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Body, matter, and signs in the constitution of meaning in mathematics

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7.1. Introduction

In a kindergarten class in Ontario, Canada, 5-6-year-old children work on the extension of a 'logical' sequence made up of 'dogs' and 'houses' (figure 7.1).

Figure 7.1. *The sequence explored by children in a kindergarten class*

In groups of two, the children draw small cards from a bag, one card at a time (figure 7.2a). Each card contains three, four or five terms, each term being 'dog' or 'house'. For example, one card shows the sequence 'house', 'house', 'dog'; another card shows the sequence 'house', 'dog', 'house', 'house'. After drawing a card, the children place it at the end of the sequence. They have to say whether or not the card extends the sequence.

In one of the groups, Chloé and Antoine work together:

1. Chloé: (*Gives the bag to Antoine*) Your turn, Antoine.

2. Antoine: Perhaps I will draw a good card! (*He chooses a card from the bag. He looks at the card*). Ha! (*He obtains a card that shows 'house, house, dog'; he places it at the end of the sequence* (figure 7.2b)). Ha ha! (*He goes through the elements of the sequence from the first term.*) Dog,

house, house (figure 7.2c); dog, house, house; dog, house, house; dog, house, house; dog! (figure 7.2d). Yes!

3. Chloé: You have it Antoine! (*She takes the bag and draws a card; then, they continue to study whether the new card extends the sequence or not*).

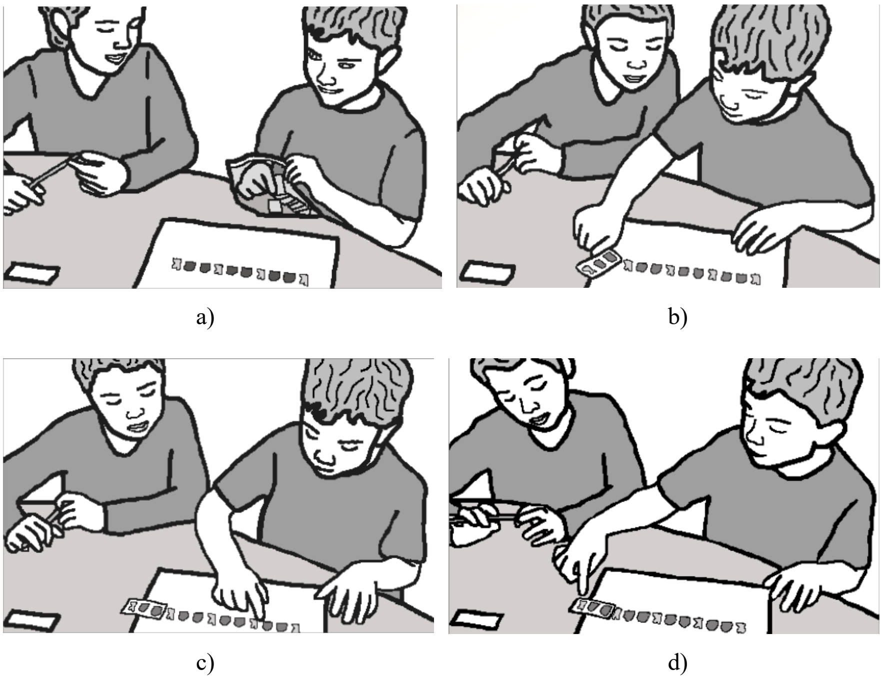


Figure 7.2. *The children exploring the extension of the sequence*

This short passage shows us a whole series of resources to which the children resort to respond to the mathematical task. It is a series of semiotic resources that includes:

– The given *signs* ('house', 'dog' and their order);

– a *semiotic-perceptual* activity;

– *gestures*;

– *language*; *and*

– *rhythm*.

