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Feeling the Slope with Augmented Reality Technology: Movements of Consciousness in the Learning of the Derivative

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Abstract
In this article, we explore how the concept of derivative is learned through a joint activity involving a tenth-grade student, an instructor, and an augmented reality application. This application, Touch the Derivative, allows users to trace a function graph with their hands and simultaneously displays the derivative function graph as they move their hands on the graph. This study is guided by the theory of knowledge objectification, which considers learning as a social, reflexive, and creative meaning-making dialectical process. The joint activity was qualitatively analyzed to answer the two research questions about the meaning of the derivative concept appearing in the joint activity and the role of contradictions. Focusing on the creative, embodied, and materialist process of becoming conscious of the function–derivative mathematical relations, in the results section we discuss the tenth-grader student's movement of consciousness and the emerging contradictions outlining the semiotic means involved. The meaning-making process of the student progressed through four interrelated layers of consciousness, which evolved dynamically through the contradictions arising as a movement beyond the opposite perspectives of the student and the instructor. We conclude the article by emphasizing the need to understand the pedagogical potential of augmented reality in terms of the activity in which it is used.

Keywords Augmented reality · Consciousness · Contradictions · Derivative · Function · Joint activity · Semiotic

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Introduction

The possibilities of overlaying virtual objects to the real world, and of interacting with them in real-time (Azuma, 1997) open new frontiers for the design of meaningful learning experiences. We consider the augmented reality (AR) artifact *Touch the derivative* (TtD) (Abu Asbe & Swidan, 2023) to fall into this stream of meaningfulness, in the sense that it has the potential to integrate the AR's affordances in the learning process. Indeed, TtD allows one to bodily interact with a given function graph; while moving a hand along the function's graphical representation, the tangent line appears augmenting one's hand and the derivative graph is created point by point simultaneously with the hand's movement. The users can feel with their own body how the slope of the tangent changes, both locally and globally, and elaborate on its relation to the derivative function graph, given the availability of semiotic resources at play in such an augmented environment.

Mathematical concepts, such as those of function and derivative, are well-acknowledged as challenging for students in mathematics education (e.g., Orhun, 2012; Rodríguez-Nieto et al., 2023; Thompson & Harel, 2021). Many students struggle connecting a function with its derivative, because they often draw the derivative in a way that looks very similar to the original function, without taking into account the essential properties of the function over specific intervals (e.g., Nemirovsky & Rubin, 1992). Additionally, students who privilege a procedural approach to mathematics can solve tasks involving symbolic representations, but struggle to interpret and sketch graphical representations of a derivative given the function or vice versa (e.g., Ikram et al., 2020).

Yet, the AR TtD artifact, as any AR pedagogical application, is insufficient to guarantee meaningful learning. It requires to be used in a suitable activity where teachers and students work together. Embedded in a suitable activity, the design of TtD may support a conceptual understanding of the derivative concept and specifically its *geometrical interpretation*, i.e., the derivative of a function at a point represents the slope of the tangent line to the function's graph at that point. In school practice, the derivative concept is usually introduced with the *analytical definition* of the limit of the average rate of the function. Both approaches are culturally shared within the community of mathematicians, but they are challenging for the students.

The complexity of a mathematical concept like the derivative function relation also emerges when looking at the historical evolution of this concept. Indeed, even though some applications of the derivative concept to solve extrema problems were already known in the 1660 s, it was only in the later part of that century that Newton and Leibniz independently developed a geometrical and analytical approach, respectively, to the derivative concept (Grabiner, 1983). Newton referred to the derivative as a fluxion, signifying a rate of change, while Leibniz viewed the derivative as a ratio of infinitesimal differences and called it the differential quotient. However, the derivative concept was formalized only one century later with further contributions by Euler and Lagrange. Indeed, in 1797,

Lagrange introduced in the polynomial expansion of a function a new term for the coefficient of the linear term that he referred to as the first derived function. After Lagrange, the derivative of a function was perceived as a function as well (Grabiner, 1983).

This short, and definitely non-exhaustive, glimpse into the historical evolution of this concept aims to invite the reader to bear in mind the richness and the complexity of elements interlacing in the learning activity presented in this contribution.[1] We will comment on a student approaching for the first time the derivative concept by means of the TtD AR application while working with an instructor. Through their mutual contributions and understandings, and the AR application, all of them shape the manner in which knowledge about the tangent and derivative is materialized. This collaborative and generative process constitutes what we refer to as *joint activity*. In such a joint activity, the voices of the learner and the instructor meet the cultural–historical ones of TtD, as together they endeavor to make meaning of the derivative concept and the derivative function relation. Such a qualitative analysis will eventually lead us to share reflections about the possibilities offered by such an augmented reality-based activity.

Theoretical Framework

This study is guided by the theory of knowledge objectification (TO) (Radford, 2021), in which *knowledge* is conceptualized as a system of forms of thinking, doing, languaging, perceiving, and symbolizing. In this view, knowledge is not considered as a psychological entity: it is considered as a cultural–historical entity; that is, an entity that has been culturally and historically constituted, and that ubiquitously shapes our understanding of, and participation in, the world. In the TO, which is inspired by neo-Hegelian philosophy (e.g., Russon, 2004), knowledge is dynamic, continuously changing; knowledge is the ideational counterpart of human cultural and historical activity and necessarily embodies and refracts the tensions of its society.

Knowledge is a dialectical system; its mode of existence resides in the continuous dialectics between *form* and *content*. Whereas form refers to a reflective general category of characteristics, content represents a singular ideal form as it becomes materialized in a specific context. By appearing in concrete activity, the content of form shows what this content *is* (its positiveness, that is, the identical relation to itself) and what it *is not* (its negativeness, that is, what is different from itself). Thus, in its positiveness, a circle appears as something that is, and that by being something, is not something else, like a triangle). In other words, *Otherness* is organically constitutive of what *Is*.

The dialectics of content and form means that knowledge is simultaneously *abstract* and *concrete*; knowledge *dis*-closes itself in the dialectics between an ideal

[1] Please note that our intention is not to offer a learning design where ontogenesis repeats the path of phylogenesis or historical development. We resort to history as a means to understand better the epistemological complexity of mathematical concepts (Furinghetti & Radford, 2008).

form and the form taking shape in human activity. Thus, in a figural sequence, F1, F2, F3, … when a student tries to find a functional relationship between the number of a figure and the number of objects in the figure (say circles), we see the material corporal actions of perceiving, pointing, and naming the objects ("Fig. 1, three circles; Fig. 2, five circles; …")—the concrete—1and the reflective general cultural category of identities and differences of positiveness and negativeness (e.g., in other sequences, figures may still similarly increase but in a different way)—the abstract.

For a student, the *dis*-closing of the cultural way of finding a functional relationship may not be clear from the start. A process of learning might need to occur. In the TO, the process of learning is envisioned as an encounter; in our example, the encounter with cultural–historical knowledge; knowledge about functions. As mentioned above, this knowledge is not a subjective entity which would arise out of the learner's actions (as constructivists proposed). Knowledge is not a subjective construction, but a cultural–historical entity. Knowing how to notice, find, and express functional relations is already there, in the learner's culture. Knowledge appears as

