ELEMENTARY FORMS OF ALGEBRAIC THINKING IN YOUNG STUDENTS[1]

Luis Radford

Université Laurentienne

This article seeks to contribute to the early algebra research field by enquiring about the types of algebraic thinking that can be made accessible to primary school students before any use of notational or alphanumeric symbolism. Results from an ongoing longitudinal study with Grade 2 students on pattern generalization suggest that early forms of embodied, non-symbolic algebraic thinking can appear in 7–8-year old students, and invite us to revisit the genetic relationship between arithmetic and algebraic thinking.

INTRODUCTION AND THEORETICAL FRAMEWORK

Inspired by theories of natural maturation, the teaching and learning of algebra has traditionally been postponed until students have acquired a relatively solid body of arithmetic knowledge. However, consonant with the Vygotskian idea that instruction precedes cognitive development, in a PME Research Forum organized by Janet Ainley (2001), Carraher, Schliemann and Brizuela (2001) argued that the learning of arithmetic need not be a prerequisite for the learning of algebra. Since then, an increasing amount of research has provided experimental evidence supporting the idea that some basic algebraic concepts can be successfully introduced in the early years (e.g., Moss & Beatty, 2006; Becker & Rivera, 2008). This article seeks to contribute to this new research field by enquiring about the types of algebraic thinking that can be made accessible to primary school students before any use of notational or alphanumeric symbolism.

The aforementioned research question is embedded in a theoretical perspective on teaching and learning—the *theory of knowledge objectification* (Radford, 2008)— in which forms of thinking are conceived of as historically and culturally constituted. Within this theoretical context, learning consists of positioning oneself reflectively and critically in historical forms of action and thinking. Functionally speaking, learning is conceptualized in terms of processes of *objectification*—i.e., activity-bound social processes through which the students encounter and grasp the historically-constituted forms of action and thinking. A central feature of the theory of objectification is that, in contrast to mental cognitive approaches, thinking is not considered something that solely happens ‘in the head’. Thinking is considered a tangible social practice materialized in the body (e.g. through kinaesthetic actions, gestures), in the use of signs (e.g. mathematical symbols, graphs, written and spoken

[1] Study funded by the Social Sciences and Humanities Research Council of Canada.

2010. In Pinto, M. F. & Kawasaki, T.F. (Eds.). Proceedings of the 34th Conference of the International Group for the Psychology of Mathematics Education, Vol. 4, pp. 73-80. Belo Horizonte, Brazil: PME.

words), and artifacts of different sorts (rulers, calculators and so on). Thus, in what follows, in the practical investigation of young students' algebraic thinking attention will be paid to the students' discourse, gestures, and artifact-use.

METHODOLOGY

Data Collection

Our data comes from an ongoing 3-year longitudinal research program conducted in an urban primary school. The data has been collected during regular mathematics lessons designed by the teacher and our research team. In these lessons the students spend substantial periods of time working together in small groups of 2 or 3. At some points, the teacher (who interacts continuously with the different groups during the small group-work phase) conducts general discussions allowing the students to expose, compare and contest their different solutions. To collect data, we use four or five video cameras, each filming one small group of students. This article focuses on one of the small groups of 3 students (Cindy, Erica, and Carl).

Data Analysis

The data that is presented here comes from a sequence of five Grade 2 lessons (7–8-years old students) on pattern generalization. In the first lesson, the students worked on the sequence shown in Fig. A. The students were asked first to draw Figures 5 and 6 and then, after answering other questions, to come up with a procedure or formula to find the number of squares in some 'remote' figures—like Figure 12, and some 'big' ones, like Figures 25 and 50. In tune with our theoretical framework, to investigate the students' algebraic thinking we conducted a *multi-semiotic data analysis*. Once the videotapes were fully transcribed, we identified salient episodes of the activities. Focusing on the selected episodes, with the support of the transcripts, we carried out a low-motion and a frame-by-frame fine-grained video microanalysis to study the role of gestures and words.

Figure 1 Figure 2 Figure 3 Figure 4

Fig. A. The first four figures of a sequence given to the students in a Grade 2 class.

