In ZDM – Mathematics Education (2023) 55:611– 626 https://doi.org/10.1007/s11858-022-01459-y

A dialogue between two theoretical perspectives on languages and resource use in mathematics teaching and learning

Luis Radford[1] · Ulises Salinas-Hernández[2] · Ana Isabel Sacristán[3]

Abstract
In this paper, we turn to the notion of networking theories with the aim of contrasting two theoretical mathematics education perspectives inspired by Vygotsky's work, namely, the Theory of Objectification and the Documentational Approach to Didactics. We are interested in comparing/contrasting these theories in accordance with the following three main questions: (a) the role that the theories ascribe to language and resources; (b) the conceptions that the theories bring forward concerning the teacher, and (c) the understandings they offer of the mathematics classroom. In the first part of the paper, some basic concepts of each perspective are presented. The second part includes some episodes from a lesson on the teaching and learning of algebra in a Grade 1 class (6–7-year-old students). The episodes serve as background to carry out, in the third part of the paper, a dialogue between proponents of the theoretical perspectives around the identified main questions. The dialogue shows some theoretical complementarities and differences and reveals, in particular, different conceptions of the teacher and the limits and possibilities that language affords in teaching–learning mathematics.

Keywords Objectification · Resources · Meaning-making · Language · Teaching–learning activity · Culture

1 Introduction

Sociocultural approaches are distinguished from other approaches in the attempts by their proponents to consider the fundamental role played by the social, cultural, and historical dimensions involved in teaching and learning. Following a Vygotskian thread, language and artifacts have received a great deal of attention (Bartolini Bussi & Mariotti, 2008; Lerman, 1996; Sfard, 2008). However, not all sociocultural approaches conceptualize the role of language and artifacts in the same way. In this paper, we explore the theoretical stances adopted in two mathematics education perspectives inspired by Vygotsky's work: the Theory of Objectification (TO) and the Documentational Approach to Didactics (DAD). In the spirit of the theme of this ZDM

issue, crossing languages and cultures is approached by drawing on the research field of networking theories (Bikner-Ahsbahs & Prediger, 2014). We are interested in comparing/contrasting the aforementioned theories in accordance with the following three main questions: (a) the role that the theories ascribe to language and resources; (b) the conceptions that the theories bring forward concerning the teacher, and (c) the understandings they offer of the mathematics classroom. We do so from a holistic perspective: instead of taking these elements separately, we seek to integrate them in relation to classroom mathematical activity. Behind this choice is the idea that language does not operate in isolation. Language is embedded in action, in what people do and say. This holistic approach to *language-resource-teacher-class- room conception* provides us with a window to appreciate what may be termed the 'cultural sensibilities' of a theory; that is to say, those foci and discursive modes that reveal (even if only partially) central features of a theory's general worldview.[1]

[1] Laurentian University, Sudbury, Canada

[2] Eindhoven University of Technology, Eindhoven, The Netherlands

[3] Center for Research and Advanced Studies, Mexico City, Mexico

[1] Theories, by themselves, have no agency. Strictly speaking we should speak about the 'cultural sensibilities' expressed and conveyed by the theory's proponents and practitioners. However, to simplify, in what follows, we will refer to theories in an agentic sense, for exam- ple as carriers of sensibilities and other features, asking the reader to

Grounded in classroom episodes, this paper is primarily conceptual in nature. In the first part, some basic concepts of each perspective are presented. The second part includes some episodes from a mathematics lesson in a primary school. In the third part, a dialogue between proponents of the theoretical perspectives is carried out around the three main questions mentioned above.

2 A brief overview of the theories

2.1 The theory of objectification

The TO finds its inspiration in Hegel's and Marx's philosophies, the work of L. S. Vygotsky and his collaborators, and Freire's concept of education (Radford, 2021a). The TO departs from student-centered pedagogies. It removes the student from the role that is ascribed within individualist approaches and leads to a redefinition of the student, the teacher, and teaching and learning. Its goal is twofold:

- to offer a precise theoretical conception of learning as a genuine collective agentic cultural-historical process; and
- to explore the practical pedagogical conditions that make genuine collective learning possible.

The TO's research questions revolve around these goals. They include questions such as the following. How is collective learning happening? How can researchers and teachers devise pedagogical actions to foster collective learning and rich forms of knowing and becoming? The TO is more than a lens through which to look at, and interpret, learning phenomena. It is also an invitation to transform pedagogical practices and to transform the classroom into a site where students can encounter different types of cultural knowledge and voices in deep conceptual ways while at the same time creating an experience of collective life, solidarity, plurality, and inclusivity.

To devise a concept of learning as a genuine, collective, agentic, cultural-historical process, learning in the TO is not conceptualized as constructing or as making, or re-inventing knowledge. Constructing, making, and re-inventing are metaphors that position the Self as the origin of cognition. Learning in the TO brings together, on an equal footing, both Self and Other. Learning is conceptualized as an *encounter* with cultural forms of mathematical thinking. This encounter has a transformative sense: in the encounter, the individuals are affected, transformed. This is why learning in the TO is not just about *knowing*, but also about *becoming*. The encounter that underpins learning occurs through processes that happen in teaching–learning activity. Two interrelated processes are distinguished: *processes of objectification* and *processes of subjectification*. Processes of objectification are defined as those social, embodied, dis- cursive, symbolic, and material processes through which the students encounter, notice, and become critically aware of culturally and historically constituted systems of mathematical thinking, reflection, and action. This encounter is sensed as a *difference*: the encounter with something that *objects* (etymologically speaking, something that is set against or that opposes) the individual. It is from the acknowledgement of the Other in this encounter that the theory borrows its name. Objectification is not the making of the Other (which is still a form of assimilation of the Other to the Self) but the encounter with the Other. Processes of subjectification are the processes of the incessant making of the subject, of the continuous creation of a singular (and unique) historical and cultural subject. These processes occur through the ways in which teachers and students position themselves, while at the same time being positioned by others against the always contested backdrop of culture and history.

The encounter with knowledge and the co-positioning of self and others are framed by the goal of mathematics education as envisioned by the TO. This theory posits the goal of mathematics education as a political, societal, historical, and cultural project aimed at the dialectical creation of reflexive and ethical subjects who critically position themselves in historically and culturally constituted mathematical dis- courses and practices, and who ponder new possibilities of action and thinking.

To understand learning and foster transformation along the lines of the aforementioned political, societal, historical, and cultural project, the classroom is investigated in terms of its activity, the *teaching–learning activity*.

Concerning language, the TO resorts to Vygotsky's, Bakhtin's and Voloshinov's work. Language is seen dialogically, hence moving us away from language as communication and stressing instead the agentic aspect of knowing and becoming. In this view, language is a means of both objectification and subjectification, a site of self-exposure, of ethic and aesthetic intersubjective experimental character, where various individuals come together expressing themselves within the limits and possibilities of cultural- historical speech genres.

2.2 The documentational approach to didactics

The DAD arises in the context of the French tradition of didactics of mathematics. It is inspired by Rabardel and Bourmaud's (2003) instrumental approach that seeks to

Footnote 1 (continued)

bear in mind that these features are predicated of their proponents and practitioners.

understand how artifacts are turned into instruments, specifically in the field of technology (Gueudet & Trouche, 2009; Trouche et al., 2020). Whereas the instrumental approach focuses on an individual's use and appropriation of artifacts, "the DAD emphasizes the dialectic nature of the teacher- resource interactions" (Trouche et al., 2020, p. 239). Such interactions "include the design, re-design, or 'design-in- use' practices (where teachers change a document 'in the moment' and according to their instructional needs)" (p. 239) with a "document" defined as resources together with schemes of usage. Thus, the DAD's central objective is to analyze teachers' activity (their "documentation work") through their interactions with resources (including the curriculum) in their daily work (Trouche et al., 2020). In this sense, the teacher develops experience (*savoir-faire*) in carrying out teaching activities with resources.

Regarding the notion of resource, the DAD draws from Adler's proposition to "think of a resource as the verb re-source, to source again or differently" (2000, p. 207). This perspective allows a wide variety of resources to be taken into account. From here on, we use the term resource to refer to both technological artifacts and to everything the teacher uses before, during, and after teaching.

The DAD relies on theoretical constructs such as *didactical situation* (Brousseau, 2005), *scheme* (Vergnaud, 1998), and *mediation* (Vygotsky, 1978). The scheme appears as a central theoretical concept of the approach. Vergnaud (1998) defined *scheme* as: "invariant organization of behavior for a certain class of situations" (p. 229). The organization consists of the following four components: (1) Goals, sub- goals, and expectations of the activity; (2) Rules of action; (3) Operational invariants—mainly *concept-in-action* (to categorize and select information), and *theorems-in-action* (proposition considered true); and (4) Possibilities of inference.

