13). Western idealist and rationalist epistemologies have conveyed the idea that thinking is something immaterial, something purely mental, bodiless. The influence of Plato's epistemology on our understanding of thinking is perhaps greater than we usually appreciate (Radford, Edwards, and Arzarello 2009).

In this article, I introduce a typology of forms of algebraic thinking based on their level of generality. The typology rests on a theoretical approach that capitalises on the results of the 1990s algebra research agenda and current research in the field (e.g., Coles and Brown 2001; Staats and Batteen 2009; Kieran 2006; Becker and

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ISSN 1479-4802 print/ISSN 1754-0178 online

DOI: 10.1080/14794800903569741
http://www.informaworld.com

Rivera 2008), and supplements it by incorporating a semiotic theoretical platform. Signs lose the representational and ancillary status with which they are usually endowed in classical cognitive theories in order to become the *material* counterpart of thought. One of the key features of this approach is that its semiotic platform opens up new possibilities for understanding algebraic signs and formulas in a nonconventional manner. Traditionally, letters and signs for operations (like +, x, etc.) have been considered *the* algebraic signs of school algebra. Alphanumeric symbolism has indeed been regarded as the semiotic system of algebra *par excellence*. Yet, from a semiotic perspective, signs can also be something very different. Words or gestures, for instance, are signs on their own – semiotically speaking, they could be algebraic signs, as genuine in this respect as letters. Of course, as I will argue later in more detail, this does not mean that they are equivalent or that we can simply substitute the ones for the others. What makes semiotic systems unique and unsubstitutable is their *mode of signifying*. There are things that we can signify and intend through certain signs, and things that we cannot. Try to put Pablo Neruda's famous poem "Canción Desesperada" ("Desperate Song") in an algebraic formula, and you will see how hopeless the task is.

In the first part of this article, I argue that the mathematical situation at hand and the embodied and other semiotic resources that are mobilised to tackle it in *analytic* ways characterise the *form* and *generality* of the algebraic thinking that is thus elicited. My claim is based not only on semiotic considerations, but also on new theories of cognition that stress the fundamental role of the context, the body and the senses in the way in which we come to know. In the second part, I present some concrete examples through which the typology of forms of algebraic thinking is illustrated.

The 1990s algebra research agenda

During the discussions held in the 1980s and 1990s (see, e.g., Bednarz, Kieran, and Lee 1996; Sutherland et al. 2001), it was impossible to agree upon a minimal set of characteristics of algebraic thinking. There was, however, a more or less general consensus concerning two aspects. Algebra deals with objects of an *indeterminate* nature, such as unknowns, variables, and parameters. Furthermore, in algebra, such objects are dealt with in an *analytic manner*. What this means is that, in algebra, you calculate with indeterminate quantities (i.e. you add, subtract, divide, etc. unknowns and parameters) as if you knew them, as if they were specific numbers (see, e.g., Kieran 1989, 1990; Filloy and Rojano 1984a, 1989; Cortes, Vergnaud, and Kavafian 1990; for some epistemological analysis, see Filloy and Rojano 1984b; Puig 2004; Radford and Puig 2007; Serfati 1999).

Of course, one way or another, algebraic objects have to be designated. The general tendency in the 1980s and early 1990s was to associate school algebra and algebraic thinking with the use of letters[1]. Even if at the time the idea was not universally shared (Linchevski 1995; Mason 1996; Balacheff 2001), it prevailed nonetheless, and is still very strong in current research on the teaching and learning of algebra[2]. Although I do believe that it is impossible to practise abstract algebra (e.g., Galois Theory) without some sort of sophisticated notations, I do not think that algebra and algebraic thinking can be reduced to the use of letters. As John Mason pointed out some years ago, "the manipulation of symbols is only a small part of what algebra is really about'' (1990, 5). Letters, indeed, have never been either a necessary or a sufficient condition for thinking algebraically. For instance, in his *Elements*, Euclid used letters without thinking algebraically. Conversely, Chinese and Babylonian mathematicians thought algebraically without using letters (Radford 2006).

What I am suggesting here, then, is this: algebra is about dealing with indeterminacy in analytic ways. But instead of conceding alphanumeric symbolism the exclusive right to designate and express indeterminacy, I am claiming that it is only one of the several semiotic forms equipped to accomplish it. This is true of the practices of elementary algebra (some examples will be provided in subsequent sections) and of advanced algebra as well – even if in the latter, alphanumeric symbolism becomes more salient.

But before I go further, let me reassure you that my idea is not to challenge the power of symbolic algebra. Rather, I am trying to convince you that it is worthwhile to entertain the idea that there are many semiotic ways (other than, and *along with*, the symbolic one) in which to express the algebraic idea of unknown, variable, parameter, etc. I deem this point important for mathematics education for the following reason. Ontogenetically speaking, there is room for a large conceptual zone where students can start thinking algebraically, even if they are not yet resorting (or at least not to a great extent) to alphanumeric signs. This zone, which we may term the *zone of emergence of algebraic thinking*, has remained largely ignored, as a result of our obsession with recognising the algebraic in the symbolic only.

