Radford, L. (in press). Early algebra: Simplifying equations. In *Proceedings of CERME 2022*. CERME.

Early algebra: Simplifying equations

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Placed within the early algebra research field (Blanton et al., 2017; Cai & Knuth, 2011; Kieran, 2018; Kilhamn & Säljö, 2019), this article focuses on young students' understanding of basic algebraic ideas around equations. The article seeks to contribute to the field by shedding light on Grade 3 students' meaning-making processes underpinning the simplification of equations and the algebraic operations involved. In the first part, I present a theoretical conception of algebraic thinking. I also describe two non-alphanumeric semiotic systems that played an important role in the students' dealings with algebra. In the second part, I discuss two episodes of students simplifying $ax + b = cx + d$ *equations.*

Keywords: Algebra, equations, isolating the unknown, semiotics.

Algebraic thinking

One of the most enduring problems with which mathematics educators have been confronted is the problem of characterizing algebra and clarifying what makes it different from arithmetic. Two main solutions have been suggested. One consists in equating algebra with the *use of letters*. The other consists in conceiving of algebra as focused on *operations* rather than on results. While the first solution offers a very narrow conception of algebra—impeding teachers from recognizing algebraic thinking in activities based on types of mathematical representations different from letters—the second one offers a very narrow conception of arithmetic, which becomes demoted to simple computation.

In previous work (Radford, 2014) I have suggested three elements to characterize algebraic thinking:

(1) *Indeterminacy of magnitudes*: algebraic thinking involves *indeterminate* magnitudes. These can be unknowns, variables, parameters, etc.

(2) *Denotation*: the indeterminate quantities involved must be named or symbolized. This symbolization can be carried out in several ways. Alphanumeric signs can be used, but not necessarily. The denotation of indeterminate quantities can also be symbolized by means of natural language, gestures, unconventional signs, or even a mixture of them.

3) *Analyticity*: algebraic thinking (a) calculates/operates with indeterminate magnitudes *as if they were known* and (b) treats the mathematical relations featuring determinate and indeterminate magnitudes (equations, formulas, expressions, etc.) in a *deductive* manner.

Simplifying equations

Drawing on the aforementioned conception of algebraic thinking, in what follows, I report on the results of a teaching-learning activity in a Grade 3 class (8-9-year-old students). The activity was based on the use of two non-alphanumeric semiotic systems: a *Concrete Semiotic System* (CSS) and an *Iconic Semiotic System* (ISS) through which students could translate simple word-problems into linear equations.[1] The CSS is comprised of material objects: a) paper envelopes that each contain the same unknown number of cardboard cards; b) cardboard cards, and c) the equal sign. The envelopes played the role of unknowns while the card played the role of concrete numbers (constants). The ISS is derived from the CSS: it replaces concrete objects with iconic drawings { , , =, ↑}. The additional "arrow" sign replaces *actions* performed on concrete cards or envelopes of the CSS during the process of simplifying equations. The students could substitute the arrow by simple lines indicating that a card or envelope (or sets of) are removed. The range of problems that can be formulated in natural language and translated into the CSS and ISS is very limited, but it is enough to ensure that young students have their first encounter with algebraic thinking.

The research question that this paper seeks to address is about the identification of the teacher and students' meaning-making processes underpinning the understanding of algebraic techniques of isolating the unknown in one-unknown linear equations. Following the methodology of the theory of objectification (Radford, 2021), the data analysis involves a multimodal investigation of teaching-learning activity where students work in small groups and participate in collective discussions.