Each of these resources operates at a different level of meaning. Thus, in the passage above, Antoine places the card at the end of the sequence (figure 7.2b). Then, after exclaiming 'Ha ha!', he draws on linguistic meanings: he names the terms. Naming does not work alone. It is supported by successive *indexical gestures*: Antoine names and points to the elements of the sequence, one after the other, from the start. These gestures help Antoine organize his reading of the sequence. They also help to hold his attention. There is more. Antoine's gestures are also deployed in *synchrony* with *words* and an important *perceptual activity* (see figure 7.2c). These semiotic resources do not 'say' the same thing (Radford 2004); they do not signify in the same way.

Because they come from different semiotic *modalities*, together, the aforementioned resources make the child's activity a *multimodal semiotic activity* (Radford *et al.* 2009).

We might ask the following question: to understand the child's production of mathematical meaning, is it necessary to linger on this multimodal semiotic activity? In mathematics education, the question of meaning has often been studied with a focus on the modalities of writing or orality and sometimes both. For example, the work of Raymond Duval (1995, 1998) and Duval and Pluvinage (2016) focuses above all on pupils' written production. The question is therefore entirely relevant: is it worth the trouble of dwelling on *other* semiotic modalities? What is there to be gained?

These questions and other similar questions made their appearance in the domain of mathematics education some years ago. They gave rise to a different conception of the mind – one that relies on the body and that has been termed the *embodied mind*. Often, the ensuing theoretical perspective is simply called *embodiment.*

The central point in this perspective is that thought is not purely a mental phenomenon. On the contrary. To function, thought has recourse to a series of semiotic modalities, the study of which can shed new light on the pupil's production of mathematical meaning. In the preceding example, we see the crucial role of rhythm in the actions of Antoine and his teammate in extending the sequence. Antoine builds rhythm through a series of speech intonations and tonalities, through a cadence of words and gestures, and also through silent pauses. He pauses each time the pattern 'Dog, house, house' is finished, and the same pattern recommences. When Antoine arrives at the end of the extended sequence, he hears himself say the repeated pattern. The solution to the task is not found in the logical movement of thought alone, but also in rhythmical bodily actions. It is this multimodal activity that makes it possible to produce mathematical meaning. It reveals to us the *sensual way* in which the production of meaning is accomplished. It is through the complex, emerging, always new organization of a series of signs of a different nature – the mathematical signs, gestures, kinesthetic action, perceptual semiosis, language and its prosody – that Antoine manages to give a *meaning* to the sequence and that he is able to say whether or not a particular card extends the sequence.

We have been able to observe how tasks of this type become difficult, indeed impossible to conduct, as soon as we forbid the young child from pointing or speaking or from (re)visualizing the terms of the sequence. We have also seen this with students at the end of secondary school when they have to solve a system of linear equations with two unknowns. From the moment they were forbidden to write, students failed to solve the system of equations. Everything happens as if thought needs to cling to signs of different modalities in order to come to life.

Embodiment, understood therefore as a theoretical approach that invites us to reconceptualize the way in which human beings think, a way that includes the body or the flesh and the materiality of the world (signs, artifacts, etc.), has become a relatively important sub-domain of research in the teaching and learning of mathematics. In 2009 there was a special issue on this theme in *Educational Studies in Mathematics* (Edwards *et al.* 2009) and, more recently, a chapter dedicated to it in the *First Compendium for Research in Mathematics Education* (Radford *et al.* 2017).

The arrival of embodiment in our field of research has led, however, to a series of theoretical and practical problems. It has become necessary to clearly reformulate not only our understanding of thinking, but also its relationships with the body, the flesh and the senses. It appears that, without taking a clear position on this subject, analysis of multimodal semiotic activity may remain superficial, limited to *exhibiting* gestures, words and symbols in the situation studied. The problem is that simply exhibiting the different facets of embodiment may not be enough to reach convincing explanations. It is here that a theoretical framework or a precise theoretical approach becomes vital.