Sensuous cognition

My claim about a diversity of semiotic forms for dealing with algebraic indeterminacy rests on a perspective on thinking that is squarely at odds with the mental conception of thinking that informed most of the 1990s research on mathematics education. Within this mental conception of thinking, signs were often considered 'symptoms' of mental activity – hence the distinction between internal and external representations. Drawing on Vygotskian psychology, from the semiotic-cultural perspective advocated here, the question of the relationship between signs and thought is thematised in a different way. First, signs are considered in a broad sense, as something encompassing written as well as oral linguistic terms, mathematical symbols, gestures, etc. (Arzarello 2006; Ernest 2008; Radford 2002a). Secondly, signs are not considered as mere indicators of mental activity. In contrast, signs are considered as *constitutive* parts of thinking. In more precise terms, within this semiotic-cultural perspective, thinking is considered a *sensuous and sign-mediated reflective activity embodied in the corporeality of actions, gestures, and artifacts.*

The adjective *sensuous* refers to a conception of thinking that is inextricably related to the role that the human senses play in it. Thinking is a versatile and sophisticated form of sensuous action, where the various senses *collaborate* in the course of a multi-sensorial experience of the world (Radford 2009a). This multi-sensory characteristic of cognition has been emphasised by philosophers like Arnold Gehlen (1988) and Maurice Merleau-Ponty (1945), and at its heart is the idea of the important role that the body plays in the way we come to conceptualise things. As Gallese and Lakoff recently contended:

> the sensory-motor system not only provides structure to conceptual content, but also characterizes the semantic content of concepts in terms of the way that we function with our bodies in the world (Gallese and Lakoff 2005, 455–6).

In tune with such views, some researchers in our field are paying attention to the embodied nature of mathematical cognition. This is the case with Ferdinando Arzarello and the Torino Team in Italy, Rafael Núñez and Laurie Edwards in the USA, Michael Roth and the CHAT group in Canada, the Uniban research team in Brazil, etc. To mention a brief example, the Uniban team is investigating the role of gestures in blind children. Here gestures and tactility come to play a crucial role in understanding mathematical concepts (see the research conducted by Solange Ali Fernandes and Lulu Healy with blind children [Ali Fernandes 2008]).

Of course, tactility and other sensorial mediated processes are also important in non-impaired children. Ricardo Nemirovsky has suggested that instead of being mere mental processes, understanding and imagination of mathematical concepts are literally embedded in perceptuo-motor action: the "understanding of a mathematical concept spans diverse perceptuo-motor activities" (Nemirovsky 2003, 108), so that in this regard, "understanding is . . . interwoven with motor action" (Nemirovsky 2003, 107).

However, thinking encompasses still much more than that. Thinking is an activity that, although performed by an 'I' and the 'I's body', is ubiquitously drawing on culture's kit of patterns of meaning-making as well as on historically-constituted concepts of an ethical, political, scientific, and aesthetic nature. Thinking is bound to the context and the culture in which it takes place. This is why it is more accurate to say that thinking in general, and algebraic thinking in particular, is a cognitive historical *praxis* mediated by the body, signs, and tools.

Learning as objectification

From an educational perspective, the main question is: how do the students acquire fluency in such cognitive cultural-historical *praxes*? How do they become acquainted with the historically-constituted forms of action, reflection and reasoning that those *praxes* convey? Since mathematical forms of reasoning have been forged and refined through centuries of cognitive activity, they are far from trivial for the students. It is the historical density of such praxes, sedimented now in compact, systemic, and highly abstract formulations, that is the basis of what Vygotsky intended with his famous distinction between 'quotidian' and 'scientific' concepts – regardless of how unfortunate Vygotsky's choice of terms was.

Reflective acquaintance with cognitive historical praxes and their concomitant forms of action and reasoning is what learning consists of. And, as I submitted elsewhere (Radford 2008a), it can be theorised as *processes of objectification*, that is, those social processes through which students grasp the cultural logic with which the objects of knowledge have been endowed, and become conversant with the historically-constituted forms of action and thinking.[3]

Working within this theoretical framework, where semiotics, culture and history are driving principles, in recent years my collaborators and I have been busy implementing classroom holistic activities that can offer students the opportunity to reflect algebraically and to get acquainted with some basic ideas of algebra in different contexts – equations, pattern generalisation and, recently, graph interpretation (Radford 2000, 2002b, 2003, 2009a, 2009b; Radford, Bardini and Sabena 2007). Our goal has been to try to understand what I previously referred to as the *zone of emergence of algebraic thinking* and forms of algebraic thinking elicited by our activities.

Let me pause this theoretical discussion here and turn now to some short examples that come from our first longitudinal research project – a project that we conducted from 1998 to 2003, and during which we accompanied four classes of students as they went from Grade 8 to Grade 12, i.e., until the completion of high school at age of 17–18. The examples will illustrate the typology of algebraic thinking that is suggested in this article and, at the same time, give an idea of our approach[4]. At the end of the article, comments on the meaning of the typology from a developmental perspective will be provided.

Some classroom results

The students' first contact with algebraic symbolism occurred when they were in Grade 8. In Grade 9 we decided to start with an activity that was intended as a means to revisit the concepts learned in the previous year. In the introductory part of the activity, the students, working in groups of three, had to draw Figure 4 and Figure 5 of the sequence shown in Figure I and find out the number of circles in Figures 10 and 100.[5] In the second part of the activity, the students were asked to write a message to a student of another Grade 9 class indicating how to find out the number of circles in any figure ("*figure quelconque*", in the original French), and then write an algebraic formula for the number of circles in Figure *n*.

Factual algebraic thinking

Usually, the students start counting the number of circles in Figures 1, 2, and 3, and realise that, in sequences like the one shown in Figure I, the number of circles increases by the same number each time. However, as the students quickly notice, this recursive relationship between consecutive figures is not really a practical way to answer the question about 'big' figures, like Figure 100.