In Grade 2 the students started being familiarized with the isolating-the-unknown procedure using the CSS (Radford, 2017). At the beginning of the teaching-learning activity that I investigate here (which was the first Grade 3 activity on equations), the teacher organized a general discussion around the equation $3 + x = 7$. (Of course, no alphanumeric symbolism, was used in Grades 2 and 3). The students discussed various solving procedures: trial and error, comparison of terms (more on this below), and the isolating-the-unknown procedure. In Grade 3 the isolating-the-unknown procedure was not yet the students' first choice. The teacher had to ask, referring to what they had learned in Grade 2: "What do we mean by isolate? If I tell you, I'd like to isolate the envelope . . . " Cyr, one of the students, answered: "Does that mean like putting it alone?" When the teacher asked Cyr to articulate the idea, Cyr went to the blackboard and removed one card after another from each side of the equation, showing the procedure. The isolating-the-unknown procedure remained *shown* with actions rather than articulated with words. The teacher rephrased Cyr's actions: "If you remove one [card] on this side, what do you do?" Cyr answered: "I remove another one from there (the other side of the equation). Isolating-the-unknown procedure was a key aspect in the systematization of algebra conducted by Arab mathematicians in the 8th and 9th centuries (Al-Khwārizmī and others; see Oaks & Alkhateeb, 2007). It involves operations with known and unknown magnitudes to simplify equations. Mathematicians called these simplifying operations *al-gabr* and *al-muqābala,* and it is from the former that our modern term algebra borrows its name. By working with Cyr and by thematizing actions through language, the teacher strives to enable the students to reach a deeper level of understanding of the ideas underpinning the algebraic procedure. In the next sections, I discuss the work of one small group, focusing on two $ax + b = cx + d$ equations.

[1] Here is an example of a simple word problem: "Sylvain and Chantal have some hockey cards. Chantal has three cards and Sylvain has two cards. Their mother puts some cards in three envelopes and makes sure to put the same number of cards in each envelope. She gives one envelope to Chantal and two to Sylvain. Now the two children have the same number of hockey cards. How many hockey cards are inside each envelope?" (Radford, 2017, p. 18).

The equation $2x + 1 = x + 6$ in the CSS and the ISS

The two equations were given in the ISS. They are translations of a story in which two children have cards and envelopes. Each envelope has the same unknown number of cards, and both children have in total the same number of cards (see footnote 1). The exercise of translating stories of this type into the ISS was done in Grade 2 and continued in Grade 3. In this section I discuss the students' dealings with the equation $2x + 1 = x + 6$ (Figure 1.1) and in the next section I discuss the equation $3x + 1 = 5 + x$ (Figure 1.2).

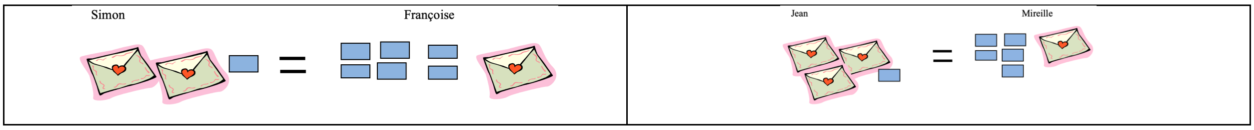

Figure 1. The equations $ax + b = cx + d$ as presented to the students in the ISS

Using a kit of envelopes and cards, the students were asked to make an equation and solve it, then draw their procedure. The idea was, hence, to have the students solve the equation first in the CSS, then using the ISS. The students made the equation in the CSS (Figure 2.1). Then, they drew the equation in the ISS. Elsa says: "We must remove that (*she circled the card on the left side of the equation*) so that there are just envelopes, do you remember? (*Then she removes one card on the other side)* 1, 1." *(Figure 2.2)*. The answer is found by the *comparison method* (i.e., the students compare the equal to the equal and associate the remaining parts of the equation: in this case, one envelope on the left side is equal to the envelope on the right; hence, the other envelope is equal to the five remaining cards).