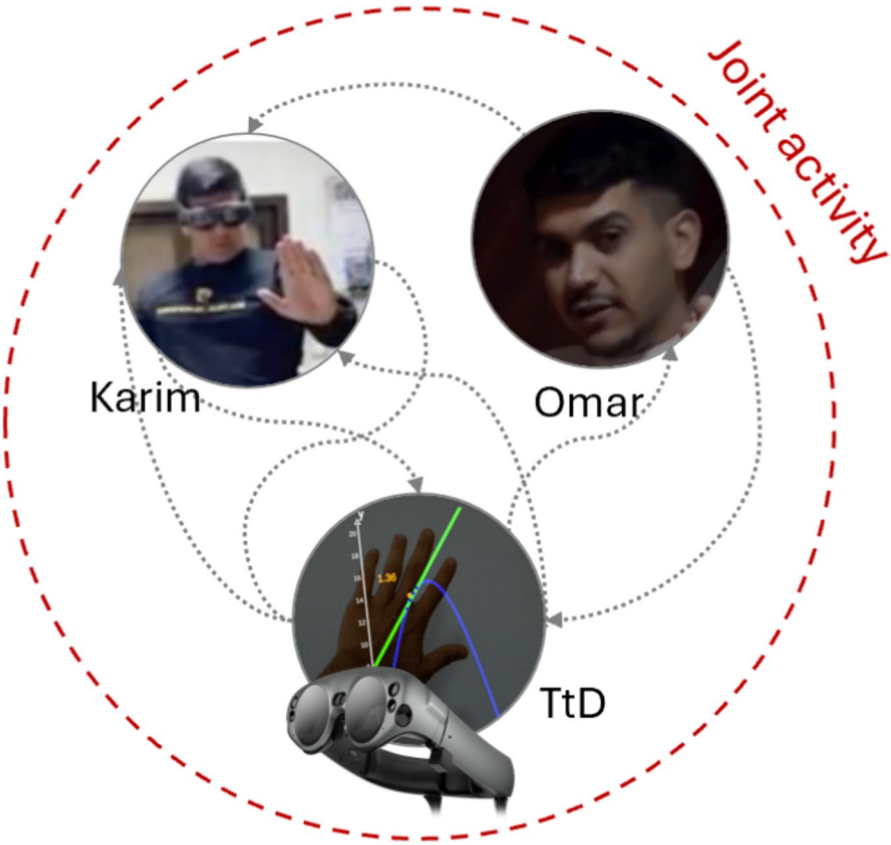

Fig. 1 The analyzed joint activity in which the voices of the student, Karim, the instructor, Omar, and the AR artifact interlace to make sense of the function-derivative relation

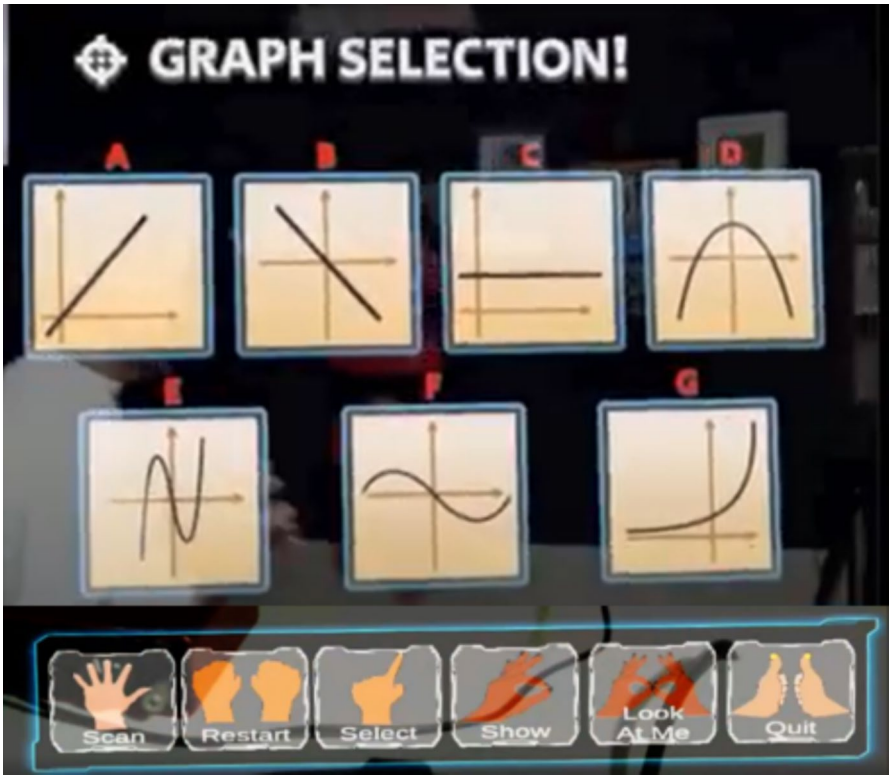

Fig. 2 The first screen shown by the TtD application

something that the learner is not, as something that, being different, presents itself as Other, as something that *objects* the learner. This encounter with what objects the learner is what in the TO is called a process of *objectification*. It is a process of encountering cultural intelligence; a process that involves making sense of the world and producing new meanings. As we strive to understand something, we challenge ourselves, change, and *become aware* of things we were unaware of before. It happens not in a solitary activity, but in a social, embodied, and material way. It happens in human activity; this activity also shapes the humans themselves.

In the dialectics between form and content that human activity realizes, mathematics appears. For mathematics resides in the activity of its enactment, and this activity is the theoretical category through which how learning occurs is explained in the TO. Artifacts are considered as key elements of activity. Sociocultural research, in particular the one inspired by Vygotsky's (1978) work, has shown the important role played by artifacts (Bartolini Bussi & Mariotti, 2008). The TO follows this line of theorizing by recognizing the role of artifacts in human cognition. Artifacts are considered as constitutive and consubstantial parts of thinking. We think with and through cultural artifacts. This viewpoint departs from cognitive approaches that conceive artifacts as facilitators, amplifiers, or mere enhancers of cognitive activity.

Artifacts change our cognitive activity and our modes of knowing; they play a *cognitive role* in our ways of knowing, as they are part of the way we think and come to know, but they also play an *ontological* role by being part of the enactment of mathematics, and an *epistemological* one, as they are tools we think with and through. Furthermore, since mathematics appears through the activity of its enactment, artifacts used in activity become part of the *presencing* of mathematics – i.e., the way mathematics appears and is experienced. In other words, artifacts are part of mathematics as a material practice, and, in doing so, they come to play an ontological role in teaching and learning. Artifacts are bearers of *cultural–historical voices* that merge with the voices of teachers and students (Radford, 2012).

According to the TO, learning involves actively producing meanings as the conceptual objects are made available by teaching–learning activity and its artifacts. Processes of objectification are meaning-making processes out of which cultural objects become progressively noticed. Such processes evolve through interrelated *layers of consciousness* (Russon, 2004), and they can be broken down into two main components. The *phenomenological component*, called *noticing*, is the one through which we encounter things and pay attention to them while often experiencing surprise, awe, but also a sense of challenge. This phenomenological component is entrenched in a second component—a *semiotic* one—through which we actively and creatively try to give meaning to the things being noticed. In both components, individuals resort to different semiotic means such as words, symbols, actions with artifacts, rhythmic speech, and gestures available in the universe of the discourse. These means are termed *semiotic means of objectification* (Radford, 2003).

From this theoretical perspective, learning turns out to be a dynamic and situated social, phenomenological, semiotic, and material process occurring in human, material activity (Radford, 2021). In this article, we will focus on the analysis of an activity involving a student, Karim, and an instructor, Omar, as they engage in the meaning-making process of the derivative concept embedded in an AR artifact. We intend to explore how the voices of Karim, Omar, and the AR artifact meet and interlace in a joint activity to make sense of the historical-cultural mathematical knowledge they encounter (Fig. 1).

In this making-sense process, contradictions play a central role. In the theory of objectification, a contradiction is said to occur between two entities when these entities enter in a mutual transformative relation. Contradiction, indeed, is not merely an opposition or an external difference between two entities, A and B, where each remains distinct and unaffected by the other. Rather, their difference—$A \neq B$—actively shapes both entities and forms the basis of the contradiction. Within this framework, contradiction is conceived as a dynamic process that transforms the opposition between A and B into something new, ultimately giving rise to an entity that integrates both (Radford, 2025). As Macherey puts it,

> The contradiction (*Widerspruch*) distinguishes itself from the opposition (*Gegensatz*) in that it is not a fixed relation between distinct and antagonistic terms but the irresistible movement that discovers in each of these elements the truth of the other and thus produces them as moments of a unique process in which they appear as inseparable. (Macherey, 2011, p. 121)

Therefore, the research questions we will answer with this study are:

1) How does the meaning of the derivative concept appear in the joint activity of a student, an instructor, and the TtD artifact?
2) What is the role of contradictions in the meaning-making processes involved?

As previously remarked, the role of the AR artifact in such a learning process is not just that of providing stimulus to the student or facilitating his cognitive processes. We shall thematize it as a distinct voice that embeds the cultural–historical intelligence associated with the concepts of function, tangent, and derivative and assumes several roles throughout the activity that will be presented in detail in what follows.