The DAD offers a cognitivist perspective on teaching–learning practice, which, while considering the sociocultural context in which teachers and students find themselves immersed, focuses on the analysis of *personal competences* (e.g., technical or social competences) that are organized in hierarchical systems and develop through life. Vergnaud (1998) pointed out that "these competences rely […] upon relevant categories from which to select the information available, deal with it, and generate from [it] plausible goals, subgoals, actions, and expectations" (p. 228). In the context of competence development, language and symbolic representations such as graphs, diagrams, or symbols, are considered as *mediators* that accompany and make knowledge explicit when it is reproduced in words and symbols used in conventional systems. Vergnaud (1998) highlighted that "explicit mathematics does not have the same cognitive status as implicit mathematics contained in schemes. It is only explicit knowledge that can be discussed, argued, proved or disproved" (p. 231).

Concerning language, we can distinguish two ways in which language is used. One use is for dealing with concepts that the theory is building; for example, language "for naming [as researchers] what the teacher develops from these resources" (Trouche et al., 2020, p. 239). Another one is the cognitive role of language assumed in the theory (e.g., analyzing the way teachers themselves name their resources and how this impacts their own practice and the development of classroom activities). However, language, so far, has not been the focus of interest, nor an exclusive category of analysis in research with the DAD where its interaction with other resources can be seen (language as a resource). Rather, it has been used to shed light on how resources are implemented and integrated (language of the resources).

Concerning culture, the DAD seeks to specify which resources have the potential to "re-source" teachers' activity and which are already integrated into teachers' activity. In this context, classroom analysis can be placed around the use of resources, related to the interactions that occur between students and teachers with the different resources and their implications both in the development of activities and in the fulfillment of previously established objectives.

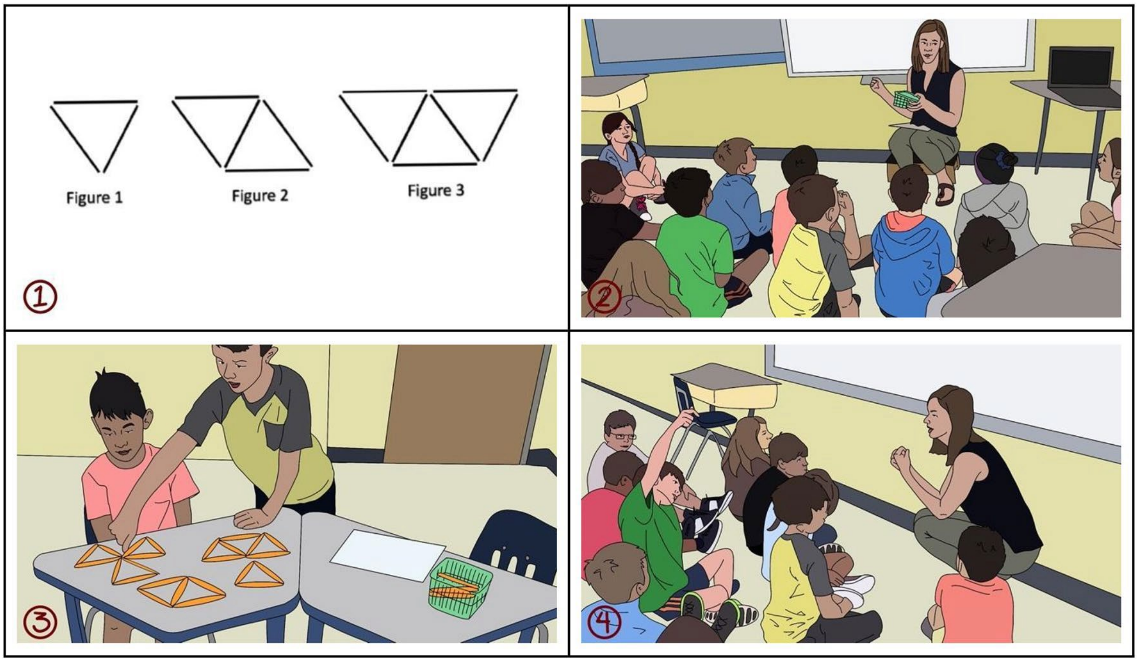
Furthermore, the classroom is the space that allows the development of the teachers' professional activity, which in turn is related to the institutional possibilities and constraints.

3 The mathematics lesson

This section includes some episodes from a Grade 1 mathematics lesson, which serve as background to carry out, in the next section, a dialogue between proponents of the theoretical perspectives.

From a methodological viewpoint, the lesson was selected by the authors from a pool of available recorded lessons, bearing in mind that the lesson should allow discussions about theoretical contrasts concerning the roles of language, resources, and the teacher, as well as conceptions of the mathematics classroom. The selected lesson was designed by three teachers and a team of researchers, based on the curricular exigencies of the school board and its timetable. Working on a full transcript of the lesson and the video recording, the authors of this paper jointly selected the episodes and pictures to be used, and worked together in the description of the episodes. Within the framework of networking theories, this methodology ensures optimal conditions to allow the concerned theories to interact as evenly as possible with the same empirical data, each theory allowing the interpretation of the data from its own principles.

Table 1 Some parts of the lesson



The mathematics lesson took place in a Grade 1 class (6–7-year-old students) in Sudbury, northern Ontario, Canada, and was part of longitudinal research dealing with the investigation of pedagogical conditions to foster collective learning. Four classes were involved. The mathematics tasks were codesigned by the research team and the teachers. In this paper we focus on the first of a sequence of lessons whose content was sequence generalization—more specifically, the reproduction and generalization of a figural sequence (see Table 1.1). (In what follows, Table m.n means Table m, cell n counted from left to right, and top to bottom.) The students would work in small groups of two or three; they would be provided with wooden sticks to reproduce the given terms and to produce the fourth figure, which would require a mathematical generalization. The lesson evolved as follows (times are approximate).

The first part (0–2 min): a general discussion in front of a whiteboard in which the teacher engages with the students in a dialogue about what they are going to do in small groups in the rest of the lesson (see Table 1.2).

The second part (2–16 min): in small groups, the students reproduce the sequence and produce the fourth term of the sequence (see Table 1.3).

The third part (18–40 min): a general discussion in front of the board; the teacher tries to introduce the key words by which the students should be able to access a cultural scientific understanding of the sequence.

The fourth part (40–50 min): the students return to their desks and reproduce the figures using pens. The wooden sticks are removed from the desks.

The fifth part (50–58 min): general discussion about what the children have done. There is also a discussion about what the children think they learned and a discussion about the activity itself, asking whether the children enjoyed the activity or not (see Table 1.4).

4 Some episodes

The episodes shown here come from the second and third parts of the lesson (first small-group work and first general discussion).

4.1 Episode 1

This episode happens during the second part of the lesson (student small-group work).

1. Ophé: Perfect. (*She* reproduces Fig. 2 *of the sequence. Meanwhile, Harold tries to reproduce* Fig. 4. *He places the stick from above, then he cannot figure out how to continue. He tries first on top, then at the bottom* (see Table 2.1)).
2. Harold: Uh … How do you... ?... The stick doesn't reach!
3. Ophé: What?... Because... Wait! Put one there (see Table 2.2) and then one there (see Table 2.3)
4. Harold: Oh yeah!
5. Ophé: That one down there, look! (*Making a sequence of four two-handed gestures*) It's a pizza (see Table 2.4),