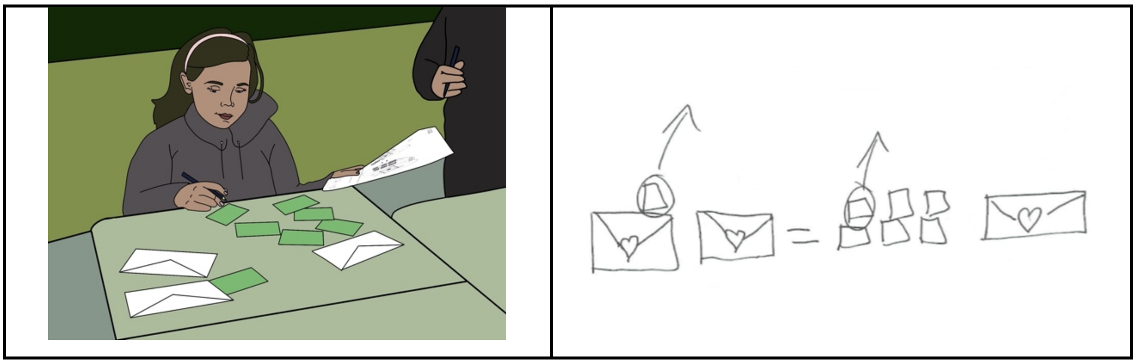

Figure 2. Solving the equation $2x + 1 = 6 + x$ in the CSS and the ISS

The teacher arrives and asks the students to explain their procedure. The students construct again the equation in the CSS. They remove one card from each side of the equation. The teacher says: "You are in the process of isolating! . . . How many envelopes do you want on one side?" Puzzled by the question, the students look at each other. One moment ago, Elsa mentioned the idea of having envelopes on one side. The teacher's intervention pushes the conversation further. On the one hand, the teacher acknowledges that the students are in the process of isolating the unknown. On the other hand, she raises a question that deals with something that has not been considered by the students. It is this unconsidered aspect of the simplification of the equation—a mathematical operation that would lead from $2x = x + 5$ to x equal to something—that puzzled the students.

1 Teacher: You want to know how many cards there are in ONE envelope (*she points to the envelope several times when she says ONE*) . . . First of all, you did this (*she removes a card from*

each side) you removed a card . . . Okay, what happens now? There are 2 envelopes (*pointing to the envelopes on one side of the equation*), then (*pointing to the objects on the other side of the equation*) 1 envelope and 5 cards.

2 Cora: We counted all these (*points to the cards*). It's 5. So, it (*pointing to 1 of the envelopes*) should have 5 too (*see Figure 3.1*).