The AR Artifact Used in the Study

Touch the Derivative (TtD) is an AR artifact designed to enable students' enactment in the meaning-making of the function–derivative relationship (Abu Asbe & Swidan, 2023; Bagossi et al., 2024). TtD offers an interactive experience for students to move their hand along the graph of the function, while such movement generates the derivative function graph in real time.

The TtD has two screens (Fig. 2). The first screen offers seven types of function graphs for students to explore (A) linear increasing; (B) linear decreasing; (C) constant; (D) concave-down quadratic function; (E) cubic; (F) trigonometric; (G) exponential function. At the bottom of this screen, a legend of gestures and their respective roles (Fig. 2, lower part) are shown; they are available gestures for interacting with the AR artifact interface.

On the second screen, students can explore the relationship between a function and its derivative graph. Once a function graph is selected from the first screen, a Cartesian system appears, displaying the chosen function graph. The graph is represented by a series of blue points connected by short blue segments. When the students move their hand along the graph, as they touch the blue points, the corresponding green point of the derivative graph is generated (Fig. 3a). To create the derivative function graph, students are asked to move their hand along the function graph from left to right (as a convention), keeping it as close to the graph as possible. As the students move their hand along the graph, their hand is augmented by means of a green straight line appearing along it and indicating a tangent line (Fig. 3a). The derivative function graph is created point by point in green as long as the students keep their hand close to the graph. When the student's hand is removed from the function graph, the straight line turns red, and the creation of the derivative graph stops (Fig. 3b).

Students can use an open hand, a semi-open hand, or even a finger to surf the graph. What really matters is the touching of the hand with the points on the function graph. Specifically, the application TtD detects such contact by calculating the mean distance between two points located on the user's index. As a result, the tracing of the graph does not necessarily mirror the precise orientation

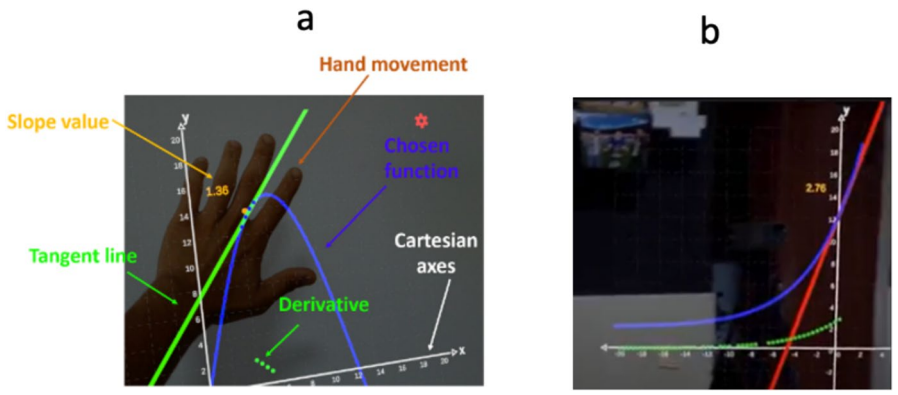

Fig. 3 The second screen of TtD: **a** The tangent and the derivative are created simultaneously with the hand movement along the graph; **b** when the hand is not touching the graph, the tangent turns red, and the creation of the derivative stops

of a tangent. This design choice may be viewed as a limitation, as it can lead to an intuitive mismatch between the user's bodily motion and the mathematical notion of the derivative. However, it may also present a pedagogical opportunity. The teacher might encourage students to examine and refine their hand movements, using any mismatches as a starting point to discuss which types of motion align with tangent behavior and which do not.

In addition to the Cartesian system, graphs, and factual hand movement, the second screen also includes a yellow number juxtaposing the student's hand and signifying the value of the tangent slope (Fig. 3a).

The use of an artifact such as TtD introduces a specific division of labour in the joint activity. In interacting with the TtD, the students are encouraged to make sense of the graphs, points, axes, etc. that the TtD produces. Since the students are wearing a headset, the teacher or the instructor cannot have direct access to what the students are seeing. This issue can be overcome by streaming what the students are seeing on an external monitor. In this way, the instructor can see a projection very close to the vision of the students. This configuration makes it inevitable that the instructor cannot directly intervene in the students' activity but he can guide the students, not only in using the TtD itself (how to move the hand, how to have a clear picture of all the elements displayed), but also in directing attention to the relevant elements that should be noticed and connected, and in reminding the goal of the task. Such a task, delivered orally by the instructor, consisted of coming up with conjectures about the relationships among the blue graph, the green graph, the student's hand movement, and the yellow number. When the student interacts with the function graph, the cultural–historical voices of the TtD become present through its various roles (cognitive, epistemological, and ontological) in the joint activity.

Cognitive Role

The cognitive role of TtD lies in how it mediates and transforms students' engagement with mathematical ideas through embodied, perceptual, and exploratory activity. Cognition is not limited to internal mental processes but is conceived as socially and materially mediated (Radford, 2012). In this view, TtD becomes more than a support tool; it becomes part of thinking, creating new possibilities for conceptual engagement through designed interaction.

The TtD was intentionally built around two core design principles that structure how students can think through action. First, TtD presents the function graph, the derivative graph, and the student's hand movement simultaneously, allowing for real-time visual and kinesthetic comparison. This layout may enable students to generate conjectures about the relationship between a function and its derivative by observing how the derivative graphs co-evolve with their own gestures. Second, by integrating bodily movement into the task, TtD allows students to "feel" the slope behavior, encouraging an embodied sense of how the function changes, not just seeing or calculating the slope, but physically enacting it.

The artifact deepens this embodied cognition by embedding semiotic cues directly into the interaction. The appearance of a green tangent line—visible only when the hand is close enough to the graph—reinforces the idea of tangency as contact at a point, while also introducing slope as both a local concept (slope at a specific point) and a global one (the derivative as a function). In this way, students are invited to co-ordinate gestures, visual feedback, and spatial reasoning in a task that prompts active exploration rather than passive reception.

The TtD's explorative structure further amplifies its cognitive potential. Students can interact with seven carefully chosen graphs (Fig. 2), each designed to highlight particular mathematical features; the constancy and directionality of slope in linear graphs (A–C), points of inflection and extrema in curves (D–F), and the smooth variation of slope in an exponential function (G). These selected experiences may encourage students to engage in iterative reasoning, i.e., formulating, testing, and refining their conjectures by moving back and forth across different scenarios.

Epistemological Role

The epistemological role of TtD lies in the way it reconfigures the nature of mathematical knowing, moving away from traditional, abstract, symbolic understandings toward a more embodied, enacted, and mediated form of meaning-making. It stresses the idea that TtD and artifacts in general are not merely about what is known, but about how knowledge itself is formed and legitimized through interaction with artifacts (Radford, 2012). In this case, TtD does not simply support the student in accessing cultural knowledge about derivatives; rather, it becomes a constitutive part of the knowing process and the manner in which it unfolds.

"Artifacts become imbricated in the way we think and come to know" and "knowing becomes knowing-with-tools" (Radford, 2012, p. 286). The epistemological shift introduced by TtD is exactly that: the student comes to know the derivative function not through traditional symbolic manipulation, but through a multimodal interaction involving gesture, vision, motion, and feedback. The concept of the derivative is no longer something merely abstract to be understood, but something that is performed, enacted, and experienced.

Furthermore, the artifact embodies historically sedimented mathematical ideas, such as the concept of tangent, and presents them in a dynamic, real-time visual form. This aligns with Radford's notion that artifacts carry "historical voices" that participate in classroom meaning-making. Thus, TtD's epistemological role is also to bring those cultural and intellectual histories into interaction with the learner's present activity.

Another crucial point is that this interaction with TtD shifts the criteria for what counts as understanding. In this mediated context, understanding the derivative is no longer demonstrated through algebraic derivation, but through the ability to navigate the graph, co-ordinate hand motion with visual slope feedback, and interpret the emergence of the derivative graph. In this way, TtD transforms not just how the user learns, but what it means to "know" the derivative.