Table 2 Ophé and Harold in small-group work

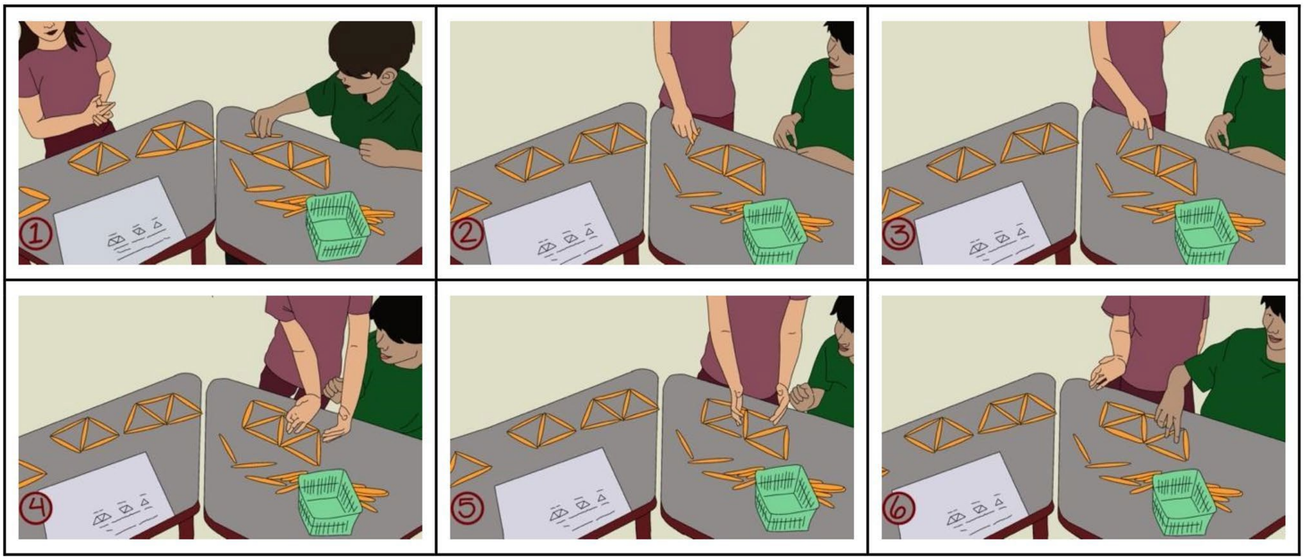
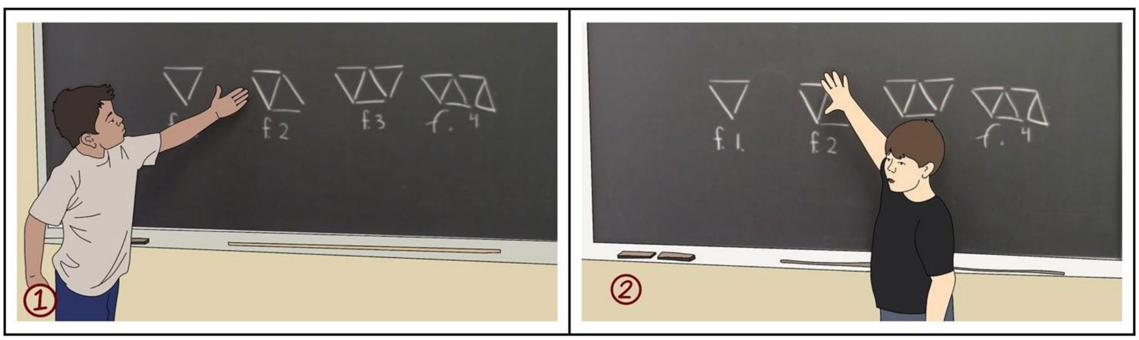


Table 3 Explaining the reproduction of figures



pizza upside down (Table 2.5), pizza, pizza upside down...

6. Harold: (*Rephrasing, he makes a sequence of four fin- ger gestures)* That's inverted, not inverted, inverted, not inverted (see Table 2.6).
7. Ophé: For me it's inverted...

The teacher comes to discuss with Ophé and Harold:

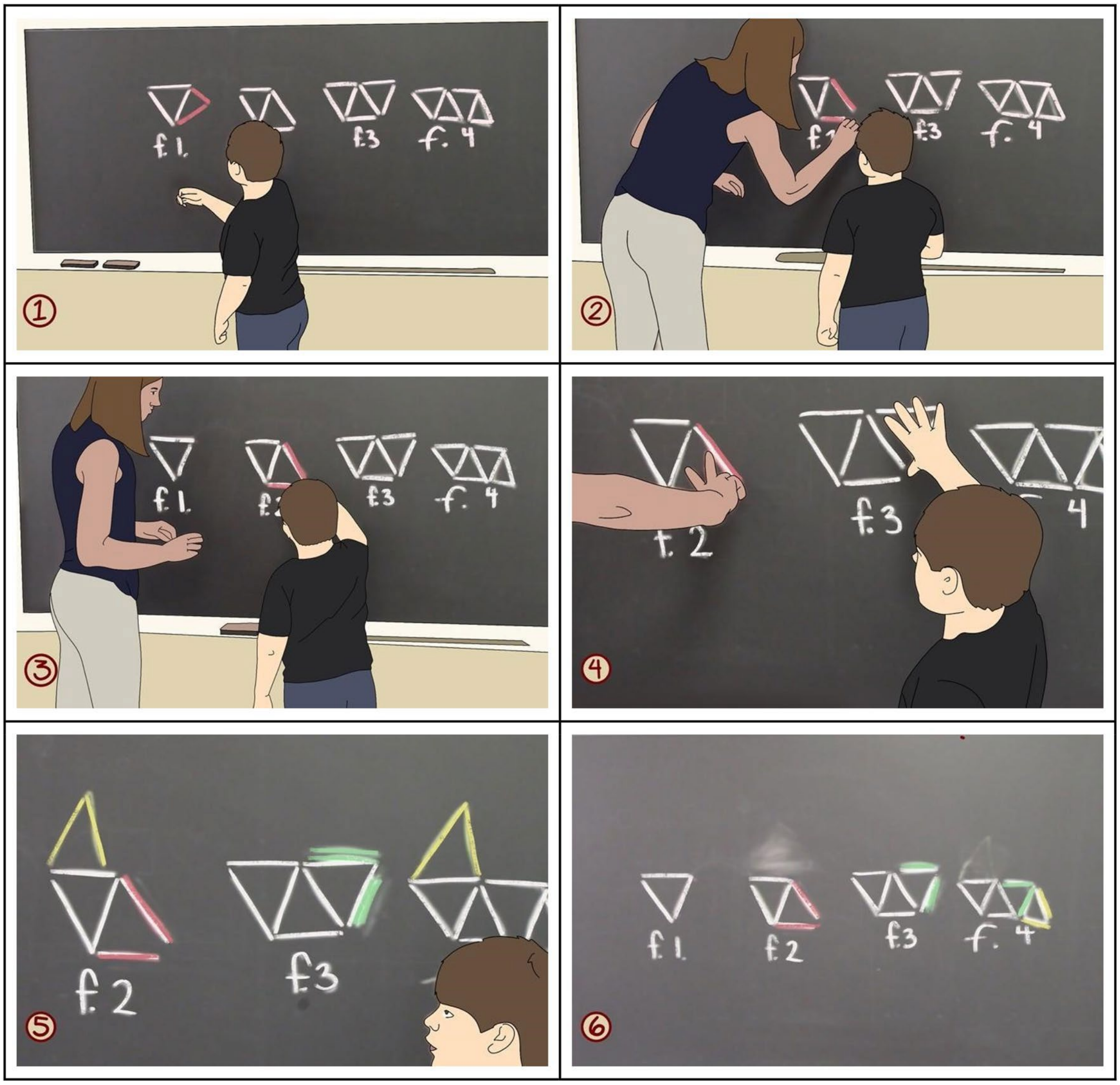
8. Teacher: Okay, this is your figure?
9. Harold: Yes
10. Teacher: (*Noticing that the figures are placed in reverse order*) Ah! You did it backwards! That's 1 2 3 4... Could you change it? Like... look, this is 1, come here, put it up here...

The next episodes happened during the first general discussion with the students sitting in front of the blackboard. Jacob drew the first four figures. Then, the teacher engaged the class concerning the procedure required in reproducing the figures.

4.2 Episode 2

11. Roland: My way was that... the whole time I saw that there was plus 1, plus 1, plus.
12. Teacher: Okay, go to Fig. 1 and then explain. What is it that is always repeated?
13. Roland: For Fig. 1 there was 1, then for Fig. 2 there was 1 more.
14. Teacher: One more what? Explain with words. [Was it] sticks, triangles?
15. Roland: Sticks. There was 1 more, like you would put 1 more, with the sticks.
16. Teacher: Okay. From Fig. 1 to Fig. 2, there was 1 more stick. (*Talking to the whole class*) Raise your hand if you agree... What changes from Fig. 1 to Fig. 2?
17. Roland: (*In a slightly exasperated tone*) There was plus 1! (Table 3.1)

Table 4 From adding triangles to adding sticks



18. Teacher: Roland, that's okay (*Noting Roland's frustra- tion*). Okay. Do you want to explain, Otis?
19. Otis: I saw 1 more triangle. We added another... (*He shows Fig. 1 embedded in Fig. 2;* Table 3.2)...
20. Teacher: Another what?
21. Otis: Triangle.
22. Teacher: And how do you add your triangle? That's what you have to explain to understand.
23. Otis: There and there, you take another Fig. 1 and place it like that.
24. Teacher: You take Fig. 1 and place it there like that. If I have my eyes closed, I'll say I'll take the figure and place it here like this, will I understand it?