In one of the groups (formed by Jimmy, Dan, and Frank), working on the sequence shown in Figure I, the students imagined the figures as divided into two rows:

1. Dan: (*referring to Figure 1*) Well … *(pointing to the top row)* 2 on top; there, there is 3 on the bottom …
2. Jimmy: [Figure] 2, there are 3; [Figure] 3, there are 4.
3. Dan: wait a minute. Ok *(he makes a series of gestures as he speaks; see four of the six gestures in Figure II*), Figure 1, 2 on top. Figure 2, 3 on top. Figure 3, 4. Figure 4, 5.
4. Jimmy: Figure 10, it will be 11 …
5. Dan: … 11 on top, and 12 on the bottom.
6. Jimmy: Every time there will be one more in the air.
7. Frank: [Figure] 100? 101, 102 …
8. Dan: 203.

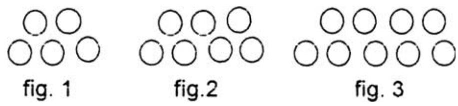


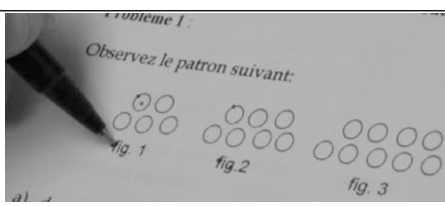
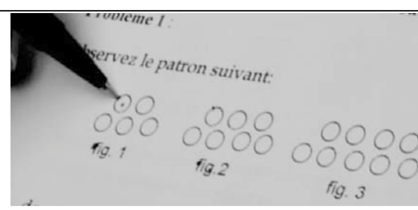
Figure I. The sequence of the introductory pattern generalisation activity in Grade 9.

As the students' dialogue suggests, the generalisation was accomplished in two steps. In the first step (lines 1–3), the students conceived of the figures as divided into two lines, and, drawing on perceptual observations made on the first three given figures, they were able to objectify a *regularity:* a relationship between the number of the figure and the number of circles in its rows.

The grasping of the regularity is not enough, however, to ensure the generalisation. The regularity has to be generalised. And this is what the students accomplished in the following turns where they came up with a formula to find the number of circles in Figures 10 and 100. Indeed:

- in turns 4 and 5 the observed regularity of perceptually available figures was *generalised* to Figure 10, a figure that is not in the students' perceptual *field*;
- line 6 contains a partial linguistic formulation of the general structure of the figures, as perceived by the students: "Every time there will be one more in the air", i.e., for all figures of the sequence, there is always one unmatched circle on the bottom row;
- in line 7, Frank resorted to the objectified pattern structure in order to calculate the number of circles in Figure 100.

The students are equipped now with a formula to answer questions about Figure 1000, Figure 1 000 000, or whatever particular figure you may have in mind.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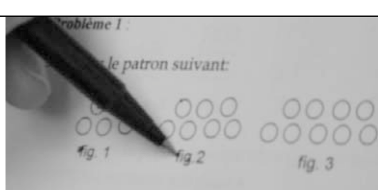	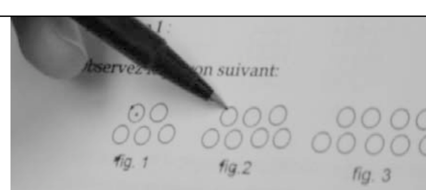
" Figure 2	3 on top"

Figure II. Dan makes a sequence of pointing gestures coordinated with words in a first process of objectification (reconstruction from the video data).

Now, I am talking about a formula, yet there are no letters! That's true. The algebraic formula consists, rather, in a piece of embodied action. We can call it – borrowing an expression from Vergnaud (1996) and changing it slightly – an *in-action-formula*.

A 'formula' of this concrete form of algebraic thinking can better be understood as an embodied 'function' or 'predicate' with a tacit variable: indeterminacy does not reach the level of discourse. It is present through the appearance of some of its instances (1, 2, 3, 4, 5, 10, 100). It remains an empty space to be filled up by the eventual uttering of particular terms. We call this type of situated and concrete form of algebraic thinking that operates at the level of particular number or facts *factual*[6].

Despite its apparently concrete nature, factual algebraic thinking is not a simple form of mathematical reflection. On the contrary, as can be seen in Fig. II, it rests on highly evolved mechanisms of perception and a sophisticated rhythmic coordination of gestures, words, and symbols. The grasping of the regularity and the imagining of the figures in the course of the generalisation results from, and remains anchored in, a profound sensuous mediated process– showing thereby the multi-modal nature of factual algebraic thinking[7].

Let us turn now to the second part of the Grade 9 activity.

Contextual algebraic thinking

In the introduction I suggested that the mathematical task at hand and the social sign-mediated processes of perception and generalisation can inform us of the form and generality of the algebraic thinking that is thus elicited. What kind of algebraic thinking will now be generated? The task requires that the students go beyond particular figures and deal with a new object: a *general* figure. Indeterminacy must now become part of explicit discourse. Our question is: How will the students build the formula? In the absence of gestures and rhythm, to which linguistic mechanisms will the students resort?

In fact, in being asked to write a message, the students were invited to enter into a deeper level of objectification than the one of action and perception characteristic of factual algebraic thinking. Writing makes one render explicit things that may have remained on what neuropsychologists call the area of proto-attention, or what Husserl used to call the horizon of intentions (Husserl 1954).

In Grade 8, writing a message that involves this new object 'general figure' proved to be very difficult. As we reported in previous work (see, e.g., Radford 2000), the students often used particular figures (like Figure 12) as examples to convey a *generic* idea, or used particular figures in a *metaphorical* sense to talk about the still unutterable generality (Radford 2002a). Sometimes the message was not complete. Here is an example: "You add 1 [circle] on the top and 1 on the bottom."