Ontological Role

The ontological role of TtD concerns how the artifact participates in constituting mathematics itself as a lived, enacted, and situated activity, rather than as a fixed body of abstract truths. According to Radford (2012), mathematics does not reside solely in theorems, formulas, or symbols; instead, it "resides in the activity of its enactment" (p. 287). Artifacts like TtD are not just tools for representing or accessing mathematical knowledge; they are integral to the very being of mathematics as it is performed in the classroom. In this sense, TtD plays an ontological role by co-creating the space where mathematical objects—like the derivative function—come into existence. The derivative, in this context, is not presented as a pre-defined formula waiting to be discovered, but rather as something that emerges through the co-ordinated, embodied interaction between the student, the instructor, and the AR artifact.

The artifact mediates a process in which mathematics is performed—much like music exists in performance rather than on sheet music. As the user moves his hand along the graph and sees the tangent and the derivative curve emerge, the mathematical object (the derivative) is brought into being through activity. Thus, mathematics happens in the moment, and TtD is a co-participant in that happening. Moreover, TtD does not just shape the appearance of mathematics but redefines what mathematics is ontologically in this learning context. It becomes a collective, multimodal event involving the student's body, gestures, the augmented reality interface, and the historical concepts embedded in the TtD. This fits with Radford's idea that mathematics is not a solitary, purely mental pursuit but a "polyphonic" event—a confluence of teacher, student, and artifact voices.

Method

The Learner and the Learning Session

In this study, we aim to investigate the ways in which the meaning of the derivative concept appears in the joint activity of a student, an instructor, and the AR application. In particular, we aim to investigate the role of contradictions in the involved objectification meaning-making processes.

To achieve this, we conducted a case study focusing on the interaction of a 10th grader (15-year-old) Israeli student named Karim, an instructor, Omar, and TtD. Karim was asked to create multiple function graphs using TtD and come up with conjectures about the relationships between the blue graph and the green points. Omar contributed to the joint activity by giving instructions on how to use the application and helped Karim when he got stuck by requesting specific actions, and asking for explanations or interpretations of what the student was doing or seeing.

By the time of the learning session, Karim, who studied mathematics according to Israel's Ministry of Education's high-level curriculum, had already learned the linear and quadratic functions, as well as how to read graphs in the Cartesian system. He was familiar with the slope concept in relation to linear functions but not yet with the derivative concept. The instructor was present throughout the entire learning session. Such session lasted for approximately 1 h and 10 min and it took place in an informal setting.

Data Collection and Data Analysis

All the activity was video recorded with a camera focused on Karim. What Karim was seeing through his headset was streamed on a wider screen so that the instructor could always track what Karim was seeing. The stream was also recorded. The two video clips of the session, the one focused on Karim and the one showing what he sees through his headset, were synchronized so that they could be watched simultaneously. The video was then transcribed and translated from Arabic into English by the first author.

In accordance with our theoretical assumptions, we analyzed the joint activity of the instructor, student, and TtD artifact in three rounds. In the first round, we watched the videos multiple times and we conducted a multimodal semiotic analysis of the learning session following the methodological paradigm outlined by Radford and Sabena (2015). The multimodal semiotic analysis focuses on signs, language, artifacts, and the body (such as gestures, actions, and perceptual activity with the TtD) in the processes of objectification of the function–derivative relationship. This analysis allows us to examine the instructor and student's joint process

of meaning-making, as they interact with the TtD and how the cultural–historical knowledge of derivatives is manifested.

In the second round, we analyzed the video and transcript diachronically. Elaborating on the multimodal semiotic analysis of the first round, we identified four intertwined layers of consciousness (Hegel, 1977, as re-interpreted in neo-Hegelian terms, e.g., Hyppolite, 1974; Russon, 2004; Stanguennec, 1997): (a) *Sense-certainty*: the most immediate form of consciousness, where knowledge is based on direct sensory experience: "I feel this." For example, when Karim said "Every point I touched appeared here" while pointing to those elements, this was coded as sense-certainty because the awareness is focused on direct sensory action and its impact; (b) *Perception*: it refers to our dealing with objects in terms of perceptual similarities and differences, moving beyond mere sensory input: the negation of "feeling this." Karim claiming that "The green points are the same like the blue line but on the x-axis" was classified as perception because it indicates the recognition of the two objects, green points and blue line, and the categorization of them based on their similarities and differences; (c) *Understanding*: based on the dialectics of content and form, understanding refers to our attempt at making meaning of what appears in front of us. The conjecture elaborated by Karim "this number is the inclination of the hand" was coded as understanding because it shows the awareness of the relationship between the tangent value and the hand inclination; (d) *Cultural-theoretical thinking*: continuing the dialectics of content and form, cultural–theoretical thinking refers to the dialectical (and hence transformative) movement of personal and cultural meanings – an often contradictory and unresolvable movement through which we seek to understand the world. Theoretically oriented consciousness, in contrast, not only involves an understanding of the underlying relationships governing the objects but also requires theoretical justification and inquiry. Therefore, utterances and gestures like "as I move in this direction [traces a concave down parabola from the left until the maximum point] the number decreases" were coded as cultural–theoretical thinking consciousness. Indeed, here the learner engages in inquiry processes, performs experiments, observes results, and draws conclusions which can be interpreted by mathematicians as theoretical ideas (e.g., in this episode the second derivative test: when a function is concave down, the slope of the tangent line to the curve decreases as moving from left to right). Eventually, the video was split into episodes according to the prevailing layer of consciousness.

In the third round of analysis, we focused on the analysis of contradictions within the episodes identified during the second round.

Results

In this section, we present the multimodal semiotic analysis of four episodes of the learning session showing the four aforementioned layers of consciousness and highlighting the role of contradictions throughout such episodes of the joint activity. In reporting the transcripts, we will refer to the original line numbering of the entire learning session. The activity was often characterized by long pauses, which we will refer to as [..n..], where n denotes the seconds of silence.

Sense-Certainty Consciousness of the Function–Derivative Relationship

At the beginning of the session, Omar instructed Karim on how to maintain the tangent line green. Karim was not precise in maintaining the line green continuously and Omar asked him to ensure that the line stayed green. Such indication supported Karim in producing the tangent line and endowing it with meaning as a line that touches the curve at one point.

Karim selected the increasing linear graph (A) and moved along it slowly (Fig. 4a and 4c). After a few seconds, the line turned red (Fig. 4b), indicating that Karim's hand was far from the function graph.

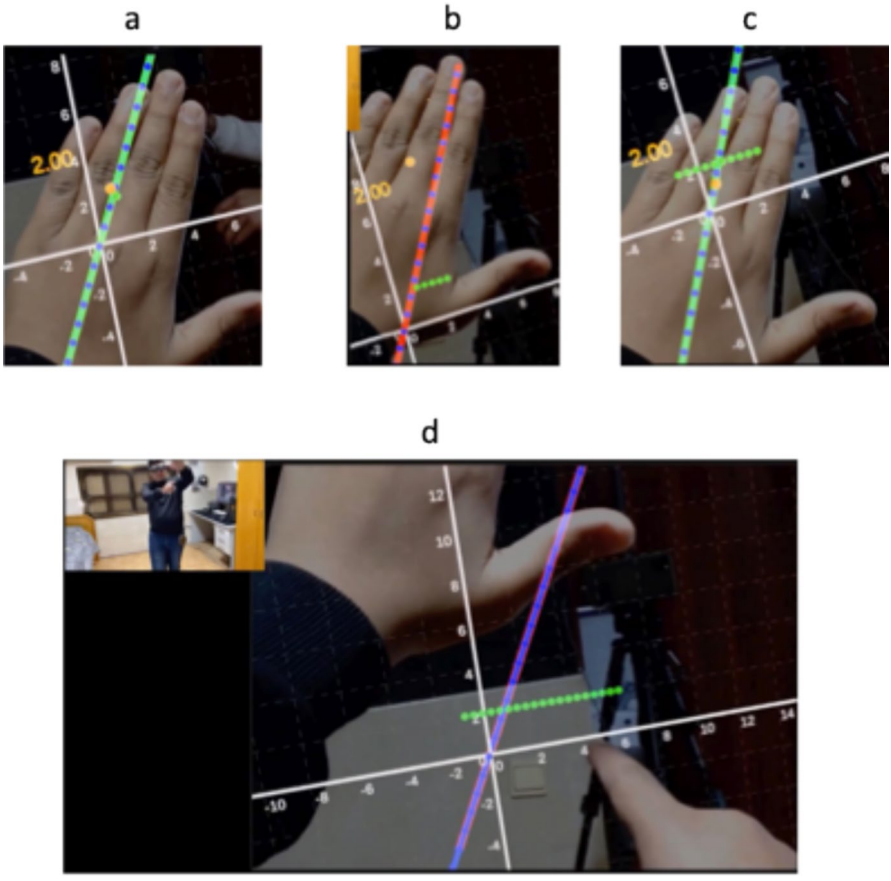

Fig. 4 **a** Karim touched the function graph as he moved along it; **b** the red tangent line indicated that Karim had moved far from the graph; **c** Karim moved along the function graph, creating the derivative; **d** Karim pointed to the derivative graph with his right hand while moving along the function graph with his left hand