RESULTS AND DISCUSSIONS

The teacher spent the first day discussing with the students how to extend the sequence to Figures 5 and 6. The next day the class tackled the questions about Figures 12 and 25. The questions were promptly answered. Thus, talking about Figure 12, with ease Cindy said: "12 plus 12, plus 1". Cindy's calculations reflect a perception of the figures as being made up of two horizontal rows; the two numbers "12" designate the number of white squares on each row, while the number "1" designates the dark square. Although such a perception of the figures may seem 'natural' to us, for second graders it is not. For second graders it entails what I have termed elsewhere *the domestication of the eye* (Radford, 2009): a lengthy process in the course of which we come to see and recognize things according to "efficient" cultural means. It is this process that converts the eye (and other human senses) into a sophisticated intellectual organ—a "theoretician" as Marx (1998) put it. Capitalizing on the teacher's efforts and class discussions of the previous day that led to efficient ways of perceiving the figures, referring to Figure 25, Erica continued:

1. Erica: Cindy! Um... Okay, What is 25 plus 25?
2. Cindy: Euh…
3. Erica: (Smiling) After that, you add one!

Before going further, it is worth noticing that asking the students to find the number of squares in 'remote' figures like Figures 25 was far from trivial. The arithmetic knowledge of our Grade 2 students was, at the time, very limited. Although they had some acquaintance with multiples of 10 and their additions, they were able to make *systematic* additions only up to 25. This is why our question about Figure 12 was at the very limit of their calculating capabilities. Our question about Figures 25 was definitely beyond them (hence Erica's question in Line 1). But instead of being a hindrance, not knowing how to make additions beyond 25 was a good thing. The design of the activities was based on the limits of students' arithmetic to promote the emergence of algebraic thinking. Indeed, by exploiting the students' limits of arithmetic thinking, the design of the activity aimed at favouring the students' *awareness* of calculation methods. And here the calculator proved to be of great importance. To help the students deal with 'remote' figures, the teacher made calculators available to the students. But, *before* finding the actual number of squares in Figure 25 or other 'remote' figures using the calculator, she asked them to come up with an idea of how to find the total. This pedagogical strategy induced in the students' activity an important shift from the numeric *qua* numeric to a *rule* or *calculation method*. It was within this context that the students tackled the following question concerning a 'big', unspecified figure. Here is the question: "Pierre wants to build a big figure of the sequence. Explain to him what to do." Surely enough, the students chose 'big' *particular* figures. Carl suggested considering Figure 500; Cindy preferred Figure 50:

4. Carl : How about doing 500 plus 500?
5. Erica: No. Do something simpler.
6. Carl: 500 plus 500 equals 1000.
7. Erica: plus 1, 1001 [...]
8. Cindy: No, 50 plus 50, plus 1 equals 101.

It might be worth asking now whether or not there is something algebraic in these responses. Let me note first that to answer the question about Figure 12 Erica and her team did not go from Figure 4 to Figure 12, building figure after figure. To deal with Figures 12 and 25 and other 'remote' figures, the students accomplished a *generalization*. This generalization was, I want to claim, *algebraic* in nature. In fact, the students' generalization is based on a *rule* or *formula* that, even if is not explicitly formulated, is shown *in action*. This in-action "formula" can better be understood as an embodied predicate (e.g. "12 plus 12, plus 1"; "50 plus 50, plus 1") through which the students can now *express* the number of squares in any *particular* figure. The students' in-action-formula attests to a shift in focus away from the numeric. Their focus is on numbers, of course, but in an algebraic manner, however simple this manner is. This is why Erica was not stopped by not knowing the result of 25 + 25. Whatever it is, you have to add 1 (the dark square). For the students' emerging understanding, what matters is not only the result, but also the calculating method or formula.

To better understand the students' algebraic thinking let us note that the formula is built on a *tacit variable* (the number of the figure) present only through some of its instances (e.g., "12", "25", "50"). This variable does not reach the level of symbolization—not even the level of discourse: there are no words in the students' vocabulary to name it. The variable remains implicit or, to be more specific, *intuited*—something whose presence is only vaguely adverted through particular instances, like clouds anticipating a storm. The variable is expressed in an *indexical* manner: its instances point to something that is in *adventus*, that is to say, to-come.

Beyond intuited variables

The following days, the students explored similar sequences. On the fifth and final day of our pattern generalization teaching-learning sequence, the teacher came back to the sequence shown in Fig. A. To recapitulate, she invited some groups to share in front of the class what they had learned about that sequence in light of previous days' classroom discussions and small group work. Then, she asked a completely new question to the class. She took a box and, in front of the students, put in it several cards, each one having a number: 5, 15, 100, 104, etc.

Each one of these numbers represented the number of a figure of the pattern shown in Fig. A. A student chose one of the cards at random and put it into an envelope, making sure that neither she nor anybody else saw the number beforehand. The envelope, the teacher said, was going to be sent to Tristan, a student from another school. The Grade 2 students were invited to send a message that would be put in the envelope along with the card. In the message the students would tell Tristan *how* to quickly calculate the number of squares in the figure indicated on the card. The number of the figure was hence *unknown*. The challenge offered to the students was to make calculations on this unknown number. Would the students be able to generalize the rule that they had objectified when working with 'remote' particular figures and engage with calculations on this *unknown* number? In other terms, would our Grade 2 students be able to go beyond intuited variables and the corresponding elementary form of algebraic thinking? Let me dwell on what happened in Erica's group. In an episode that lasted 30 seconds, Erica started making a suggestion:

9. Erica : You can do the number… (she makes the gesture shown in Fig. B, Pic. 1) the same number… as at the bottom (she makes the pointing gesture shown in

Pic. 2), after on the side you put another one (she makes the pointing gesture shown in Pic. 3).