In Grade 9, the students felt much more comfortable with this level of generality. The following message is paradigmatic of what the students wrote: "You have to add one more circle than the number of the figure in the top row, and add one more circle than the top row to the one on the bottom."

Of course, this procedural sentence can be seen as a *formula*. But it is very different from the one discussed in the previous section. Here, rhythm and gestures have been replaced by key descriptive terms – 'top,' 'bottom.' These terms are what linguists call spatial *deictics*, that is to say, words with which we describe, in a contextual way, objects in space. The indeterminate object variable is now explicitly mentioned through the term 'number of the figure.' However, although different from factual algebraic thinking, both in terms of the way indeterminacy is handled and the semiotic means by which the students think, the new form of algebraic thinking is still contextual and 'perspectival' in that it is based on a particular way of regarding something[8]. The algebraic formula is indeed a *description* of the general term, as it was to be drawn or imagined. This is why we term this form of algebraic thinking *contextual*. Here is another Grade 9 example: "# of the figure + 1 for the top row and the # of the figure + 2 for the bottom. Add the two for the total."

Let us turn now to the last part of the Grade 9 activity.

Standard algebraic thinking

Expressing the formula in algebraic standard symbolism was much more difficult than expressing it in words, both in Grades 8 and 9, although, of course, there was some progress from one year to the next. The results mentioned in the previous section shed some light on the nature of these difficulties: previously, the students could resort to a range of semiotic resources, like pointing and iconic gestures, deictics, adverbs, etc. Those rich semiotic resources do not have a place in the alphanumeric-based algebraic formulas. In short, there is a drastic change in the mode of designation of the objects of discourse.

How, then, to designate the number of circles in a figure, in the highly-condensed semiotic system of alphanumeric signs? From an ontogenetic viewpoint, direct 'translation' is not something on which we can count, as we cannot count on direct translation from our native language to a new one we are just starting to learn. Direct translation presupposes that you already know the target language. In the case of the standard alphanumeric algebraic language, the situation is even worse, as this language is not even 'natural.' Our standard algebraic language is *artificial*. Historical analysis shows that its construction was preceded by a good deal of efforts that ended up in dead ends and failures (Høyrup 2008; Serfati 2006).

In Grade 8, the students often resorted to particular examples. Thus, dealing with the sequence shown in Figure III, Dan and his group (in Grade 8, the group was formed by Dan, Frank and Sara), illustrated the formula through the case of Figure 100:

1. Dan: You add 3 on top, and 1 at the bottom.
2. Sara: That's true if you go by the [form of the] figure.
3. Dan: You add 3 on top, and 1 at the bottom. Let's say that n equals 100. It would be 100 ... you add 1, it would be 101 [on the bottom row] ...
4. Frank: (Interrupting) and 103 [on the top row].

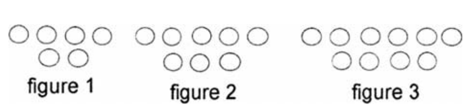

Figure III. One of the sequences the students investigated in Grade 8.

In other cases, the students often resorted to formulas that, superficially, look to be algebraic, in particular because they contain letters. Thus, in the sequence shown in Figure III, several students in Grade 8 produced the formula $n \times 2+4$. However, despite its appearance, the formula is *not* algebraic. It was instead obtained by trial and error. Dan and his group first tried $n \times 2+1$, then $n \times 2+2$, etc. until they obtained $n \times 2+4$, which seemed to work in the few cases in which they tested it. This procedure is not based on an *analytic* way of thinking about indeterminate quantities – the chief characteristic of algebraic thinking. This procedure does not even reach the sophistication of pre-algebraic arithmetic methods such as 'false position.' It is rather a kind of arithmetic naïve induction[9].

To counter these inductive arithmetic procedures, in designing the classroom activity, we added a question in which the students were asked to provide a formula for calculating the number of circles on the top row of Figure n. They were then asked to find a formula for the total number of circles in Figure n. Establishing a functional relationship between the number of the figure and the number of circles on top of the figure proved very difficult. Dan and his group suggested using two letters:

> Dan: (*Noticing that each figure has two more circles than the previous one*) It's plus 2 [to obtain the number of circles in the next figure], plus 2 [to obtain the number of circles in the next figure], plus 2 ... Unless we put 2 letters ... What we would do is ... the top row would be n, and the bottom row would be like b. After that, you do $n+b+2$.

In this case, the letters n and b do not designate the number of circles in the top and bottom rows of Figure n. Actually, the number of the figure is not even taken into account. The formula, rather, expresses a vague recursive relationship.

Another Grade 8 group suggested the 'cascading formula' shown in Figure IV.

The first line corresponds to the number of circles on the bottom row. The result is called 'w'. This is expressed in the second line, where it is also said that you still have to add 2 to get the number of circles on the top row. This last number is called 'x', as indicated in the third line of the formula. Finally, in the last line, the students are saying that you still have to add the numbers represented by 'w' and 'x' to obtain the total of circles in Figure n. Not bad, although still a bit far away from the standard way to write formulas within the alphanumeric semiotic system of algebra. Not bad, even if the use of several letters and their inter-connected meanings is not fully clear for the students. As one of the students from this group said to the other two members, "You mix me up with all your letters!"

The first example (Dan's) is interesting in that it shows that, although these students were able to produce an inductive formula that looked like an algebraic one (i.e., '$n \times 2+4$'), they did not produce the expected algebraic formula '$n+3$' for the top row of Figure n – even if the formula '$n \times 2+4$' seems much more complex.