44	0:02:37	Omar	Try to keep the color green [..20.., *Karim moves his hand along the function graph*] What do you see?
45	0:03:01	Karim	I see the function, and the points I touched appear here [Fig. 4d]

Omar noticed that Karim's hand was moving far from the function graph and asked him to keep his hand close to it. Karim complied with Omar's indication, and, as a result, the line changed from red to green. This suggested that Karim distinguished between two situations: when the line is green (the hand touches the graph) and when it is red (the hand does not touch the graph). Karim then spent 20 s moving his hand along the function graph, observing the creation of green dots [44]. After this process, Karim demonstrated sense-certainty consciousness as he produced a tentative meaning to the relationship between the two graphs: "The points I touched appear here" [45], indicating the line on which they appeared with his finger (Fig. 4d).

Two minutes later, Karim repeated the experiment moving his hand along the same linear function graph (graph A). This time, Karim's hand kept the tangent line green. This action suggests that Karim is becoming conscious that the straight line should touch the function graph to be green.

57	0:05:01	Karim	[*Karim moves his hand along the function graph and observes the creation of green dots,* Fig. 5a] Every point I touched, appeared here [*points to the green dots,* Fig. 5b]
58	0:05:18	Omar	[..13..] There is a number here, what does it mean?
59	0:05:21	Karim	2 [*the yellow number in* Fig. 5a]
60	0:05:49	Omar	Karim, go back and tell me what you see
61	0:05:57	Karim	The entire function [*makes a circular movement with his hand,* Fig. 5c], and the line is straight [*moves diagonally along the function graph with his finger,* Fig. 5d]. The points I was going through [*on the function graph*], each point I went through appeared on this straight line [*moves horizontally along the derivative graph with his finger,* Fig. 5e]
62	0:06:13	Omar	Ah ha
63	0:06:16	Karim	There is a number 2.00

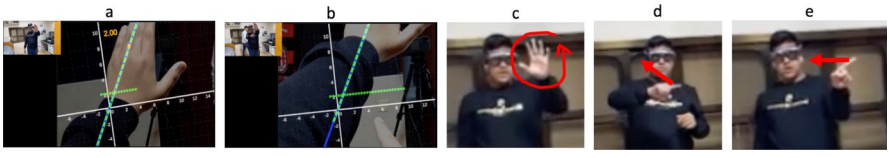

Fig. 5 **a** Karim moved his hand along the function graph and observed the creation of green dots; **b** pointed to the green dots; **c** made a circular movement with his hand; **d** moved diagonally along the function graph with his finger; **e** moved horizontally along the derivative graph by his finger

Even when Karim repeated the same actions as he did in the previous episode, the meaning he ascribed to the green dots remained within the sense-certainty consciousness. Omar drew Karim's attention to the yellow number (Fig. 5a), which signifies the tangent slope value. Omar seemed to want to draw Karim's attention to the connection between the function and the green points by using the tangent slope value. However, Karim's response did not align with Omar's intent and so an opposition between two perspectives—semiotic and phenomenological—emerges.

To move beyond this opposition, Omar asked Karim to step back from the AR screen, potentially to help him better see the details shown on the AR screen. At this point, Karim noticed the function graph, the dots on the graph he went through with his hand movement, the emergence of the green dots corresponding to the touching of his hand and appearing on a straight line [61], and the value of the yellow number [63]. While Karim focused his attention on the blue points on the function graph, he used the metaphorical expressions "going through" and "Each point I went through" [61]. These utterances echo the dynamicity of his interaction with AR through a perceptual action, suggesting that Karim is still in the layer of sense-certainty.

The interaction mentioned above must be understood dialectically: Karim's response and understanding must be understood not as separated from the instructor's question and the information displayed by the TtD (as if these were mere external supports of Karim's ongoing subjective experience), but as made up of the instructor's voice (who asks Karim to discursively formulate what he is seeing) and the Ttd's voice that responds to Karim in its own Cartesian language. In other words, the way of interacting with Omar and the TtD shapes and gives content to the voice of Karim: after interacting with TtD, he uses metaphorical expressions built on the idea of movement to think about what is happening in joint activity.

This dialectical interaction helped Karim notice a relationship between the function graph and the green points. Karim noticed that each point on the blue function graph appeared on the function graph made of green points: "Each point I went through appeared on this straight line" [61]. Of course, between the mathematical culturally appropriated expression of the tangent idea and Karim's sensuous experience of seeing and feeling, there is still a gap. It is precisely the link between the function and its derivative that is at stake here, the link to which the instructor is trying to call Karim's attention, but Karim only described what he saw on the screen (sense-certainty). In other words, the opposition between the semiotic perspective of Omar and the phenomenological perspective of Karim is not resolved within the episode but serves as a generative tension that may catalyze conceptual development in subsequent interaction.

Perceiving that Both Functions Have the Same x-Value

The opposition between Omar's and Karim's perspectives, prompted the instructor to inquire directly about the relationship between the function and derivative graphs.

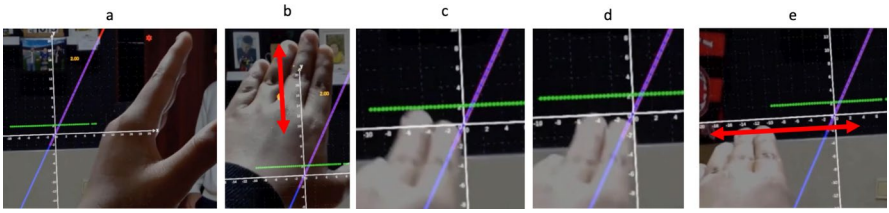

Fig. 6 **a** Karim made a gesture similar in shape to the function graph; **b** moved his hand vertically on the y-axis; **c** pointed to the green points; **d** moved his hand vertically between the two graphs; **e** moved his hand horizontally on the x-axis

69	0:06:24	Omar	Can you tell me if there is a relationship between the blue and the green line? The blue and green points
70	0:06:41	Karim	The blue [*makes a gesture similar to the function graph,* Fig. 6a] points are on [*moves his hand toward the y-axis*] the *y* [*moves his hand vertically on the y-axis,* Fig. 6b], and the green [*points to the green points,* Fig. 6c] are the same [*like the blue – moves his hand vertically between the two graphs,* Fig. 6d] but on *x* [*moves his hand horizontally on the x-axis,* Fig. 6e]

In reacting to Omar's prompt, Karim categorized the function and derivative graphs, highlighting their differences and similarities. Karim's utterances, "the blue points are on the *y*-axis" and "the green (points) is on the *x*-axis," along with his gestures (Fig. 6a, b, and e) indicate his recognition of the green points as a constant function (on the *x*) and the blue line as a non-constant function (on the *y*). In regard to the similarity, when Karim said, "the same (like the blue)" and made a vertical hand movement between the two graphs (Fig. 6d), it indicates his recognition that the two graphs share the same *x*-value.

The awareness that the function and its derivative graphs share the same *x*-value, but different *y*-value moves beyond direct sensory actions and the phenomenological perspective he maintained so far. This moment reflects a transition in Karim's perspective: from a phenomenological engagement with what appears on the interface, toward a structural mapping of how the graphs are aligned in relation to the co-ordinate axes. The dynamic process of moving beyond the opposition between Omar's and Karim's perspectives which leads to the creation of a new viewpoint in which Karim integrates a semiotic perspective into the phenomenological one, is what we call contradiction. In particular, such a contradiction here determines a transition to another layer of consciousness, i.e., perception, for Karim. Although Karim does not yet articulate the functional dependency between the graphs, this structural mapping lays the groundwork for future objectification of the mathematical relationship between a function and its derivative.