3 Teacher: How do you know?

4 Elsa: We are going to remove (*she removes 1 envelope from the left side; see Figure 3.2*).

5 Teacher: You're removing 1 envelope?

6 Elsa and Cora: Yes. (*Elsa removes an envelope from the other side as well; Figure 3.3*).

7 Teacher: Why did you choose to do that?

8 Cora: Because this (*the sides of the equation*) must be equal.

9 Elsa: because we must remove; because there must be only 1 envelope left (*she takes the envelope that is left*)

10 Teacher: Is it okay to remove 1 envelope and then 1 envelope? Is your equation still equal?

11 Cora: Yes!

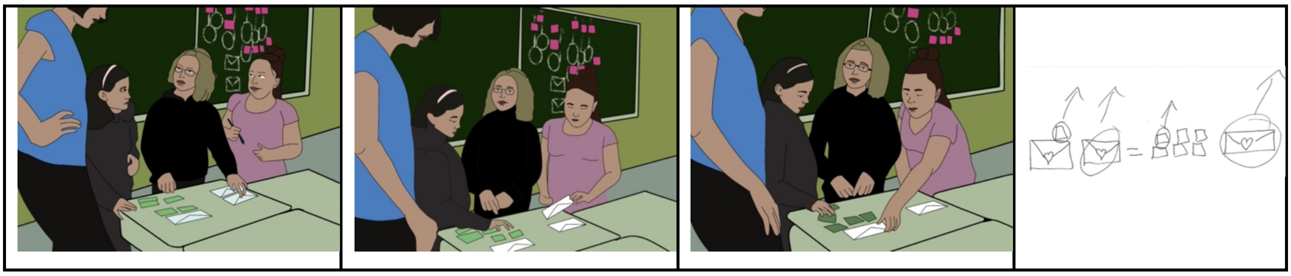

Figure 3. The students and the teacher discussing the equation $2x + 1 = x + 6$

In Line 1 the teacher starts simplifying the equation as the students did. She says: "First of all, you did this" and removes one card from each side. Then, in an encouraging tone, she asks "What happens now?" In Line 2 Cora resorts to the comparison method, but the verbal articulation of ideas leaves important relations unaccounted for. These are the relations that the teacher asks for in Line 3. In Line 4 Elsa starts removing one envelope from each side. The teacher wants to make sure that the students understand the idea behind the "removing" operation. So, in Line 7 she asks for reasons. In Lines 8 and 9 the students offer two answers: Cora's focuses on the conservation of the equality between both sides of the equation; Elsa's focuses on the idea of ending up with one envelope. In Line 10 the teacher wants again to make sure that there is a clear understanding of the actions that are carried out to simplify the equation. When the teacher leaves, the students come back to the equation in the ISS and remove one envelope from each side (Figure 3.4).

So far, the isolating-the-unknown procedure has necessitated the application of a key operation: *removing* equal things from both sides of the equation. In the next equation an additional mathematical operation is required. Let's turn to the students' investigation of this equation.

The equation $3x + 1 = 5 + x$ in the CSS and the ISS

The students tackle the equation $3x + 1 = 5 + x$. They construct the equation in the CSS and, instead of solving it with the help of concrete materials, they draw the equation.

Cora starts by removing one envelope from each side. After that, she removes one card from each side (Figure 4.1).

12 Elsa: You only removed 1, but there must be only 1 envelope left. That's a problem. (*They think for a while; then Elsa continues*). Four [cards], but there's not another envelope here (*points to the right side of the equation*).

13 Cora: There are 4 cards left, that's 4, we must remove these cards (*she circles the 4 remaining cards on the right side of the equation*) . . . And here (*she points to 1 of the remaining envelopes on the left side of the equation*) there are 0 [cards].

14 Elsa: Yes but look! If there is 0 [cards] in the envelope, this (*pointing to the envelope on the right side of the equation*) will be 4 and this (*pointing to an envelope on the left side*) will be 1 [meaning perhaps 0]. But the 2 [envelopes] must have the same exact (*she points to the drawing*), the 2 [envelopes] must have the same number [of cards].

15 Cora: (*Explaining the idea again*) We removed that (*the 4 cards*).

16 Elsa: Then, there are 0, but there must be some cards [in the envelope].

17 Cora: Why?

18 Elsa: Here you have to remove this, here you remove this (*points with her pen to her drawing*) and you can't remove that [the 4 cards on the right side], because there are not 4 other [cards] here [on the left side] that you can remove . . .

Here, the students find themselves in a new situation. While in the previous problem, removing the same number of cards and envelopes was sufficient to isolate the unknown, in this problem the "removing" operation is not enough. They end up with two envelopes on the left side of the equation and four cards on the right side. They cannot continue removing envelopes for, as Elsa notes in Line 12, there are no more envelopes to remove on the right side. And "That's a problem." Cora suggests removing the four cards on the left side, which will lead them to zero cards. She then assigns zero cards to one of the two envelopes on the left side, which means that there are four cards in the other envelope. Elsa points out two problems with Cora's suggestion. First, she argues that all envelopes must have the same number of cards (Line 14). Second, simplifying entails removing the same things *on both sides* of the equation (Line 18). This requirement or condition is violated.

The students reach an impasse. "On est en train de se chicaner pour la réponse" ["We are having an altercation over the response"]. They tried to call the teacher, but she was busy discussing with another group. I was videotaping this group; I removed my headphones and went to talk to the students. I suggested that they use the concrete material (envelopes and cards). The students constructed the equation again and proceeded to remove one card and one envelope on each side.

19 Elsa: There are still 2 envelopes left (*see Figure 4.2*).

20 Mia: Then, there are 2 (*pointing to 2 cards*) here (*pointing to 1 of the envelopes*) and 2 (*pointing to the 2 remaining cards*) here (*pointing to the other envelope; see Figure 4.3*).