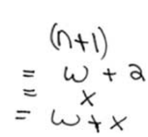

Figure IV. A Grade 8 student's formula using two letters.

The complexity of the formulas cannot be judged by the number of involved terms only; the complexity of the formula should also be judged in terms of the mode of designation of the objects of discourse.

The second example is interesting in that it unveils some of the tremendous difficulties that the students have to face when using letters to intend to say what they perfectly know how to express in natural language. This problem is much more complex than a simple translation. As Glaeser remarked, "the urge to give an immediate meaning to every intermediate result has to be resisted" (1999, 154). Meaning, indeed, has to be put in abeyance.

In Grade 9 we still found some formulas that resembled those produced in Grade 8. But more typical of Grade 9 were the formulas shown in Figure V (these formulas correspond to the sequence shown in Figure I).

These formulas bear a closer resemblance to those of the standard symbolic algebra. Yet, the signs still keep the embodied and perspectival experience of the objectification process observed in Grade 8. We easily recognise in the term '$n+1$' the reference to the top row, as we recognise in the term '$n+2$' the reference to the bottom row. In Dan's group, for instance, this embodied manner of symbolising was made very clear:

1. Dan: No, no, well, it's that ... $n+1$ is the top row ...
2. Frank: (*Interrupting*) Yes, I know.
3. Dan: $n+2$ is the bottom row.

As is clear from Figure V, the students add brackets to distinguish carefully between the rows. This is why, I want to suggest, the formula is an *icon*, a kind of *geometric description* of the figure. In other terms, the formula is not an abstract symbolic calculating artefact but rather a *story* that narrates, in a highly condensed manner, the students' mathematical experience. In other words, the formula is a *narrative.* And it is the narrative dimension of the students' iconic formulas that very often makes it possible to infer from the formula the sequence to which it corresponds (see Figure VI).

That which previously was distinguished through pointing gestures and linguistic deictics is now distinguished through the effect of signs and brackets. It is precisely this 'perspectival' nature of the formula that leads many students to argue that brackets cannot be removed. Otherwise, they argue, it would be impossible to know what the terms of the formula mean. Yet, this is precisely what constitutes the force of algebra – the detachment from the context in order to signify things in an abstract way. The mode of designation has to move to a different layer where signs borrow their meaning not from the things they denote but from the *relational* way they mean within the context of other signs.

The narrative meaning of iconic symbolic formulas became even clearer when a fifth class was added to our project. As our project progressed, other teachers became interested in it and, to the extent that we could, we included new classes. The

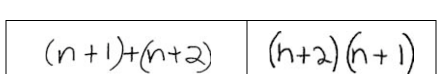

Figure V. Left, the formula produced by Dan's group in Grade 9. Right, a variant of it produced by another Grade 9 group.

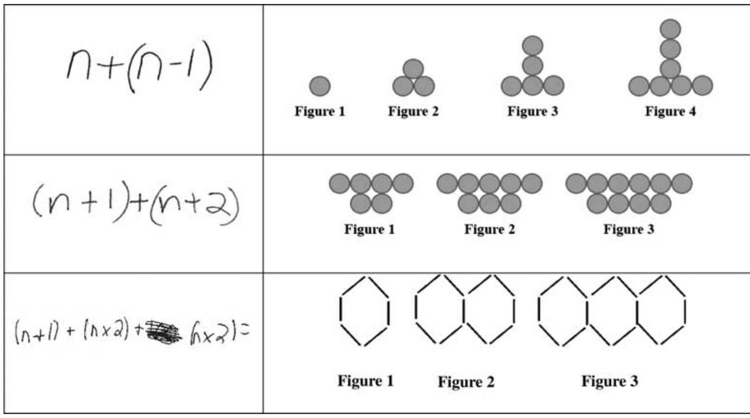

Figure VI. Formulas as narratives. Instead of decontextualised calculations, the formulas narrate the manner in which calculations have to be carried out in close relationship to the geometry of the figures and position of their parts.

fifth class regrouped Grade 8 students who were recognised as having difficulties in following the rhythm of 'regular' math classes. Dealing with the pattern shown in Figure VII (left) one group of students produced the formula shown in Figure VII (right).

The formula does not have the usual linear organisation of standard algebraic formulas. Rather, signs signify in a spatial manner: as the students explained to us, the top 'R' means that there are as many toothpicks on the top of the figure as the number of the figure. The 'R' placed on the bottom of the formula means that there are as many toothpicks on the bottom of the figure as the number of the figure. The lateral 'R' means that there are as many vertical toothpicks on the top of the figure as the number of the figure, but not really. There is an extra toothpick to be accounted for, placed at the right end, signified by the lateral sign '1.' The '+' signs mean that you have to add all of those things.