10. Carl: And then at the bottom he will have the same number of light squares (he makes the pointing gesture shown in Pic. 4), at the top the same number of light squares (he makes the pointing gesture shown in Pic. 5), and a dark one (he makes the pointing gesture shown in Pic. 6).

Fig. B. Pictures1-6. Erica's and Carl's gestures.

As the previous dialogue shows, the fact that the number of the figure was unspecified did not impede the students in thinking of and talking about the figure in a mathematical way. Through the linguistic expression "the number", the students engaged with the variable in an explicit manner. The definite article "the" qualifies the noun "number" making it specific even if it is unknown. But the students' engagement with the variable was not only linguistic. The body also played a fundamental role, as shown by the students' fierce recourse to gestures. Indeed, in line 9, Erica says: "You can do the number…" and points to an imaginary place where she would find the bottom row of the unspecified figure. Then, she says: "The same number… as at the bottom", pointing now a bit higher to the imaginary place of the top row. Then, pointing to a spot on the right side, she finishes the sentence saying "after on the side you put another one." Drawing on Erica's idea, Carl immediately offered a recapitulation that was accompanied by a set of three gestures on the table with a pronounced movement of the arms and the whole upper part of the body. The unspecified figure thus became an object of consciousness and imagination through language but also through an impressive array of pointing gestures (Erica) and arm motion (Carl). Rather than a mental process, mathematical imagination appears here as something definitely visceral (Nemirovsky & Ferrara, 2009). To sum up, from the intuited form in which it appeared in the students' previous activity, the variable has now been objectified in an explicit way and has entered the realm of the students' universe of discourse. In so doing, the students have reached a new layer of generality (for a detailed discussion of layers of generality, see Radford, 2010a).

However, in contrast to what the students did when dealing with "big" particular figures, like Figure 50, here the students did not produce a formula. Indeed, instead of something similar to Cindy's formula "12 plus 12, plus 1", the students produced a *spatial description* of the unspecified figure. As Carl said: "at the bottom he will have the same number of light squares; at the top the same number of light squares and a dark one." As a result, there are no explicit *operations* with the unknown number.

When the teacher came to see the group's work, Carl explained the message they were working on, using an example—Figure 50. He said: "You do 50, plus 50, plus

1," to which the teacher responded: "Excellent! That would be a good example. But what if Tristan finds another number?" Erica continued:

11. Erica: It's the number he has, the same number at the bottom, the same number at the top, plus 1. . .
12. Teacher: That is excellent, but don't forget: he doesn't have to draw [the figure]. He just has to add… So, how can we say it, using this good idea?
13. Erica : We can use our calculator to calculate!
14. Teacher : Ok. And what is he going to do with the calculator?
15. Erica : He will put the number…(she pretends to be inserting a number into the calculator; see Fig. C, pic. 1)… plus the same number, plus 1 (as she speaks, she pretends to be inserting the number again (Pic. 2) and the number 1 (Pic. 3))…
16. Teacher: (Repeating.) The number, plus the same number, plus 1! Do you think that Tristan would be able to find the total like that?
17. Cindy and Carl: Yes!
18. Teacher: Very good. I will go to check on the other groups now.

Fig. C. Pictures 1-3. Erica using the calculator to imagine calculations on the variable

In Line 12 the teacher makes the subtle distinction between drawing and calculating. The formula can be derived from the students' general description of the figure, but it is not equal to it. An algebraic formula does not include terms such as "top" and "bottom". In Line 13, Erica suggested using the calculator and mentioned the sequence of calculations to be carried out in order to find the total. Naturally, the use of the calculator is metaphorical. In the students' calculator, all inputs are specific numbers. Nevertheless, the calculator helped the students objectify the analytic dimension that was apparently missing in the new layer of generality. Through the calculator, calculations are now performed on this unspecified instance of the variable—the unknown number of the figure.

Since our Grade 2 students were still in the early stages of writing and since writing an explanation of a few sentences was taking them far too long, from Day 2 on we did not ask the students to write explanations and justifications of their work, but rather to use a digital voice recorder to record them. Our Grade 2 students ended up practicing oral algebra— perhaps like early Renaissance students before the invention of the printing press and the spread of writing as a social phenomenon. At the end of the lesson, several groups were invited to come forward and record in this way the message to Tristan. Erica summarized her group message as follows:

> Hi Tristan. You put the number at the bottom (she makes a gesture as if pointing at the imaginary bottom row) the same number on top (as if pointing to the imaginary top row), plus 1. Afterwards, you use the calculator and (making gestures as if using the calculator keyboard) you insert the number plus the same number plus 1, and after you press equal and it will show you what it is.