21 Cora: There must be 1 envelope!

22 Elsa: (*She removes 1 envelope and moves the cards to the other side of the equation; see Figure 4.4*)

Figure 4. Discussing the solution of $3x + 1 = 5 + x$ in the CSS

In Line 20 Mia suggests an idea. However, the idea is not taken into consideration by the other students. Perhaps because the idea is not framed within the kind of actions that the students recognize as legitimate in solving the equation. Yet, we see in Figure 4.4 that Elsa, in despair, removes one envelope and transfers the cards to the other side of the equation, placing them underneath the envelope, twice breaking the "do the same on both sides" rule. Not finding a convincing way to proceed, Elsa (like Cora in Line 13) steps outside the boundaries of the algebraically thinkable that they have established so far. The situation once again became very tense as we saw in Line 18. Elsa says that they are still altercating and laughs. Cora says: “OK. We’ll do it again!” They remove one card and one envelope from each side of the equation.

23 Elsa: There are 4 [cards]. We must have just 1 envelope remaining. So, we must remove 1 [envelope]; we don’t have a choice (*she removes the envelope*).

24 Cora: Yes, but if we remove 1 … we must remove something else (*she points to the other side of the equation*).

They discuss for a while and come back to the simplified equation ($2x = 4$). After looking attentively at the 4 cards and the 2 envelopes, Elsa says that she has an idea:

25 Elsa: Wait, wait. Here's my idea. Because we have 2 [cards] here (*with each hand, she takes 2 cards from the bunch of 4 cards; then, she slowly moves the 2 hands holding the cards and puts them in front of each of the envelopes; see Figure 5.1. When the cards arrive at their destination, she says*) 2 in each envelope.
She immediately starts the explanation again: she slides the 4 envelopes as she did before, on one side of the equation. She says:

26 Elsa: Separate this [the 4 cards] into 2 (*as she says this, she separates the envelopes; see Figure 5.2. She then slides them in front of each envelope*); there are 2 in each envelope.

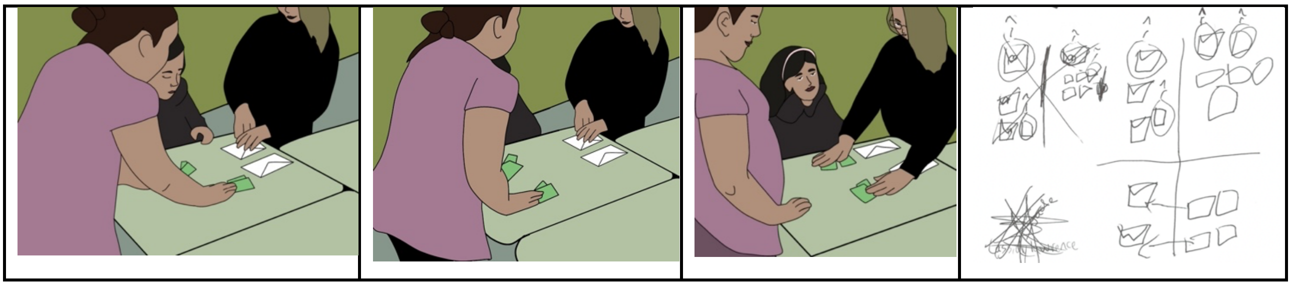

Figure 5. Finding (again) how to solve the $2x = 4$ equation

Elsa’s demonstration is followed by Mia’s reaction:

27 Mia: This is what I said before, but you, you were . . .
28 Elsa: (*completing Mia’s sentence*) . . . altercating!
29 Mia: . . . you said, no, no . . .
30 Elsa: I am sorry, Mia!

Cora makes the equation again and goes through the steps to isolate the unknown. When she reaches the equation $2x = 4$, she says:

31 Cora: We are going to separate . . . (*and slides the 2 cards towards 1 envelope and 2 cards towards the other envelope; see Figure 5.3*).

Mia is right in arguing that she suggested long before (Line 20, Figure 4.3) that each envelope has two cards. However, her suggestion was not articulated in terms of a separation of cards. In Elsa’s case, the solution appears first in an embodied way: “Wait, wait. Here's my idea. Because we have 2 [cards] here . . . 2 in each envelope.” The few uttered words are accompanied by a complex set of grabbing and sliding actions that remain unqualified linguistically. The linguistic articulation appears when she starts again the process of solving the problem. She says: “Separate this into 2, there are 2 in each envelope.” Although the importance of the kinesthetic dimension that accompanied the problem-solving procedure does not disappear, the thematic articulation in language is much more sophisticated. The new mathematical operation is named “to separate.” This new operation is a precursor of what will later be known as the algebraic operation of division. The Arab mathematicians had a term for it: *al-radd*, decreasing the coefficient of the unknown to 1.