Understanding the Relationship Between the Tangent Value and the Hand Movement

Karim's response in [70] prompted the instructor to ask about the meaning of the yellow number [95], which signifies the value of the tangent slope.

95	0:14:30	Omar	And this number? [..3..] Here it was 2, before it was varied

The instructor also differentiates between the slope of a linear function, which is constant "here it was 2," and the tangent slope of a nonlinear function graph "before it was varied." Cognitively speaking, the TtD offers Karim various graphs to experiment with, each one leading to specific contents of the general mathematical form of the derivative. Although the instructor's remarks intended to provide Karim with clues about the sought-after relationship, the question about the meaning of the yellow number posed a challenge for Karim. To resolve this challenge, Karim chooses a linear graph with a decreasing slope (Graph B). Karim adjusted his position and traced the function graph once (Fig. 7a) before moving to another location in the room (Fig. 7b).

118	0:18:07	Omar	I saw you did a nice movement, I saw you did like this [*Omar imitates the hand movement as Karim did,* Fig. 8] while you were working. And before I saw you doing another form
119	0:18:16	Karim	Yes, in order to be able to go through, as I go through the line
120	0:18:25	Omar	Ah ha
121	0:18:30	Karim	I do not know if it is correct, maybe this number is the inclination of the hand
122	0:18:36	Omar	I do not know, maybe yes maybe not, you can say what you want, you can examine another function if you want

In a first moment, Omar pointed out the change in the yellow number's value [95]. After Karim completed the trace of the function graph, the instructor noticed that he had a nice movement and differentiated it from the other hand movements Karim had done so far [118]. It seems that the instructor found in the changes in his hand movement a potential for Karim to recognize the cultural meaning of the yellow number, which here has a negative value, but previously had a positive value. Karim

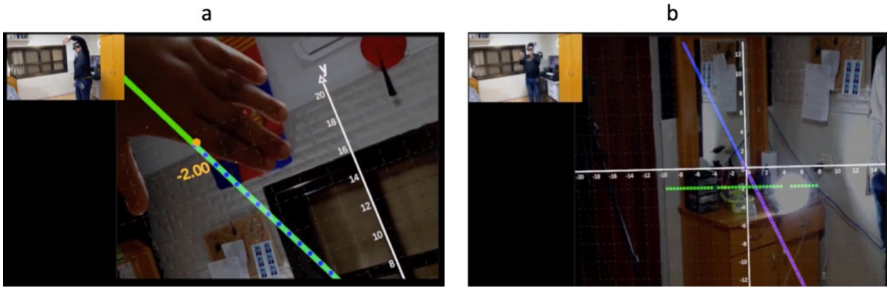

Fig. 7 **a** The initial position of Karim in the room; **b** the second position in the room

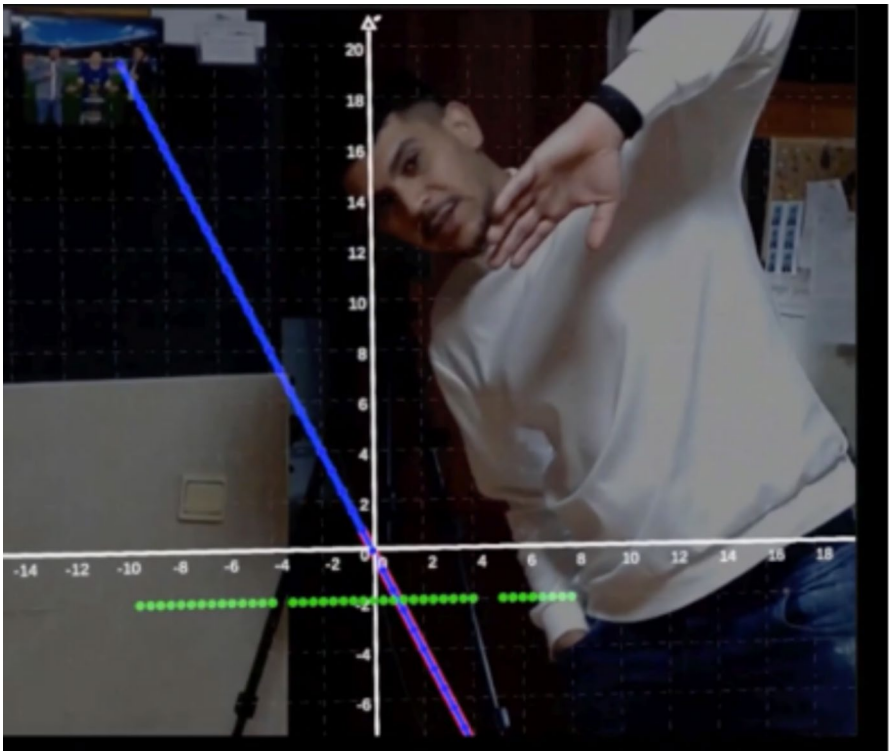

Fig. 8 Omar imitated the hand movement as Karim did

confirmed that he did a different hand movement, but with a different rationale "in order to be able to go through, as I go through the line." Here we can notice an opposition between Omar's perspective, which highlights the relationship between the yellow number and the hand movement by underlining changes in these two elements, and Karim's kinetic perspective which adapts his hand movement to the shape of the graph. The teacher's semiotic perspective, where gesture and number are interpreted as signs of an underlying mathematical relationship, and the student's phenomenological experience, where gesture is primarily a tool for navigation.

Fourteen seconds later, Karim hesitantly linked the yellow number and his hand: "maybe this number is the inclination of the hand" [121]. Such a connection reveals a contradiction, that is a movement beyond Karim's kinetic perspective that now integrates conceptual elements. This utterance marks a transitional moment in which the embodied and symbolic layers of the activity begin to converge. It seems that the instructor was satisfied with this revelation and immediately invited Karim to examine another function. He sustains the contradiction as a generative space for exploration, encouraging further investigation rather than offering a definitive explanation. In doing so, the contradiction is kept alive as a productive force that can support Karim's gradual objectification of the mathematical concept.

Karim chose to explore the constant function graph (graph C). He went along the graph with his hand, and soon after he completed the trace action, he said:

134	0:19:31	Karim	The number is 0.00 [..5..] I expect this number is the inclination of my hand [..5..] the line I go along with my hand, when I go along this line in this way with my hand. [Fig. 9a] [..6..] The number was 0000, the inclination of my hand was constant. [Fig. 9b] [..6..] My hand didn't tilt like this [Fig. 9c] or that. [Fig. 9d]

Karim focused his attention on three elements: keeping his hand touching the function graph, i.e., identically with the tangent idea, the inclination of his hand, and the yellow number. When Karim went along the graph, he made efforts to keep the line green. His attempts to keep the line green suggested that Karim was becoming aware of the meaning of the tangent idea as a line touching the function graph. It seems that Karim's hand inclination fostered the linkage with the yellow number, which helped him to endow it with the meaning "the yellow number is the inclination of my hand." Karim was not only satisfied with connecting his hand inclination and the yellow number, but he also justified this linkage by differentiating his hand inclination: "The inclination of my hand was constant [..6..] my hand didn't tilt like this or that."

This utterance shows that the opposition that was initially between acting and interpreting, is now reorganized by Karim. Rather than simply performing gestures to complete a task, he now reflects on those gestures as expressive of mathematical structure. This transformation illustrates the evolution of an opposition into a contradiction as a movement in consciousness, where different forms of knowing are brought into relation, not collapsed or replaced.

To verify his conjecture, Karim selected the increasing linear function graph (graph A) and soon after, the decreasing linear function graph (graph B).