The message was divided into two parts. In the first part Erica tells Tristan about the *aspect* of the figure. In the second part, Erica indicates the *calculations* to be performed. It seems that knowing how the figure looks is a prerequisite to making the calculations. Indeed, Erica imagines entering the numbers in the calculator in the order that the students imagined the unspecified figure (from bottom to top, then the dark square). The meaning of the terms in the formula is hence derived from the *spatial configuration* of the figure. The situated, spatial sense of the unknown numbers in the formula seems to constitute the limits of our Grade 2 students' algebraic thinking. Yet there was one group that overcame this limit. When their turn came to record the message for Tristan, they produced the following message:

> Hello Tristan, um… we are going to show you a strategy to figure out the sequence. Um, you have to find the number. If the number you grab is like 50 or 40 or something, you have to do like the number times two and after plus 1, and you will see what it equals to.

Here the addition of the unknown number with itself is turned into a multiplication by two. The spatial meaning of the unknown is overcome.

CONCLUDING REMARKS

Algebraic thinking does not appear in ontogeny by chance, nor does it appear as the necessary consequence of cognitive maturation. To make algebraic thinking appear some pedagogical conditions need to be created. Of course, there are certain limits to what can be accomplished. In this context, this article explored the question of the kind of algebraic thinking that can be made accessible to Grade 2 students. Our exploration was conducted within the framework of the theory of objectification and its concept of thinking—i.e., a culturally constituted form of reflection and action situated or materialized in the body, signs, and artifacts. Our exploration was also based on the epistemological view that variables (as well as other algebraic objects) can genuinely be expressed through signs other than the alphanumeric ones of conventional modern algebraic symbolism (Radford, 2010b). This view is fully compatible with the historical development of algebra. Even more importantly, this view makes room for the investigation of non-symbolic forms of algebraic thinking—an endeavour that is of great importance if we are to honour and understand young students' algebraic activity. As shown by our analysis, two basic forms of algebraic thinking arose as the students engaged with the classroom activities. In the first one, elicited in dealing with 'remote' figures, the students encountered the concept of variable in an *intuited* form: the variable was expressed through some of its particular instances as part of an *in-action formula* (e.g., "12 plus 12, plus 1", "50 plus 50, plus 1", etc.). The design of the Tristan problem allowed the students to go a step further and to deal with variables in an *explicit* way, thereby reaching a more sophisticated form of algebraic thinking. Here the calculator proved to be extremely useful. Its usefulness, however, was not limited to producing the numerical answers that were beyond the limited arithmetic knowledge of our students. Its primary usefulness resided instead in the conceptual frame it made available, so that the students could objectify the calculations to be performed and come up with embodied algebraic formulas. The embodied, non-symbolic forms of algebraic thinking evidenced in this paper suggest that early forms of algebraic thinking can appear in 7-8 year old students, and invite us to revisit the genetic relationship between arithmetic and algebraic thinking. Instead of the traditional linear development in which algebra appears as a generalized arithmetic, their ontogenetic relationship seems to be much more complex. Our ongoing longitudinal investigation should enable us to better understand this relationship and begin to document the consolidation of our Grade 2 students' algebraic thinking and its evolution into more sophisticated forms, including the dawn of symbolic algebraic thinking.

References

Ainley, J. (2001). Research forum: Early algebra. In *25th PME Conference Proceedings* (V.1, pp. 129-159).

Becker, J., & Rivera, F. (ed.) (2008). Generalization in algebra. *ZDM - The International Journal on Mathematics Education,* 40(1).

Carraher, D., Schliemann, A., & Brizuela, B. (2001). Can young students operate on unknowns? In *24th PME Conference Proceedings* (V.1, pp. 130-140).

Moss, J., & Beatty, R. (2006). Knowledge building and knowledge forum. In *30th PME Conference Proceedings* (Vol. 4, pp. 193-199).

Nemirovsky, R., & Ferrara, F. (2009). Mathematical imagination and embodied cognition. *Educational Studies in Mathematics, 70*(2), 159-174.

Radford, L. (2008). The ethics of being and knowing: Towards a cultural theory of learning. In *Semiotics in mathematics education* (pp. 215-234). Rotterdam: Sense Publishers.

Radford, L. (2009). *The eye as a theoretician*. Laurentian University, Pre-print Series, 1 2009 (www.laurentian.ca/educ/lradford/).

Radford, L. (2010a). Layers of generality and types of generalization in pattern activities. *PNA - Pensamiento Numérico Avanzado*, 4(2) (in press).

Radford, L. (2010b). Algebraic thinking from a cultural semiotic perspective. *Research in Mathematics Education*, 12(1), 1-19.