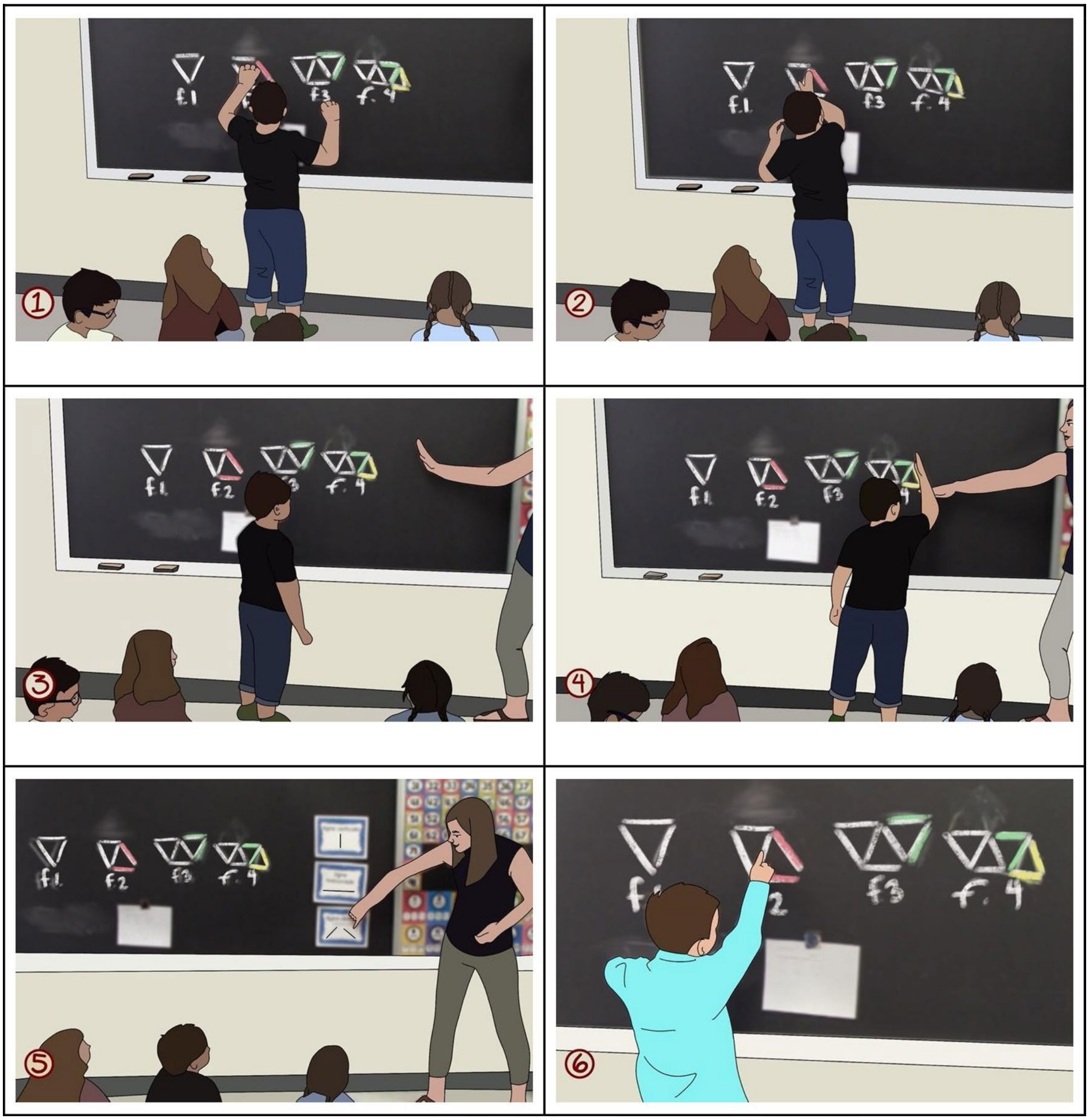
4.3 Episode 3

The reproducing procedure is not well articulated linguistically quite yet. In this episode the teacher tries to make the procedure more apparent.

Otis goes to the blackboard and adds a triangle to the right of Fig. 1 and says, "And this is Fig. 2" (see Table 4.1).

25. Teacher: Okay, like here (*She erases the added chevron in Fig. 1, and, following Otis's idea, colors in red in Fig. 2 the chevron that Otis is adding; see* Table 4.2), what you mean is you put this here? How many sticks did you put here?
26. Otis: 2 (see Table 4.3).

Table 5 Hands, colors, and language in making sense of the reproduction of figures



27. Teacher: Okay. The 2 [sticks] there in pink, okay. In Fig. 3, what did you do?
28. Otis: We had Fig. 2 and then we added 1 more (*With a full open hand gesture, he points to the right triangle in Fig. 3*).
29. Teacher: 1 more? In Fig. 2, you said you added 2 sticks! (see Table 4.4).
30. Otis: I added 2 sticks.
31. Otis: Ava thought we were [adding] here (*He adds a green chevron to Fig. 4*), but look, Fig. 2 does not have that (*Adding a chevron to Fig. 2*) (see Table 4.5)...

Table 4.6 shows the result.

4.4 Episode 4

So far, the teacher and the students have worked towards making apparent a regularity: two sticks are added at the end of the last part of the previous figure to obtain the follow- ing figure. Then, the teacher turned to the introduction of a vocabulary that could facilitate the counting of sticks in each figure. After that, Harold, Flora, and other students offered some ideas, Otis asked to go to the blackboard.

32. Otis: So, we always saw 2 more (*With the left hand, he points to the 2 added sticks; see* Table 5.1) and there [we saw] another triangle (*Making a gesture along the left side of the right triangle, see* Table 5.2).

33. Teacher: Yes! Look! (*Talking to the class*) Did you see?... Did you see Otis's hand? It was standing like this (see Table 5.3). You did that, look at that! Do you agree that it's like lines? We studied lines at the beginning of the year! (see Table 5.4)... When you put them in a certain way, the lines have names. Look carefully. It [the added chevron] has lines.
34. Student 1: Horizontal!
35. Teacher: Oh! horizontal lines... Do you see any horizontal lines?
36. Class: Yes!

The teacher stuck papers with the names of the lines on the blackboard (see Table 5.5). Then, she and the students spent some time recognizing and counting the horizontal and diagonal lines in the figures (see Table 5.6).

5 What do the theories make apparent in this lesson?

5.1 The TO

At a general level, what we see in the lesson is whether collective learning is occurring and how. Indeed, not all learning is collective (even if people are learning in the same classroom). During our research, we have found that two elements are required to make learning a collective endeavor: first, there needs to be a common, well identified learning *object*, and second, concerted forms of action and interaction. Both together define what we call the *joint labor* of teachers and students (Radford, 2021a). In the Grade 1 les- son, the learning *object* is a cultural-historical mathematical manner in which to think, talk, and deal with sequence generalization. The concerted forms of action and interaction appear in

(a) shared forms of the production and circulation of mathematical ideas; and
(b) strong community-oriented forms of human collaboration.

These two forms glue together, so to speak, teachers and students; they *interlink* the deeds of teachers and students and *propel* these deeds in the same direction—the direction of the common learning *object*. These forms are created, re-created, refined, expanded, pondered, and modified continuously by teachers and students, each day. So, what we see in the Grade 1 lesson is how these two intertwined forms of production of mathematics ideas and forms of human collaboration appear and come to frame the way the learning *object* of activity is collectively encountered.

In the Grade 1 lesson, to promote the aforementioned concerted forms of action and interaction, the students are organized in small groups that, at certain points, gather together to discuss the ideas they have generated. However, it is not unusual that the students tend to work individually, even if they are in a small group. This is why, during the second part of the lesson, the teacher reminds the class to work *together*: "make the figure with your partner; it's one figure for two persons." She reminds the students to "think and talk with your partner." We also see, during the general discussion, the tremendous effort the teacher makes to actively involve the students in the production and circulation of ideas through which knowledge is encountered.