From iconic formulas to symbolic ones

One of the important didactic problems is to implement classroom activities that will allow the students to endow their formulas with new abstract meanings. In more precise terms, the problem is to transform the iconic meaning of formulas into something that no longer designates concrete objects. For instance, the formula $(n+1)+(n+2)$ mentioned previously (Figure V), has to be seen in a new light. The narrative dimension of formulas has to collapse (Radford 2002c). The embodied meaning of the formulas does not disappear. It rather gives rise to a more abstract

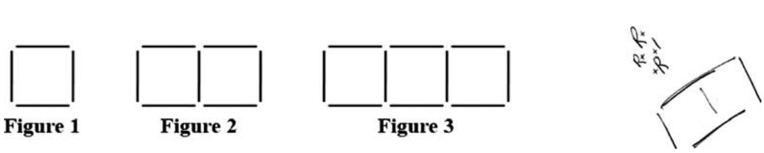

Figure VII. Left, a toothpick sequence. Right, an algebraic symbolic formula that includes its diagrammatic 'user guide' or *mode d'emploi*.

one. Thus, in addition to signifying the sum of circles in the top and bottom rows, the terms of the formula have to be considered in relation to the signs that they contain. Resemblances and differences – these key aspects of signification in general (Radford 2008b) – must no longer be exclusively based on spatial and contextual considerations (such as 'top' and 'bottom'). In the new form of signifying, there is a shift in focus: attention has to be directed now to morphological differences, i.e., differences in terms of *letters* versus *numbers.* In short, meaning must become *relational*.

The search for the pedagogical actions allowing the students to objectify this abstract form of signifying became one of our goals, both from a theoretical and a practical viewpoint. Our strategy was based on *comparing* and *simplifying* formulas. Here is an example that deals with the sequence of squares shown in Figure VII.

The previous day, the students produced several formulas. At the beginning of the class, the teacher asked for some examples. The students mentioned two, that were written as $r \cdot 3+1$ and $(r+1)+r \cdot 2$, where r stands for the rank or number of the figure.

1. Teacher: I would like to compare these formulas and to see where they come from. Brian, do you want to explain the first formula to us?
2. Brian: (*Going to the blackboard*). Ok, yesterday we saw that the first figure only has 1 toothpick at the bottom (*he points to the bottom of Figure 1 on the blackboard*) and the second figure, there were 2, third figure, there were 3. So, we added the bottom and the top, and then we saw that, in the first term, there were 2 [vertical toothpicks] (*points to the vertical toothpicks of Figure 1*) and Figure 2 has 3 (*points to the vertical toothpicks of Figure 2*) therefore, it's always [the rank or number of the figure] plus 1. So we did the bottom plus the top plus the rank plus 1. And then we saw that … Well, we discussed a lot, and we saw that … it was the rank, rank times 3 (*points towards the first term of the formula*) because it has the bottom, the top and the vertical. There was, there was, plus [one] …
3. Teacher: So you say that this (*pointing to the bottom row of the first square and colouring it with blue chalk; see Figure VIII, pic. 1*) is one r; this is another r (*pointing to the top row of the first square and colouring it with blue chalk; see pic. 2*); and this is the third r (*pointing to the left vertical side of the first square and colouring it with blue chalk; see pic. 3*) and there remains another one [toothpick] (*pointing to the second vertical line of the first square; pic. 4*). So, (*pointing to the formula*) r times 3 … I have three r here (*pointing successively to the coloured sides of the first square*) plus another one in each term (*pointing the uncoloured right vertical side of the first square*). (*Then, the teacher repeated the same set of sequence of pointing gestures on Figure 2, see Figure VIII, pics. 5–8*). This is the explanation of the formula. Now, Ron, would you please explain the second formula?

Ron went to the blackboard and explained the various elements of $(r+1)+r \cdot 2$. After that, the teacher encouraged a discussion about the formulas. Sandra – a student sitting at the end of the classroom – argued that both equations work, but the first one was simpler. The teacher summarised the difference as follows:

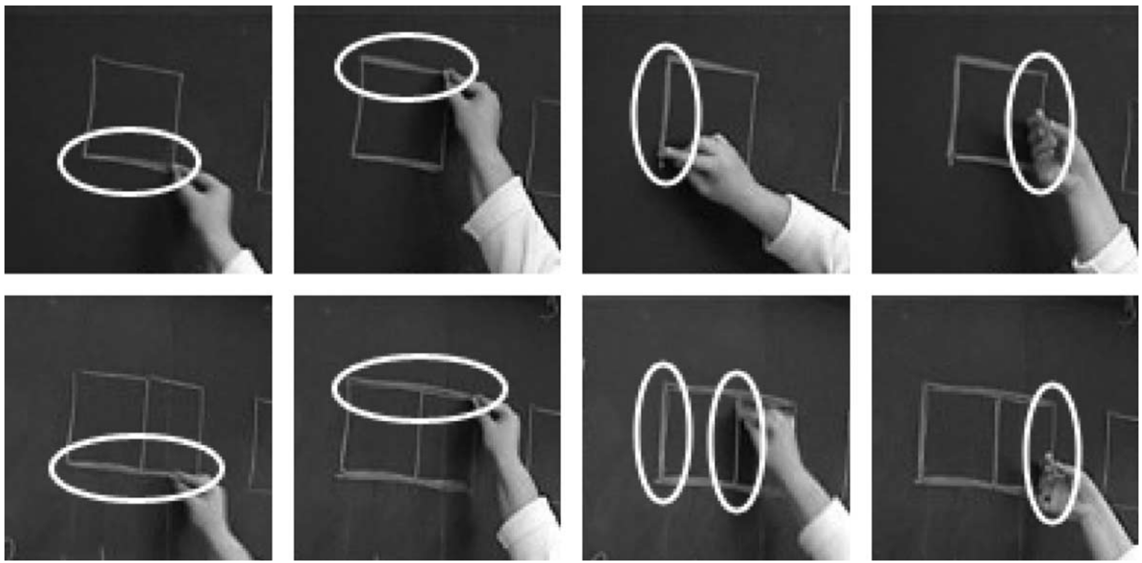

Figure VIII. Pictures 1–4 (top) show the teacher's effort to relate the terms of the formula $r \cdot 3 + 1$ to the various parts of Figure 1. Pictures 5–8 (bottom) show the same effort but this time the focus is on Figure 2. The teacher makes apparent for the students the new way of signifying through a subtle coordination of gestures, words, drawings and coloured segments.