In this chapter, we address the question of embodiment in the formation of meaning in mathematics. We focus in particular on the formation of meaning in its relationship with the body, matter and signs. The first section of this chapter deals briefly with some historical and contemporary approaches to the relationship between thinking and body. This short survey intends to throw some light on some contemporary understandings of embodiment and their challenges. We shall see, for instance, that, by focusing on the body only, some contemporary approaches fall into a new form of subjectivism – a form of radical empirical subjectivism. We opt for a different theorization of embodiment: a monistic position articulated in the seventeenth century by Benedict of Spinoza (2010) and developed later by Karl Marx (see Fischbach 2014), Lev Vygotski (see Clot 2015), Alexis Leontiev (1976) and Evald Ilyenkov (1977) from a dialectical materialist perspective. This 'materialist Spinozism' allows us to arrive at a cultural-historical sensuous understanding of cognition (Radford 2013, 2014, 2015). We return to this idea in the second section of this chapter to explore how, phylogenetically speaking, at the end of the Late Middle Ages and the start of the Renaissance, the historical emergence of algebraic symbolism led to a series of transformations in meaning. Algebra moved from a practice organized around *orality* to a *visual* semiotic practice deployed around the written sign. The transformations of meaning went hand in hand with a transformation of the human senses, especially perception. We advance the idea that the historical transformations of the human senses should be understood in light of the emergence of a visual culture *par* *excellence* – the culture of the Renaissance. The phylogenetic analysis of the second section of the chapter gives way in the third section to an ontogenetic analysis related to an algebra lesson in a Grade 6 class of a primary school. In this lesson, to encounter symbolic algebra, the pupils go beyond the practice of algebra organized around orality and the use of material objects, to reach a new practice of algebra based on visual-symbolic thinking.

7.2. Body, matter and thought

7.2.1. *From Antiquity to the Middle Ages*

The question at the center of this section – that of the relationship between body, matter and thought – is a question that has been the source of much debate throughout Western history. In general, in each historical period, we find divergent views in the conceptions that were made about the epistemic role of the body and matter. In Antiquity, for example, Plato, whose conception of the world reflected the position of a noble class that disdained manual labor, considered the body as a nuisance or an obstacle to the acquisition of true knowledge. In his famous *dialogue*, *Phædo*, one of the characters, Simmias, is invited to decide who, among all types of men, is able to attain true knowledge. Was it not he, asked Socrates

> 'who pursues the truth by applying his pure and unadulterated thought to the pure and unadulterated object, cutting himself off as much as possible from his eyes and ears and virtually all the rest of his body, as an impediment which by its presence prevent?' (Plato 1961, p. 48, 65e-66a)

In such an aristocratic conceptual framework, it was not therefore through sight or touch that one could access true knowledge. Aristotle, however, articulated a more worldly epistemology. In *Metaphysics*, sight, he said, 'is the sense that particularly produces cognition in us and reveals many distinctive traits of things' (Aristotle 1998, p. 4, Book Alpha, 980a).

In the Middle Ages, to a large extent, the body and the flesh were associated with the corrosive omnipresence of sin. The flesh was 'during this period an expression and an externalization of the debased human condition, carrying the hereditary imprint of sexuality and death' (Bierfnoff 2002, p. 23). In its conception of the body as the vehicle of sin, the Middle Ages gave a preponderant place to sight. A famous anonymous text of the thirteenth century, *Ancrene Wisse* (*The Nun's Rule*), reviews the five senses, starting with sight. The text tells us, in fact, that it was through sight that Eve sinned.

> 'And it is written of Eve, mother to us all, that sin entered into her for the first time through her eyesight, that is, "Eve looked on the forbidden apple, and saw it fair, and began to take delight in beholding it, and set her desire

upon it, and took and ate of it, and gave of it to her lord [Adam]".' (Morton 2000, p. 22)

The author goes on to say to his nun readers: 'When thou lookest upon a man thou art in Eve's case; thou lookest upon the apple' (Morton 2000, p. 22). But sight, linked to vision (of which one of the understandings is 'cognition' in the sense of knowing), used wisely, can also lead to salvation. It is precisely sight that is the medium of contemplation (*contemplatio*), which can lead to the 'union with God' (Gunn 2008, p. 161). *Hugues de l'abbaye de St. Victor*, in the twelfth-century text, *In Salmonis Ecclesiasten homiliæ XIX*, tells us that contemplation 'is that acumen of intelligence which, keeping all things open to view, comprehends all with clear vision' (Petry 2006, p. 90).