The common *object* of learning and the concerted forms of action and interaction provide a fluid framework for the unfolding of the processes of objectification and subjectification. These processes are investigated through a multimodal semiotic analysis. In the Grade 1 lesson, the processes of objectification reveal the progressive awareness of ordinality, spatiality, and the implied mathematical variables that underpin the production of a formula (a co-variational outlook of the sequence). The processes of subjectification reveal how the students co-position themselves in the encounter with knowledge.

5.1.1 Ordinality

The co-variational outlook of sequences entails first that the student pay attention to *ordinality*: the sequence is made up of ordered terms; this order is not arbitrary. The teacher realizes that the students do not reproduce them in the expected order. In Line 10, she invites the students to rearrange the terms. A few moments later, she tells the class: "It's better if you make [Figure] 1, 2, 3, and 4. [It won't work] if you do Figure 4, Figure 2, Figure 3, and Figure 1. Make them in the order. Do you understand? It has to be *clear* when you look at them." The "clear" the teacher refers to must be understood in the sense of a cultural-historical rationality with which the students are starting to become acquainted.

5.1.2 Spatiality

Second, there is the problem of *spatiality*: Figure 1 goes into a specific *position*. While Ophé pays attention to it, other students do not. The left–right order of the terms, indicated through names under them (Figure 1, Figure 2, etc.) positions the sequence and the subject that sees the sequence in a space. In this space, we could say that the subject *sees* the sequence and that, in return, the sequence *sees* the subject. When seen from this implied perspective, Figure 1 looks like an *inverted* triangle (or "a pizza," as Ophé suggests in Line 5). All figures start with an inverted triangle. The problem of spatiality appears also in the position of the parts that constitute a figure. In some cases, the last "triangle" is added on top, as opposed to the right, of the figure. This is what Otis mentions in Table 4.5.

5.1.3 Variables and the step-by-step procedure

To reproduce the first terms of the sequence and to generalize it to produce the fourth term, the classroom as a collective is confronted with the problem of generating a *step-by- step procedure*. We are interested here in understanding the epistemological complexity involved in this endeavor. The generation of the *step-by-step procedure* should facilitate later on a focus on *numeric* variables (i.e., "the number of the figure," and "the number of wood sticks added to go to the next figure"), the foundation of which, after some abstractions and generalizations, may take the students a few years to go through, will appear as the formula $T_n = 2n + 1$. Although the students are not there yet, they are already encountering an instrumental rationality that is conveyed by the teaching-learning activity. There is a *cultural way of knowing* about sequences that requires the students to account, in an explicit way, for the steps by which the sequence's terms are built. Not every account is deemed right. There is a *normative* dimension in coming to know about sequences. Knowing about sequences (and about many other things) in a mathematical and scientific way is based on the idea of *calculability*—one of the chief aspects of instrumental rationality of Western thought (Weber, 1992) and contemporary mathematical thinking (Radford, 2021b). In Lines 19–29 we see how the teacher invites Otis to be more explicit about what has to be added to the terms. We see in other parts of the lesson that the discussion moves from adding triangles to adding wood sticks, which brings the class closer to thinking about the *number* of sticks that are added or the number of sticks in a figure. At any rate, what is added has to be made explicit (see Lines 22–23). In short, ordinality, spatiality, and the building of a reproducing procedure are key elements of the students' process of objectification.

As is usually the case, in the Grade 1 lesson, the processes of subjectification go hand in hand with those of objectification. For instance, we see how, in Lines 13–18, the student is co-positioned by the teacher in the classroom mathematics practice. Raymond is asked to give reasons to enter into the world of reason (Bakhurst, 2011). In these lines, Raymond responds to the teacher's invitation, and learns what rea- sons are expected to be given. However, the teacher is also positioned by the students. When the students produce the figures in an unexpected order or spatial position, the teacher comes into the realm of praxis and makes herself heard; she takes position vis-à-vis what the students say and do.

To recapitulate, our attention in the mathematics lessons is directed to making room for collective learning to happen. We look at the processes of objectification and subjectification in light of the joint labor of teachers and students that aims at revealing the learning *object*. We reach here one of the inner contradictions of teaching-learning activity: how can the students engage in an activity directed towards an *object* that they are not aware of? The students are in a situation similar to that of the sculptor who sees the sculpture appear from the stone as she carves it. While the sculptor draws on a range of cultural-historical sculpting traditions and artifacts to achieve her goal, teaching-learning activity draws on a path imagined by the teacher towards the grasping of the learning *object*. The way this activity unfolds really cannot be predicted exactly since activity is movement (movement of ideas, and meanings that the students and the teacher produce together, woven with emotions, social relations, and ethical stances). Hence, the appearing of knowledge is an unpredictable event. Like the appearing of the sculpture, the unpredictability of the appearing of knowledge in the classroom and the wonder of its realization are what make collective learning an aesthetic event.

5.2 The DAD

The DAD focuses on the activities that take place in/outside the classroom as a product of the competences developed by teachers around the design and implementation of les- sons through the use and design of *resources*. These competences are analyzed through a cognitive perspective of Piagetian orientation in which knowledge is considered a form of adaptation (accommodation and assimilation), and where both actions and representations play a central role in the cognitive development of the individuals, as inferred from their actions.

A starting point for the analysis of activities is to consider teachers as designers of their lessons, where possible learning paths are planned through interaction with resources. Trouche et al. (2020) highlighted two aspects of this con- sideration: (1) it is related to looking at teaching as design, and (2) "that any understanding of teacher as designer must include a conscious/deliberate act of designing, of creating 'something new' (e.g., combining existing and novel elements) in order to reach a certain (didactical) aim" (p. 241). In this line of thought, the DAD has turned to Vergnaud's concept of *scheme* (Vergnaud, 1998), which offers the possibility of analyzing the set of professional situations that correspond to the same objective of the activity. In the con- text of the teaching-learning of mathematics, professional situations correspond to mathematical didactical situations (MDS) and problems (e.g., organization, implementation, and communication) that are dealt with. The MDS in the lesson presented in this paper is: "how to work in a Grade 1 class with the reproduction and generalization of terms of a figural sequence"; a MDS that was agreed upon during task co-designing. The *scheme* involved is one related to the use of *resources*; that is, it is a *scheme of usage*. Naturally, it is not just one resource used by the teacher, but a set of resources interacting with each other, among which are the wooden sticks, the gestures, the blackboard, and the information sheets. Thus, the set of resources used by the teacher is called her *resource system*. And it is through these resources that the teacher tries to cope with the didactic situation.

5.2.1 Schemes of usage

To analyze the MDS, the notion of scheme is used regarding the design (by the teacher) and the use of *resources* (by the teacher, but also by the students). This allows us to account for the invariant organization of the teacher's behavior to achieve the activity's objective; that is, the competencies she has developed to deal with the MDS. Mathematical knowledge (ordinality, spatiality, and reproduction of the fourth term) is considered in the scheme, with the particularity that this knowledge is inferred from the actions of the teacher and the students.

Goals, subgoals, and expectations are associated with finding a way for students to reproduce the first three terms and produce the fourth term of the sequence (Table 1.1). Wooden sticks constitute the chosen *resource* for developing this task. It is expected that by working with the sticks, students might be able to recognize the essential elements that constitute the first three terms and subsequently be able to produce the following term. In turn, rules of action are established, associated with how the activity is organized. Episode 1 shows a part of this organization corresponding to the work in teams of two or three students. This team- work, organized by the teacher, promotes the development of *schemes* in the students, which are also related to the use of wooden sticks. Here we can distinguish the following operational invariants: "students must manipulate in teams the wooden sticks (*resources*) to find Term 4," "students must keep placing the wooden sticks with a specific orientation," and "students must follow an order." Episodes 2, 3, and 4 correspond to the next activity stage: a general discussion. In this discussion, the coordinated use of different resources appears. The *blackboard* is one of the leading resources during this stage. Its use, in coordination with the *use of gestures* (Tables 4, 5) and the incorporation of *information sheets* (Table 5.5) allows the teacher to accomplish the following: attract the students' attention, address the diversity of results obtained by the different teams during their teamwork, and work on the transition from informal to more formal language.

In the discussion, the teacher seeks, from the beginning,
not to lose sight of the importance of the order of the tri- angles (see Lines 12–13). As the discussion progresses, the teacher modulates her interventions on the blackboard according to the moments she considers pertinent. Two moments stand out:

when the teacher pays attention to the orientation of the figure, which must be precise (see Line 25); and
when the teacher decides to incorporate formal definitions (see Line 33).

6 Dialoguing

In this section, we present a dialogue on three main themes: the role of language, resources, and artifacts; the role of the teacher; and the conceptions of the classroom and its educational phenomena.