1. Teacher: the difference is that here (*pointing to the formula* $r \cdot 3+1$) we put together the terms that were the same and we simplified. Since I am calculating the total number of toothpicks, I can put all together (*while talking, she emphasised the words 'same', 'simplified' and 'total'*). It is exactly this that the first formula does. (*Smiling to the class, she says*) I think that you are ready for the next activity.

The previous formula $r \cdot 3+1$ looks much like Dan's formula $n \times 2+4$ discussed earlier. Yet, the difference is considerable. Brian's formula was not produced by trial and error. It was the result of an algebraic generalising process where general functional relationships were first identified (e.g., the number of toothpicks on top vis-à-vis the rank or number of the figure), then simplified. As Brian put it, "... it was the rank, rank times 3 *because* it has the bottom, the top and the vertical." The teacher capitalised on Brian's idea and, through a feast of clear and consecutive gestures that echoed Brian's timid gestures, coloured parts of the first two figures to make clear for all the students the relationship between the spatial-geometric parts of the terms and their corresponding rank (Figure VIII, pic. 1–8). After showing each one of the three *r* in Figure 1, she linked the first part of the formula ($r \cdot 3$) to the three parts she had just coloured. She said: "*r* times 3 ... I have three *r* here," followed by the crucial remark that there is still "another one in each term" (which corresponds to the constant term of the formula). Her coordinated gestures and words related very well the spatial elements of the figures with the corresponding parts of the formula. The idea of putting *together* the toothpicks on the bottom, the top and the vertical ones, led to *adding* the number of the figure several times.[10]

That day, after the general discussion, the students dealt with a sequence of houses (Figure IX). The students identified the relationship between clue elements of the figures and their rank or number:

1. Raymond: the number of toothpicks in the roof is twice the number of the figure. For the walls [which included the floor], it is twice, and another wall ...
2. Joyce: (*interrupting*) to close the space ...
3. Raymond: So, the formula is rank times 4 plus 1.

In so doing, the students entered into a new form of algebraic understanding and moved into a deep region of the zone of emergence of algebraic thinking. They moved from a referential understanding of signs (signs as referring to particular objects, like the number of toothpicks in the roof) to a morphological one —the beginning perhaps of what Kieran (1990) Kirshner (2001), Hoch and Dreyfus (2006) and others have called the structural dimension of algebra.

It is clear that the symbolic formula is no longer just iconic. Iconicity is still present, but it has receded to make room for a more concise and abstract form of signification. Naturally, the students have yet to undergo a supplementary lengthy process of objectification to become fluent with the modern form of symbolic algebraic thinking, where symbolic calculations are carried out through formal

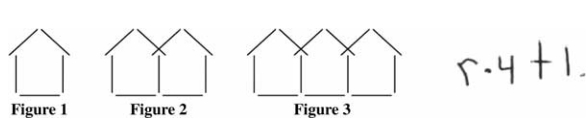

Figure IX. Left, a toothpick sequence of houses. Right, one of the students' formulas considerations only. For this to occur, new objects like x^2 and x^2+x will have to enter the universe of discourse and acquire a detached existence. It is not vain to recall here that this process was not easily achieved in the history of algebra. Thus, to distinguish magnitudes, Vieta – one of the founders of our modern algebraic symbolism – was still, in the 16th century talking about 'length', 'plane', 'solid', etc. Our modern way of referring to the now abstract monomials of algebra still reminds us of their embedded concrete beginnings. Indeed, monomials such as x^2 or x^3 read as 'x *squared*', 'x *cubed*'. Our modern language hangs behind the relics of its past, revealing thereby the monomials' original geometric-spatial origin.

Synthesis and concluding remarks

In this article, drawing on recent conceptions of thinking offered by anthropology, semiotics and neurosciences, I suggested that thinking is a complex form of reflection mediated by the senses, the body, signs and artifacts. I argued that the mathematical situation and the semiotic resources that are mobilised to tackle it in analytic ways characterise the *form* and *generality* of the algebraic thinking that is thus elicited. Focusing on the context of pattern generalisation, I suggested a typology of forms of algebraic thinking – factual, contextual, and symbolic. However, the typology should not be understood in terms of developmental stages in a 'naturalistic' sense. Classrooms are much more than the neutral spaces portrayed by Piagetian-inspired educational theory, alleged spaces to which the students idiosyncratically adapt. Developmentally speaking, the questions that we ask the students (like the patterning questions discussed in this article) are far from innocent. They are loaded with cultural and scientific values. They insinuate lines of cognitive development. Our pattern activities did not merely provide a kind of grooming context for thought to appear and evolve; they certainly had a definite influence in the students' emerging algebraic thinking[11]. The typology is not meant to be understood in a rigid hierarchical manner either. Thus, depending on the context and the problem at hand, a student can move back and forth along those forms of thinking. The typology is rather an attempt at understanding the processes that the students undergo in their contact with the forms of action, reflection and reasoning conveyed by the historically constituted praxis of school algebra.[12]

The classroom data presented here offers a glimpse of the ontogenetic journey of our students on their route to algebraic thinking. It stresses some of the challenges that they had to overcome when passing from factual to contextual to symbolic thinking. It stresses in particular the changes to be accomplished in modes of signification. While in factual thinking, indeterminacy remains implicit and gestures, words, and rhythm constitute the semiotic substance of the students' *in-action-formulas*, in contextual algebraic thinking indeterminacy becomes an explicit object of discourse. Gestures and rhythm are replaced by linguistic deictics, adverbs, etc. Formulas are expressed in a perceptual and 'perspectival' manner based on key terms like 'top', 'bottom', etc. Formulas, in short, are based on a particular way of seeing the sequence at hand.