In previous research we have found that at the precise moment of learning something, the students undergo a process where mathematical thinking becomes reorganized; what previously took many words and actions becomes reorganized and contracted: the students filter the necessary from the unnecessary and their semiotic activity becomes contracted. There is a *semiotic contraction* (Radford, 2021). Here, we see the opposite process: in Line 26 Elsa adds actions and words to signify the emerging operation. There is a *semiotic expansion* that allows her and her teammates to better notice the operation and endow it with meaning.

The students kept solving with their hands the equation in the CSS several times. It seems that seeing was not enough and that feeling with their hands and their bodies was necessary. Then, they drew their solution in the ISS. The new operation requires a sign to be expressed. Figure 5.4 shows that the students chose an arrow, which is reminiscent of the sliding action that makes the two cards correspond to each envelope. The sign is an icon of the action.

Concluding remarks

This article dealt with the topic of equations in early algebra. It focused on the way Grade 3 students dealt with some of the key algebraic ideas that underpin the simplification of equations. In the first part, I suggested that the characterization of algebra (a) as calculation with letters or (b) as focused on operations rather than on their results are both unsatisfactory. In the first case, the characterization falls short by limiting the scope of algebra; in the second case it fails by downplaying the complexities of arithmetic thinking (which is reduced to trivial calculations). Based on historical-epistemological considerations (Radford, 1995; 2001), I suggested a conception of algebra that stress the authenticity of denotating unknown magnitudes in various ways and emphasizes the analytic-deductive nature that underpins algebraic inquiries. If we know that second degree equations have at most two solutions, it is not because we guessed the solutions, it is because they were *deduced*.

Starting from these premises, the Grade 3 teaching-learning activity was didactically organized around the use of two semiotic systems: the CSS and the ISS. The excerpts analyzed here started with a classroom general discussion around different methods to solve the equation $3 + x = 7$. According to the definition of algebra suggested in the first section of this paper, the solution of equations $ax + b = c$ does not include the operation of the unknown. As a result, in solving those equations the students have not stepped yet into the realm of algebra (Filloy & Rojano, 1987). However, the investigation of the equation $3 + x = 7$ provided the students with an opportunity to continue familiarizing themselves with the isolating-the-unknown procedure that they encountered in Grade 2. In this sense the equation $3 + x = 7$ was envisioned rather as a propaedeutic step towards tackling equations of the type $ax + b = cx + d$ algebraically, something that the students did in the second part of the teaching-learning activity. We can see in Figures 3.2 and 3.3 the moment at which Elsa applies the *al-muquabāla* or removing operation that was previously applied to the constants in solving the equation $3 + x = 7$ to the equation $2x + 1 = x + 6$. The “removing” operation now acquires a new and more developed meaning. It requires seeing the unknown and the equation under a new light. It is this new aspect of the mathematical activity that leads the teacher, in Line 10, to ask two fundamental questions: "Is it okay to remove 1 envelope and then 1 envelope? Is your equation still equal?" More generally, the CSS- and ISS-based teaching-and-learning activity made room for meaning-making processes out of which the Grade 3 students to generate, in their work with the teacher, two important algebraic ideas that underpin the simplification of equations: "removing" (removing equal terms from both sides of the equation) and "separating" (i.e., reducing the coefficient of the unknown to 1), those operational ideas that Arab mathematicians referred to as *al-gabr* / *al-muqābala and al-radd,* respectively (Oaks & Alkhateeb, 2007). The emergence of these sensuous and embodied operations served as foundational blocks for the students' encounter with algebraic alphanumeric symbolism, which happened one year later, when they were in Grade 4.

Acknowledgment

This article is a result of a research program funded by the Social Sciences and Humanities Research Council of Canada / Le conseil de recherches en sciences humaines du Canada (SSHRC/CRSH).

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