147	0:20:25	Karim	[*Traces the graph*] When the function is increasing
148	0:20:27	Omar	Okay
149	0:20:29	Karim	And I make my hand this way, in this movement [*gestures with his index finger* Fig. 10b] upward on the function. On the curve. It displays 2.00. I expect that the inclination of my hand was two
150	0:20:46	*Karim selected the decreasing linear function graph and traced it*	

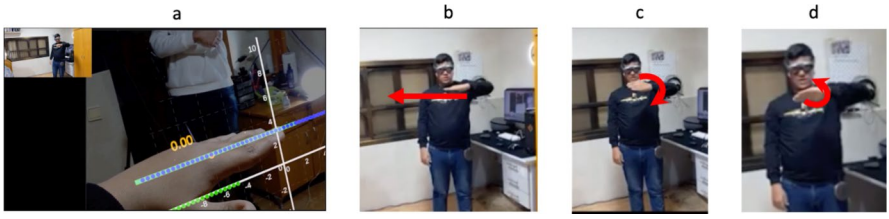

Fig. 9 **a** Karim moved along the constant function graph; **b** gestured a constant slope; **c** and **d** rotated his palm half a turn, first clockwise and then counterclockwise

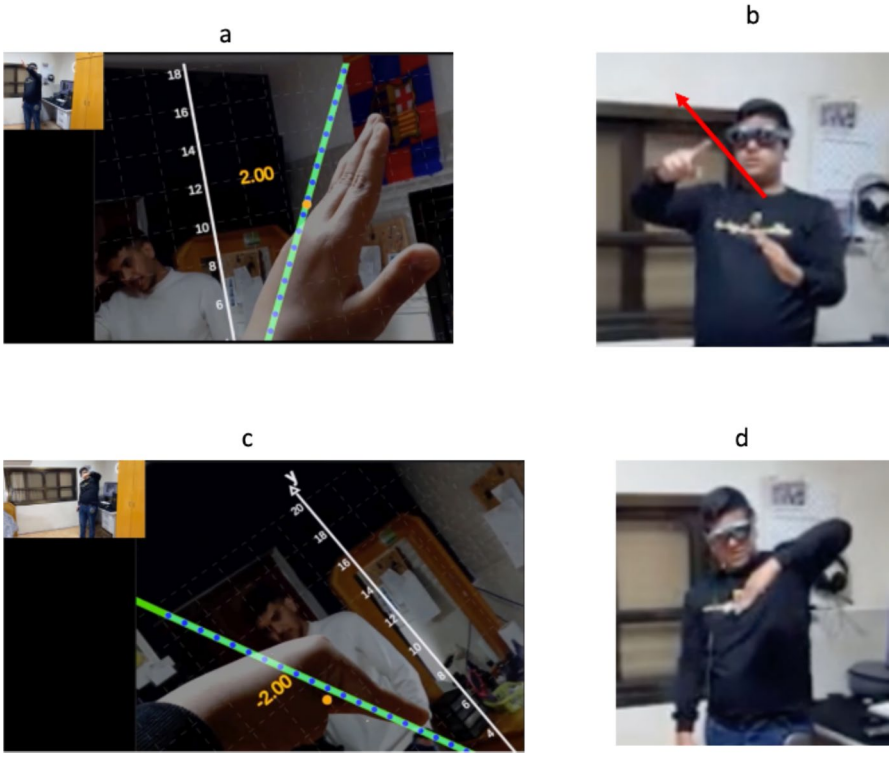

Fig. 10 **a** Karim moved along an increasing linear graph; **b** gestured upward movement; **c** moved along a decreasing function graph; **d** gestured downward movement.

151	0:21:17	Karim	Now when I go along it [*goes along the decreasing linear graph* Fig. 10c], the number becomes –2. Because the inclination of my hand is downward in this way [*gestures with his hand downward* Fig. 10d]. I expect that this is the reason why the number is negative and 2
152	0:21:21	Karim	When I did this movement as I go upward in such inclination the number was positive two. When I did the same movement but downward it becomes negative

In [149] and [151], Karim is becoming aware of the tangent idea by endowing the slope of the linear function with the meaning of hand inclination: this awareness is built by his hand, the spatial words "upward/downward" and the value of the yellow number. Karim's utterances and gestures (Fig. 10b and d) in [152] suggested that he differentiated between the increasing and decreasing linear functions and was aware of the differences and similarities between the two situations. So far, Karim has become aware that two linear graphs have the same slope, "the same movement but downward," while the value of the slope is determined according to the tendency of the graph (increasing or decreasing). Karim's utterances [149, 151, 152] and

gestures (Fig. 10a, b, c) indicate that he moved toward another layer of consciousness – understanding – in that he became aware of the actions and the rule that controls the yellow number.

Cultural–Theoretical Thinking

Throughout the joint activity, Karim has become aware of the historical meaning of the tangent idea as a slope of the curve while investigating linear functions. At this point, the joint activity took a different form. Omar's role changed from guiding Karim toward the historical–cultural meanings embedded in the artifact to observing Karim's inquiry processes. As an inquirer, Karim selected a function graph with multiple extremum points (graph E) and began his exploration from a maximum point.

166	0:22:08	Karim	The first point I touched; it [*the application*] displays 0.00, because this point is straight [Fig. 11a and b]. It displays 00 It is like that function [*the constant function*] which I go along it in a straight way

Karim started his investigation from the maximum point and the yellow number displayed a value of 0. This yellow number led Karim to connect his hand position (held horizontally) with the maximum point of the function, where "the point is straight" in Karim's words. Karim's experience with the constant graph (C) where the yellow number was always 0, "It is like that function which I go along it in a straight way," and what he experiences now with graph E, where the yellow number is 0 only at the maximum point, evolved to a conceptualization of the relationship between the tangent's slope and the yellow number. This moment reveals an opposition in Karim's reasoning: he interprets the locally flat behavior of the parabola at the extremum as being similar to the globally constant behavior of a constant function. In doing so, he generalizes from an embodied and visual experience—horizontal motion and the value 0.00—to a conceptual interpretation that collapses local flatness and global constancy. The slope being zero at a single point does not imply that the function is constant; it only indicates that the rate of change is momentarily zero. This opposition between the localized flatness of a turning point and the

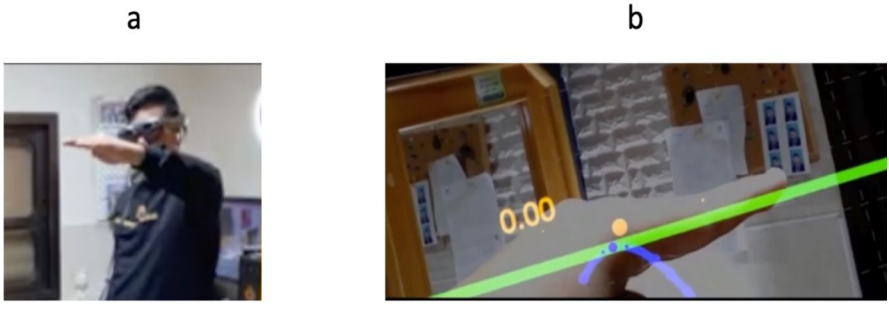

Fig. 11 **a** and **b** Karim moved along the maximum point

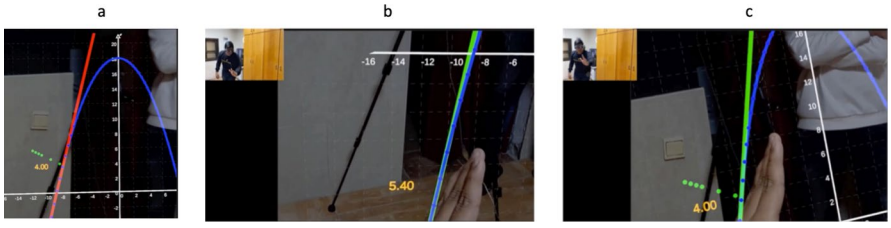

Fig. 12 **a** The graph that Karim selected; **b** Karim traced the graph from left to right; **c** the point where Karim stopped

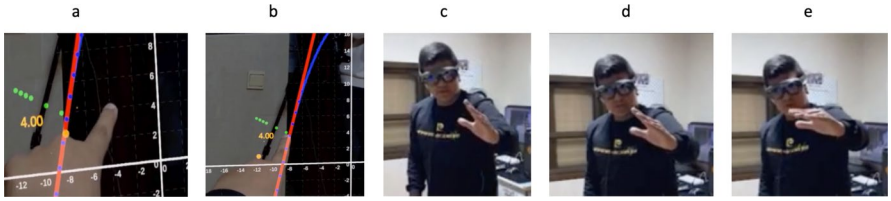

Fig. 13 **a** Karim pointed to the green point where the y-value is 4; **b** pointed to the yellow number, which is 4; **c**, **d**, and **e** traced the parabola from the yellow number until the maximum point, without considering the tangent line

structural constancy of a function may serve as a productive opening for Karim's objectification processes.