Our discussion about symbolic algebraic thinking sheds some light on the meaning with which the students endow their first alphanumeric formulas. Instead of being an abstract calculating device, formulas often appear as vivid narratives. They are icons in that they offer a kind of *spatial description* of the figure and the actions to be carried out. What I called the 'collapse of narratives' appears as an important step towards more encompassing ways of algebraic signification. The constitution of meaning after such a collapse deserves more research (see also Barallobres 2007). While Russell (1976) considered the formal manipulations of signs as empty descriptions of reality, Husserl stressed the fact that such a manipulation of signs requires a shift of intention: the focus becomes the signs themselves, but not as signs *per se*. And he insisted that the abstract manipulation of signs is supported by new meanings arising from rules resembling the rules of a game (Husserl 1970), which led him to talk about signs having a game signification.

The classroom example discussed in the last section shows how the teacher, through a complex coordination of gestures, alphanumeric formulas, and words, capitalised on the formula of one of the groups to make apparent for the whole class the idea of simplification of formulas. It was a first step, and certainly an important one in the students' ontogenetic journey.

Although I limited my account to the first two years of the 5-year journey, I hope that such an account is enough to give an idea of the students' struggles and progresses towards increasingly more encompassing forms of algebraic thinking.

Acknowledgements

This article is a result of a research program funded by The Social Sciences and Humanities Research Council of Canada/Le Conseil de recherches en sciences humaines du Canada (SSHRC/CRSH). A previous version of this paper was presented at the *Sixth Congress of the European Society for Research in Mathematics Education* (CERME 6), in Lyon, France, in January 2009. I wish to thank Heinz Steinbring for his thoughtful and stimulating reaction to my CERME presentation.

Notes

1. This point was well made by Nemirovsky in an interesting article published in 1994. Nemirovsky complained about the emphasis put on symbolic systems and the students' understanding of symbolic systems' rules: "Countless studies," he said, referring to previous research, "describe how students' mistakes related to specific 'alternative' rules." (1994, 391).
2. See e.g. the emphasis on notations in contemporary research on early algebra.
3. My use of the term objectification differs hence from other current uses where objectification is conceived of as referring to something external and objective (regardless of the culture) or as a process transformed into object. The former has been developed in epistemological research informed by Realism; the latter by the linguistic tradition. My use of objectification comes from Hegel, Vygotsky, and Husserl's phenomenological work.
4. The examples have been chosen because they are representative of the ideas discussed in the article. They are also strongly representative of what happened in the classroom – without meaning that they are representations of a kind of "average" of the students' behaviour in the sense of quantitative studies.
5. To avoid confusions, figures in the article will be numbered using Roman numerals to distinguish them from numbers that refer to figures *in the sequences* investigated by the students.
6. The adjective factual stresses the idea that this generalisation occurs within an elementary layer of generality – one in which the universe of discourse does not go beyond particular figures, like Figure 1000, Figure 3245, and so on.
7. In our current research with Grade 2 students these mechanisms of rhythmic coordination are also present, but they do not reach the subtle sensorial synchrony that we observe in older students, as reported here.

8. It still supposes a spatially situated relationship between the individual and the object of knowledge that gives sense to expressions like 'top' and 'bottom'.
9. As epistemological analyses show, algebra has never been about guessing. Algebra since Babylonian and Greek times has always been about direct procedures to answer questions and solve problems characterised by the *analytic* manner in which *indeterminate* quantities (e.g., unknowns, variables, parameters; see Radford 2001) are dealt with. The advent of algebraic symbolism in the Renaissance and a concomitant interest in the devising of *general methods* to solve problems resulted in a focus on *structures*, although the 'structural turn' was not specific to algebra. In the case that we are discussing here, the meaning of the formula $n \times 2 + 4$ does not include this analytic structural dimension. The formula was obtained by simple guessing. It includes indeterminate quantities (symbolised by '*n*'), but lacks the *analytic* component. Its justification results from a numerical match between a guessed formula and a few observed cases, a match that is *hoped* to hold for *all* numbers. It is a form of naïve arithmetic generalisation. Bills and Rowland (1999) make a similar point in their interesting distinction between structural versus empirical generalisations without being concerned, however, by the question of analyticity. It might be the case that a generalisation can be structural without being algebraic, as there are also arithmetic and geometric structures (e.g., arithmetic false position methods exhibit a sort of structural component, without including the analytical component proper to algebraic thinking). I do not have the space here to go into further details, nor do I have space to say more about the delicate distinction between algebraic and arithmetic formulas. For a further discussion of the latter point, see (Radford 2006).
10. Rhythmic gestures, in this passage, were very important. As in factual algebraic thinking, they allowed the teacher to link various visual, linguistic, and symbolic elements together. However, rhythm here is not as prominent as it usually is in factual algebraic thinking. The cognitive difference in rhythmicity in both types of algebraic thinking is a matter of further investigation. At this point, I cannot say more. I am indebted to one of the reviewers for bringing this interesting point to discussion.
11. It might not be useless to remind here that these were the reasons that led Vygotsky (1981) to argue that education is the *artificial* development of the individuals.
12. Our current research on equations suggests that the typology presented here applies also to other domains of algebra (Radford, Demers and Miranda 2009).

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