6.1 Language, resources, and artifacts

6.1.1 The DAD

In the DAD, language and resources are considered in their role of *mediators* of mathematical knowledge in the class- room; and are also important elements of the *schemes* that accompany or drive the thinking of the teacher and the students. Vergnaud (1998) states: "We have schemes to produce sentences for mathematics, as well as for other domains of human activity: we have schemes to discuss and argue, dialogue with others, give lectures, or write text. Schemes have physical, linguistic and social components" (p. 235). Thus, they become means by which to analyze the competencies that the teacher has developed throughout her professional career to face didactical situations. Particularly on the language side, language is associated with the predicative form of knowledge, which consists in the linguistic and symbolic expressions of this knowledge (Vergnaud, 2009). That is, mathematical concepts are communicated, and knowledge is made explicit through the way language, gestures, diagrams, and other forms of representation are used to communicate with students. Thus, on one hand, language is a tool for the exchange of ideas and concepts that occurs when teacher and students interact with resources, and, on the other hand, it is a way to account for the teacher's competencies based on the way she uses the resources.

For example, in the lesson presented in this paper, the wooden sticks, languaging, and discussions (between stu- dents and students with the teacher) are used according to the teacher's plan in the context of the task, which is as follows: while working in pairs, the students will reproduce the terms of a figural sequence (Table 1.1). With this, the teacher seeks to have students put into words, as well as into student-formed figures, what they understand about the mathematical knowledge required to perform the task.

6.1.2 The TO

Like other sociocultural approaches, the TO gives a prominent role to language and sign-and artifact-use. As previously mentioned, we draw on a dialogical conception of language. However, the TO's chief theoretical category is neither language nor artifacts, but *activity*—the *joint labor* of teachers and students. On the surface, the problem seems to be about a simple choice: language or activity; saying or doing. It is not. The choice is about our conception of human nature: *homo discursivus* vs. *homo laborans*. This choice was at the heart of some tensions and controversies in Vygotsky's circle of collaborators. Leontiev (2005) developed his activity theory following the latter view, contending that our ontological makeup is to labor to fulfill our needs and craft our existence. It is because we labor with others that language is created. We are not discursive beings per se. Transposing these ideas to education, a corollary of this theoretical standpoint is that to understand learning processes, attention has to be paid to the *activity* where learning is occurring; this activity *includes* language. It also includes other semiotic systems having different and very fluid forms of signification (such as gestures or rhythm). It is classroom activity that makes learning a discursive, embodied, multi- modal, semiotic experience, and not the other way around. This is why, in the TO, language can only be understood in terms of the cultural activity where it is subsumed. We argue that differentials in cultures are better grasped by look- ing into what people do (activity) than by what they say (language).

6.1.3 DAD reaction

There is a point of coincidence in the sense that for the DAD, like for the TO, neither language nor artifacts is its chief theoretical category. The difference is that, from the DAD point of view, the processes of meaning-making, where the language and resources are involved, are related to the operational characteristic of the schemes. The schemes operate in didactical situations in which the teacher and students overcome a variety of difficulties and organize a way to progress in the management of these situations. Language and resources are involved in the activity; they involve two processes: an *instrumentation* process in which resources influence the activity and thinking of the subjects; and an *instrumentalization* process in which the subjects shape the resources based on their knowledge and interests (Trouche, 2004). Both processes are guided by the teacher's didactic objectives.

Thus, language and resources mediate the activity that takes place during the lesson by being involved in the performance of tasks. Both processes are observed in Episode 1, where Harold faces the difficulty of not knowing how to make Fig. 4, while Ophé has been able to (Line 2). Har- old is in the process of instrumentation with the wooden sticks, which serve at this point to form any figure. However, it is necessary to manipulate them and place them in a certain way, as Ophé does, to obtain the figure that is sought, which is in relation to the previous figures; this corresponds to the process of instrumentalization. During the development of the two processes, argumentation and dialogue schemes are developed among students in which, through language, gestures, and the use of wooden sticks, students seek to give meaning to the task. However, it is still necessary to use formal language to give mathematical meaning to the activity. This occurs in the episodes analyzed in the following Sect. 6.2, in the context of the role of the teacher.

6.1.4 TO reaction

An important difference between the DAD and the TO can be found in terms of their *focus* of attention and their *explanatory principle*. The DAD seems more focused on how teachers and students *deal* with the problems they face. The construct of schemes of usage gives one the explanatory principle to understand how teachers and students navigate their difficulties. Language and artifacts are embedded in these schemes. In the TO, the focus is different: it is about how teachers and students *encounter* cultural knowledge. In other words, language and resources seem to appear in the DAD as cultural means that the individuals integrate into their deeds to navigate their world. In the TO, the cultural means are part of what it means to know mathematically. This is why they are not considered as mediators or resources. They are part of the very fabric of what is mathematics and mathematics thinking.

Behind these different theoretical stances there is also a difference concerning the relationship between language and knowledge. In the DAD, language plays a nominalizing role: it allows one to name things and to know about them. Of course, the same is true in the TO (where language plays the role of a semiotic means of objectification). The question, however, is to what extent knowledge can be captured through language. Language works in such a way that it creates specific determinations of reality. It delineates contours. When you say something, you renounce saying many other things. You gain something and lose something. The result is that there is a surplus of knowledge that lies beyond what is said in language (what in the DAD is called the predicative form of knowledge). In the TO, the epistemological limits of what can be captured through language leads us to conceive of learning as an unachievable process that resists being encapsulated in the form of schemes, no matter how rich they might be.

6.2 The teacher

6.2.1 The DAD

As noted in Sect. 5.2, proponents of the DAD are interested in investigating the role of the teacher in designing and implementing a lesson in the classroom. Hence, observing what happens in the classroom through the DAD lens implies analyzing how the teacher faces a professional situation. In this paper we focus on the teacher and her role in the development of the classroom activity, in particular on the teacher's use of resources, and, more specifically, the *schemes of usage* of *resources* produced by the teacher to face the didactical situation. From the DAD perspective, one general question (the didactical situation) pertaining to the lesson under discussion is as follows: how can the teacher work in a Grade 1 class with the reproduction and generalization of terms of a figural sequence? The answer is to be sought in the teacher's role to direct the development of the lesson through the use of different *resources*. As a designer, the teacher establishes which *resources* will be used throughout the lesson and the different moments at which they will be incorporated according to the lesson's stage and specific moments to modulate its development. In this way of organizing the lesson, different components of the scheme are associated, such as goals, subgoals, expectations of the activity, and possibilities of inferences (see Sect. 2.2), which are enacted during the implementation.

6.2.2 The TO

One of the main ideas in the origin of the TO was to move away from both student-centered and teacher-centered pedagogies. We remove the longstanding line that, in other approaches, separates the teacher from the students. In the TO, teachers and students work *together*, hand in hand. Both are needed in order to make mathematics appear in the classroom.

Let's have a look at Tables 4.4 and 5.4. There we see Otis and the teacher working hand in hand, almost literally. Otis makes a gesture with his hand to shape an idea and the teacher replicates with her hand Otis's gestural idea. They are working together. Behind this idea, there is a conception of mathematics as a potential way of thinking, reflecting, and doing things, and the concomitant idea of materializing this potential way of thinking mathematically by making it show itself in the classroom. We see mathematics as being a bit like music. For music to be heard, it has to be played. It then appears sensuously, through the activity of an orchestra. The same goes for mathematics. Mathematics appears sensuously through teaching–learning activity. This is what we see in the classroom episodes above. While musicians make music appear through musical artifacts, mathematics appears here through the joint labor of the teacher and the students and the wooden sticks, language, gestures, signs, rhythm, and so on. The teacher and the students 'play' mathematics together. It is interesting to note that in some languages like Spanish and Portuguese, music is not said to be played, but to be *touched*. Classroom activity is like a *kinesthetic organ* through which the students 'touch' mathematics (Radford, 2019).

In the DAD, as you point out, the teacher is seen more as a designer and an operator, maybe like an engineer; the focus is on producing a 'document'. In the TO, the teacher is viewed more like a stage actor —an artist who, in their interaction with other actors (in this case, the students), participates in the production of a 'common work' as it is coming into life.