Karim's utterance, "it displays 0.00 because this point is straight," suggests that he has moved to the cultural–theoretical thinking layer. Indeed, he became aware of the relationship between the tangent slope and the extremum point, and has justified this relationship using the language of Fermat's theorem. According to the theorem, at a local maximum point of a differentiable function, the slope of the tangent line is zero. Yet, his explanation still reflects the influence of his embodied perspective, and the contradiction between local and global behaviors remains latent, pointing toward a developmental threshold.

Karim continued his investigation and chose a parabola (graph D) (Fig. 12a), tracing it from left to right (Fig. 12b). When the yellow number became 4 (Fig. 12c), Karim stopped his inquiry and said:

180	0:25:50	Karim	In this point [*points to green point,* Fig. 13a] the number is 4.00. I expect the inclination of my hand to be 4.00. And as I move in this direction [*traces the parabola from the yellow number until the maximum point, without considering the tangent line,* Fig. 13c, d, and e] the number decreases

In this excerpt, Karim began his exploration by keeping the tangent line green until the value of the tangent became 4. At this moment, Karim pointed with his index finger to the last green point in the derivative graph (Fig. 13a) and then moved his index finger horizontally to point to the yellow number (Fig. 13b). Karim's utterances and gestures suggested that he noticed the relationship between the green dots

and the tangent slope value. In the second part of [180], Karim traced the left side of the parabola with his hand, showing a family of tangent lines with decreasing slopes (Fig. 13c, d, e). This time, Karim determined that the slope was decreasing based on his hand movement along the concave down curve without referring to the tangent line in the artifact or the yellow numbers. Furthermore, Karim's gestures and utterances [180] suggest that he became aware of the relationship between the concave down curve and the tangent line. This relationship is related to the second derivative test, namely, when a function is concave down, the slope of the tangent line to the curve decreases as moving from left to right.

At the same time, this moment reveals an emerging contradiction between two interpretive frames; Karim's embodied act of tracing the curve as a whole, and the mathematical meaning of slope as a property of a specific tangent line at a point. By describing the slope as "decreasing" through his movement along the curve, Karim is intuitively grasping a second-order property (variation in slope) without explicitly distinguishing between the instantaneous nature of the derivative and the experience of slope gradually changing as one moves along the curve. This contradiction remains implicit and does not disrupt his reasoning, but it indicates a developmental phase where Karim's embodied understanding may be stretched as he begins to coordinate local (pointwise) slope with global view. While his conclusion aligns with the second derivative test, the explanatory path he takes blends motion-based inference with formal structure, suggesting that his current way of knowing is generative, but not yet differentiated into distinct conceptual layers.

Concluding Discussion

The article aims to discuss how a student, an instructor, and an AR application, TtD, work in a joint activity to make meaning of the mathematical concept of derivative. The AR application allows direct interaction with function graphs and the creation of derivative function graphs, appearing in discrete form, as a sequence of points close to each other, on the same virtual screen. The study shows that the student became aware of the historical–cultural meaning of the derivative as a function corresponding to any x-value the slope of the function's tangent at the same x-value. The meaning-making process progresses through four interrelated layers of consciousness (Hyppolite, 1974), sense-certainty, perception, understanding, and cultural–theoretical thinking, through which Karim moved in a gradual way. The analysis also demonstrates that these movements of consciousness, which characterized the meaning-making process in the joint activity with the instructor and the AR artifact, evolved through a dialectical and materialist process.

Embedded in a suitable joint activity, the specific design of TtD, as an example of an AR artifact, has the potential to manifest the historical evolution of the derivative (going from the tangent, to the relationship between the tangent and the extreme point, and the derivative as a function), and can assist the learners in making sense of these ideas and the relationships among them. The change in color of the tangent line in the TtD artifact supports the creation of a dialectical process of demonstrating what the tangent is (its positiveness) and what is not

tangent (its negativeness). Through this dialectical process, in our example, the student became aware of the meaning of the tangent via direct embodied actions, such as moving along the function graph to keep the line green. The student's effort to maintain the green line suggests that the student is becoming aware of the tangent idea. This awareness is crucial for forming the abstract–concrete unity of the tangent idea, as genuine understanding involves a reflective consciousness of abstract ideas in their relation to concrete reality (Radford, 2021, 2024).

The TtD's responses to the student's actions (e.g., the change in line color) shape the way the mathematics appears in joint activity and how the student becomes progressively conscious of the meaning of the tangent line, thereby showing its cognitive, epistemological, and ontological roles. Also, the way in which the tangent line augments the student's hand significantly contributes to how the student thinks and comes to know. The instructor invites the student to reflect on and adjust his movements, using the discrepancies as a basis for discussion about what kinds of hand motion resemble tangent behavior and what kinds do not [e.g., 118]. In addition, the hand movement played a central role in the process of objectification of the relationship between the extremum point and the value zero of the tangent.

This process included a dialectical process of comparing similarities and contrasting differences—the dialectics between form and content. The student compared the horizontal position of the hand and the value zero of the tangent slope in two different cases: in the first case, when the function was constant, and in the second case, when the function was polynomial and had a maximum point. While Karim engaged in embodied interaction with TtD, he recalled cases previously explored and applied these cases to new scenarios. Hence, this dialectical movement, moving backward and forward, opens paths for the continuation of the processes of objectification and for developing a deep awareness of the cultural–historical ideas embedded in the artifact.

The processes of objectification of the function–derivative relationship begin early during the joint activity by endowing the functional relationship with personal meanings that are distant from the ones accepted in the mathematics culture. The consciousness of the relationship at the beginning is characterized by sense-certainty, the most immediate form of consciousness, where knowledge is based on direct sensory experience. These findings were expected as the AR artifact allows embodied interaction with the graph. However, the personal meanings of the student regarding the functional relationship evolved as the joint activity progressed until he became aware of the cultural meaning of the relationship. The evolution of the meanings is accompanied by the development of the students' consciousness, which takes different forms. While in the beginning, the student was able to reach sense-certainty, as joint activity unfolded, new layers of intelligibility involving the instructor's and the AR artifact's voices became available, leading to other forms of consciousness, for example, understanding and perceiving more theoretically (Radford, 2010). The contradictions emerging from the opposition between the different perspectives of Omar and Karim, semiotic versus phenomenological, and the strategies adopted by the instructor to move forward, were prominent in fostering such movements of consciousness.

The processes of objectification occur in joint activity, the form of which changes continuously as the participants engage with each other. An important transition occurred when the student attained a higher layer of cultural–theoretical thinking. During this phase, the dialectical processes were not driven by the instructor but by the student, who became more autonomous, creating situations to explore and ultimately providing a theoretical explanation of the cultural–historical significance of the function–derivative relationship. Karim's initial embodied meanings of this relationship were transformed as he moved toward a re-embodiment process that carries, in a sublated way, the voices of the instructor and the AR artifact, and the imprint of the joint activity. In this process, the pure singularity of the function–derivative relationship found a generalizing expression whose texture is made up of subjective and cultural–objective meanings.

Throughout the joint activity, contradictions functioned as driving elements in the student's process of meaning-making. These contradictions were not obstacles to the objectification of the function–derivative relationship, but rather productive tensions that prompted the student to reorganize and expand his consciousness. For example, the contradiction between local flatness at a turning point and global constancy in a constant function led Karim to reflect on the nature of slope and its relationship to the behavior of the function. Similarly, the contradiction between his gesture-based reasoning and the formal properties of the derivative surfaced moments where intuitive understandings became insufficient, prompting new forms of reflection. These dialectical tensions, often surfaced by the instructor, created spaces where different perspectives (phenomenological, semiotic, embodied, and symbolic) collided and were gradually reconciled through activity. Thus, contradictions did not merely accompany the learning process; they actively constituted the conditions under which the process of objectification and the emergence of theoretical understanding could occur.

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Data Availability Due to the nature of this research, supporting data is not publicly available due to ethical restrictions, as the participants involved are minors.

Declarations

Ethic Statement The Human Subjects Research Committee of Ben-Gurion University has approved this study.

Competing Interests The authors declare no competing interests.

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