6.2.3 DAD reaction

From the viewpoint of the DAD, it is also possible to see that the teacher and students work together, although this activity and the interactions between them are seen in a different way. We will take the same metaphor of the activity of an orchestra to analyze what happens at some moments during the lesson, in which there is the appearance of formal language to carry out the task, from the point of view of the orchestra conductor. We start by associating the teacher with the role of the orchestra conductor; as the one who drives, coordinates, and is in charge of empowering each musician—in fact, part of what the DAD evokes is the instrumental orchestration of the class- room activity with resources by the teacher (Drijvers et al., 2010).

After the students have worked in small teams, the teacher looks for ways to introduce key words so that the students can access a mathematical understanding of the task at hand. For this, she discusses the issues with the whole group using the board. The first point that is observed in Episode 2 is the difficulty of putting into formal words the work that the students have previously done (Lines 11–15). Faced with a moment of frustration on the part of a student (Line 17), the teacher seeks to incorporate other voices into the discussion without losing sight of the type of language she is looking for the students to use. In this search, the teacher controls her own participation at the blackboard in order to modulate the activity; for example, there are times when she erases or highlights what the students have done. This can be seen as an example of the competencies that the teacher has developed to face the didactical situation, just as the orchestra conductor develops competencies to know when to intervene in a musical performance.

6.2.4 TO reaction

In the DAD the focus is on *the teacher*'s competences, and *the students*' competences. I would argue that competences need to be seen as dependent on what we do *with* others. Competences, then, would be more than the attributes of an individual. At the April 2019 "Digital Turn in Epistemology" conference in Utrecht, reacting to a research report on competences, Ricardo Nemirovsky made the interesting remark that Lionel Messi's "competencies" are not as good when he plays with the national Argentinian soccer team as when he plays with the Barcelona team. Since we endeavor to see our classrooms as sites of collective learning, if we were to talk about teacher's competencies in the TO, they would have to be seen as the *joint* competencies of teachers and students, the competencies of the collective. Our theoretical take may be understood if we bear in mind that on our way to our collective learning project, we resist making two separations.

First, we resist making the separation between the concrete and the abstract. A consequence is that, under this assumption, a scheme cannot be seen as a kind of abstraction: an invariant that drives the teacher's thinking. Were we to talk about schemes in the TO, it would be as some- thing that always carries with it contextual elements (the concrete) and includes the Other; it would not be a *sui generis* general entity, but an entity that is concrete and abstract at the same time, where self and other merge.

Second, we resist making the classical separation between teachers and students. You were talking about orchestration. It can be argued that it is not only the teacher who orchestrates. The students orchestrate too, and in a decisive way. In the TO, we try to find a more symmetrical role between teachers and students, which has led us to the concept of joint labor. Of course, this does not mean that teachers and students do the same thing. There is almost always a division of labor in human activity. Do you remember Leontiev's (2005) example of hunters? One of the hunters keeps the fire burning while others go in search of the prey. They do not do the same thing, yet they participate in a joint labor. This is why the orchestra conductor we have in mind in our music metaphor (Radford, 2019) is the conductor you see playing *with* the orchestra, like Daniel Barenboim playing the piano in Beethoven's First Piano Concerto while conducting the Berlin Philharmonic Orchestra in 1989 after the fall of the Berlin Wall. What was just said above about the teacher does not mean that there is no planning involved. Of course, there is. And it is about how to make collective learning hap- pen. However, we do not consider the teacher as a designer *as such*. Planning is a warmup exercise, like the pianist stretching her fingers before the concert.

6.2.5 DAD reaction

We want to point out that while the design part is meaningful, it is important to emphasize that the DAD also considers as relevant what happens during the implementation of the activity where the social interactions appear, and that taking up our metaphor of the orchestra conductor, this conducting activity also involves an artistic aspect, but without leaving out the adjustments that must be made to meet the objective. It is also important to consider how the ways of organizing the activity are adapted to the situations. Thus, after the teacher worked on the regularity underpinning the figural sequence in Episode 3, Episode 4 represents the moment when the mathematical vocabulary that accounts for the sequence appears. Here again the teacher's role is to know when it is time to introduce the notions of horizontal, vertical, and oblique lines. This occurs when she notices a student making a hand gesture (Table 5.2).

6.2.6 The TO

Still, although some theoretical ideas or principles of a theory *A* can be translated into the theoretical ideas or principles of another theory *B*, there is a limit to what can be translated; something is always lost in translation.

Curiously, we reach here, from a different angle, the epistemological limits of language that we addressed in Sect. 6.1, where we were saying that language is impotent in front of the task to give a precise account of the objects of its cultural conceptual world. In this part of our conversation, we find language impotent to rise above itself to trans- late unproblematically from one theory to another. Were we able to find the perfect language that Leibniz dreamed of, a universal language capable of removing all and each one of our cultural differences, then translation would work like a marvelous mathematical isomorphism; theories would coincide exactly, and mathematics education would be a superb tautology of equal theories and practices. That this is not the case can be seen in the impossibility that we would find to satisfactorily translate the "teacher as a designer" of the DAD into the TO; or to translate the concept of scheme from the DAD into the TO. It seems that the teacher, for example, is simply not the same in both theoretical practices.

Let us turn to our next section, where we continue trying to make sense of similarities and differences between our theories.

6.3 Conceptions of the classroom and its educational phenomena

6.3.1 The TO

We mentioned above that learning in the TO is not just about knowing, but about becoming too. This double focus explains why in the TO the mathematics classroom is conceptualized as a site where, on the one hand, the students encounter culturally and historically constituted ways of mathematical thinking (the axis of knowing) and, on the other hand, have the opportunity to experience democracy, collective life, solidarity, and inclusivity through the exercise of a communitarian ethics (the axis of being) (Rad- ford, 2021a). Learning as knowing-and-becoming unfolds in the classroom as a dynamic system. One of its features is that what happens in it (i.e., teaching and learning) cannot be captured in a model. Regardless of how much you try, important and fundamental things will remain beyond the scope of a model—not because you are not good at making models, but because the nature of the world is less *rational* than we often assume. The nature of the world is to be messy. What we observe in a classroom (for instance, the way people interact with each other, the positions they take vis-à-vis knowledge) is not always the same; people do not interact with each other in a rule-guided manner. Usually, in other approaches, it is assumed that, *things being equal*, what has happened in a classroom can be generalized to other classrooms. From our ontological perspective, there are two problematic assumptions here. First, as classroom phenomena are concerned, things are never equal—teachers and students are always changing. Second, teaching and learning are deeply context sensitive. You can ask: How then do you generalize your results? The answer is: not by making the results more general, but by making them more concrete. Let me explain. From our previous research, it was hypothetically sound to expect that processes of objectification in our Grade 1 class would involve the general categories we have observed before (i.e., ordinality, spatiality, and the difficulties surrounding the step-by-step procedure). These categories were dis- cussed during the co-design of the task with the teachers. They were considered as "potential categories." However, it was simply impossible to anticipate how these categories were going to be manifested in concrete ways in our Grade 1 classroom. Their manifestation, their actual appearance in classroom activity, provides us with the possibility to see new theoretical aspects of the potential categories, becoming thereby more precise, more 'concrete'. This is what in dialectical materialism is called *the ascent from the abstract to the concrete* (Radford, 2021a). However, it can also be the case that *new* categories appear, categories that were not part of our a priori analyses. The role of rhythm in mathematics cognition is a case in point (for details see Radford & Sabena, 2015). This is why we see our results as truths-in-the-making, results to be continuously modified, revisited, trashed, and recreated. We see them as *reflective* points to help us understand the complexity of teaching–learning. These results are considered as sources of inspiration for us (mathematics educators) and teachers to engage in situations that will emerge in different ways with other students or the same students later on.

6.3.2 The DAD

Like the TO, the DAD also recognizes that teaching and learning situations are never the same, and that they are context sensitive. But it is the context that is the focus of the DAD, as it analyzes the way in which teachers organize their classroom activity, the use of resources, and the role of the students in order to create didactical situations, developing schemes with operational invariants; that is, ways of organizing the classroom activity that can be invariant across a diversity of circumstances. In other words, each teacher develops schemes of usage of the resources and orchestration of student tasks that serve as an invariant structure for future classroom implementations. These schemes can be a set of structures to follow the possibilities and constraints of the educational and institutional context (the curriculum, the available resources, and the roles of both the teacher and students in the classroom). In that sense, the DAD conceives these schemes as more than just 'inspirations' for engaging in future situations; the DAD seeks to find that which can take place in the classroom and is developed as part of the professional activity of the teacher. This activity, in turn, promotes the development of students' skills and competences, as well as their own ways of dealing with situations (previously designed by the teacher) as they use and integrate the resources placed in the classroom.

7 Concluding remarks

In mathematics education research, networking means bringing together two or more theories into a dialogue. Various strategies have been identified, such as combining, comparing and contrasting, integrating, and synthesizing theories (Prediger et al., 2008). Our dialogue falls in the comparing/contrasting category. It has focused on the contrasts that arise when we (as authors bringing to the fore our own understanding of the theories) compare our theoretical stances on three crucial aspects of sociocultural research, namely language and resource use, the role of the teacher, and the conception of the classroom. These three aspects constituted the main questions of our dialogue, as they seem well-suited to unveil what we called in the introduction the "cultural sensibilities" of a theory and respond to cultural aspects that this ZDM issue intends to explore.

Contrasting theories allows us to appreciate "the specificity of theories and their possible connections" as well as their "individual strengths" (Prediger et al., 2008, p. 171). In fact, it does more. Macherey reminds us that, "*To compare* *things … is an operation which is not absolutely free … because it measures its objects in relation to each other, in the undefined relation of their reciprocal determinations"* (Macherey, 1979, p. 224). The result is then that the "corresponding aspects" in each theory acquire new meanings, new contours that they might not have had before the com- parison. This is true of the three aspects we have discussed.

7.1 Language and artifacts/resources

We have seen how the DAD emphasizes the cognitive and epistemological nominalizing role of language which, on the side of the TO, led to asking the question of the scope of what is being said in the processes of learning. From this interrogation arises the question of the speakable and the unspeakable, what you can say in language and what remains ungraspable beyond it, beyond the realm of *logos*. The question of artifacts resulted in pointing out a difference. The DAD conceptualizes resources as everything that has the potential to resource teacher activity and participates in two well-defined processes, instrumentation and instrumentalization; resources are seen as mediators of activity. In the TO, artifacts are not seen as mediators but as part of the teacher-student activity, more specifically, part of the material texture of mathematics and mathematical thinking. We observe an interesting difference: for the DAD, artifacts are part of the resources seen in an instrumental way, while in the TO artifacts are semiotic means of objectification.

7.2 The teacher

The differences we found might enlighten the understanding of the specific orientations of the theories. The role of the TO teacher as a full-fledged participant in the joint teaching–learning activity is at odds with the cognitive role the teacher acquires in the DAD. In the DAD the teacher works *for and in response to* the student. In the TO, the teacher works *with* the student. While in the TO the teacher appears as a stage actor, in the DAD the teacher appears as a designer. The tension that arises from this difference, however, is not simply the index of a distinction; the tension is also an invitation to see the teacher in one theory through the eyes of the other theory. For a genuine dialogue to exist, Self and Other need to try to understand each other and see oneself as the other (Ricœur, 1990). How can the teachers as designers see themselves as stage actors, and vice-versa? Sect. 6.2 contains attempts to walk along the line of this enriching tension.

7.3 Conceptions of the classroom

Our dialogue around the conception of the classroom allowed us to explore some ontological assumptions that the theories make. In the TO, because of the assumed inseparability of the concrete and the abstract, what happens in the classroom is not considered as phenomena that can be abstracted and described in terms of laws or rules or schemes. The DAD makes the opposite supposition, namely that it is possible to abstract from the phenomenal world schemes with operational invariants that can be found across a diversity of circumstances. Without claiming that teachers start each day from scratch, the TO understands that what is gained from the critical analysis of a classroom is a reflective vantage point that is impossible to cast in terms of invariants. In the TO, the reflective vantage point allows teachers to envision and imagine future rich pedagogical actions. The DAD, by contrast, does seek operational invariants of the classroom activity with a double purpose: to analyze the teacher's professional activity and to study how the teacher's organization promotes the students' development of competencies and skills.

Like all theories, the ones discussed here have been shaped by their own cultural-historical educational contexts. They do not only express key sensibilities of their cultures (e.g., the societal and institutional *telos* of education and corresponding ideas about teachers and students) but, in their theorizing, they also affirm those sensibilities. The dialogue has not been seen as a mechanism of assimilation. The dialogue has been rather an opportunity for A to better know B, and for B to better know A, and in doing so for A (resp. B) to better know itself.

Acknowledgements This paper is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada. We are grateful to the reviewers and editors for their comments and critique.

Data availability statement Anonymized data generated and analyzed during the current study are available from the corresponding author on reasonable request.

References

Adler, J. (2000). Conceptualising resources as a theme for teacher education. *Journal of Mathematics Teacher Education, 3*, 205–224.

Bakhurst, D. (2011). *The formation of reason*. Wiley-Blackwell.

Bartolini Bussi, M., & Mariotti, M. A. (2008). Semiotic mediation in the mathematics classroom. In L. English (Ed.), *Handbook of international research in mathematics education* (2nd ed., pp. 746–783). Taylor and Francis.

Bikner-Ahsbahs, A., & Prediger, S. (2014). *Networking of theories as a research practice in mathematics education*. Springer.

Brousseau, G. (2005). *Theory of didactical situations in mathematics*. Kluwer.

Drijvers, P., Doorman, M., Boon, P., Reed, H., & Gravemeijer, K. (2010). The teacher and the tool: Instrumental orchestrations in the technology-rich mathematics classroom. *Educational Studies in Mathematics, 75*(2), 213–234. https://www.jstor.org/stable/40928556

Gueudet, G., & Trouche, L. (2009). Towards new documentation systems for mathematics teachers? *Educational Studies in Mathematics, 71*(3), 199–218. https://doi.org/10.1007/s10649-008-9159-8

Leontiev, A. N. (2005). The genesis of activity. *Journal of Russian and East European Psychology, 43*(4), 58–71. https://doi.org/10.1080/ 10610405.2005.11059253

Lerman, S. (1996). Intersubjectivity in mathematics learning: A challenge to the radical constructivist paradigm? *Journal for Research in Mathematics Education, 27*(2), 133–150. https://doi.org/10. 5951/jresematheduc.27.2.0133

Macherey, P. (1979). *Hegel ou Spinoza*. François Maspero.

Prediger, S., Bikner-Ahsbahs, A., & Arzarello, F. (2008). Networking strategies and methods for connecting theoretical approaches: First steps towards a conceptual framework. *ZDM – Mathematics Education, 40*(2), 165–178. https://doi.org/10.1007/ s11858-008-0086-z

Rabardel, P., & Bourmaud, G. (2003). From computer to instrument system: A developmental perspective. In: P. Rabardel, & Y. Waern (Eds.) Special issue "From computer artifact to mediated activity", Part 1: Organisational issues. *Interacting with Computers, 15*(5), 665–691. https://doi.org/10.1016/S0953-5438(03)00058-4

Radford, L. (2019). So, you say that doing math is like playing music? The mathematics classroom as a concert hall. *La matematica e la sua didattica, 27*(1), 69–87. https://rsddm.dm.unibo.it/wp-content/ uploads/2019/05/Radford-The-mathematics-classroom-as-a-conce rt-hall-MD-2019-27-1-4.pdf

Radford, L. (2021a). *The theory of objectification. A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill/Sense. https://doi.org/10.1163/9789004459663

Radford, L. (2021b). Davydov's concept of the concept and its dialectical materialist background. *Educational Studies in Mathematics, 106*, 327–342. https://doi.org/10.1007/s10649-020-09959-y

Radford, L., & Sabena, C. (2015). The question of method in a Vygotskian semiotic approach. In A. Bikner-Ahsbahs, C. Knipping, & N. Presmeg (Eds.), *Approaches to qualitative research in mathematics education* (pp. 157–182). Springer.

Ricœur, P. (1990). *Soi-même comme un autre. [Oneself as another]*. Éditions du Seuil.

Sfard, A. (2008). *Thinking as communicating*. Cambridge University Press.

Trouche, L. (2004). Managing the complexity of human/machine inter- actions in computerized learning environments. *International Journal of Computers for Mathematical Learning, 9*(3), 281. https://doi.org/10.1007/s10758-004-3468-5

Trouche, L., Gueudet, G., & Pepin, B. (2020). Documentational approach to didactics. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., pp. 237–247). Springer. https://doi. org/10.1007/978-3-030-15789-0_100011

Vergnaud, G. (1998). Toward a cognitive theory of practice. In A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity* (pp. 227–241). Kluwer.

Vergnaud, G. (2009). The theory of conceptual fields. *Human Development, 52*, 83–94. https://doi.org/10.1159/000202727

Vygotsky, L. S. (1978). *Mind in society*. Harvard University Press. Weber, M. (1992). *The Protestant ethic and the spirit of capitalism*. Routledge.