However, these medieval conceptions of the body, of the senses, and of sight in particular, which are sometimes associated with the temptation of sensual appetites and with sin, and sometimes with the path to salvation through ascetic practice for the body and the senses, are in conflict with the appearance of a new conception of nature that Roger Bacon and others promoted. Following Aristotle, these natural philosophers thought of sight and vision as the bearers of a positive epistemology: 'vision alone reveals the differences of things; since by means of it we search out experimental knowledge of all things that are in the heavens and in the earth' (Bacon, cited in (Biernoff 2002, p. 63)). Biernoff (p. 64) suggests that 'the bodily senses, freed (at least in this context) from their Augustinian association with fleshly sensuality, were able to be aligned with experimental knowledge' (see also (Kärkkäinen 2011)).

7.2.2. *Rationalism and empiricism in the seventeenth and eighteenth centuries*

These contradictory views on the body, the flesh and the senses resurged with particular force in the seventeenth and eighteenth centuries. We find there in fact, a rationalistic current that extended the medieval ascetic view of the body and an empiricist view that extended those of the natural philosophers. We cite, among the rationalists, René Descartes, for whom knowing a thing equated to having a clear understanding of that thing. And for him, this understanding was not ensured by the senses. Thus, at the end of the *Second Meditation*, to explain how the body and things outside us become known, Descartes said that we arrive at our knowledge of the body neither through the imagination nor through the senses. It is not because we can touch the body that we know about it, because true knowledge is guaranteed, according to Descartes, by the faculty of understanding (*la faculté d'entendement*; i.e., the faculty through which we perceive ideas). As he put it, we know the body because 'we conceive of it through thought' (Descartes 1637, p. 26). Thus, the knowledge of things is not to be found in the materiality of things or in sensation. Gottfried Wilhelm Leibniz made similar comments: 'all the thoughts and actions of our soul come from its own basis, but cannot be given to us through the senses' (Leibniz 1887, p. 79), so that one can 'manufacture this knowledge for oneself [arithmetic and geometry] in one's study, and even with one's eyes closed, without understanding by sight or even by touch the truths that we need' (Leibniz 1887, p. 84).

In opposition to this ascetic view of the body, there existed at the same time the path of empiricism advocated by George Berkeley, John Locke and David Hume, among others. For Hume, the ideas that we form of the world are the result of sensory impressions; that is, perceptions obtained thanks to the senses. In Hume's account, the most complex ideas are only compositions, transpositions, augmentations and diminutions of 'the materials afforded us by the senses and experience' (Hume 1921, p. 16). The only way through which an idea can have access to the mind is through the senses and sensation.

The rationalist and empiricist positions of the era can be summarized thus: for the empiricists, nothing can exist in the intellect unless it exists first in the senses. For the rationalists, however, nothing can exist in the senses if it is not first in the intellect.

Over the course of the eighteenth century, Emmanuel Kant articulated an epistemology that sought to bridge the rationalist and empiricist positions. In Kant's epistemology, in agreement with the empiricists, no conceptual object could be given to us without first passing through the senses (directly or through some form of representation). But the reverse is also true. In agreement with the rationalists, Kant stated that, without the faculty of understanding, no tangible object could be thinkable (Kant 2003, p. 93). Although embodiment and the senses play a more important role in Kant's theory of knowledge, their contributions, however, remain limited to providing the raw material to the faculty of understanding so that this faculty can be put in motion. According to Kant, knowledge is not the content of a generalized experience, since empirical data – data which passes through the body and the senses – is only thinkable because the understanding collects it and fills it with conceptual content.

7.2.3. *The body and the senses in contemporary research*

The empiricist and rationalist conceptions mentioned above are not without implications for research in education, because, as we know very well, Jean Piaget (1970) was inspired by Kant to build his genetic epistemology. And, like Kantian epistemology, Piaget's epistemology is presented as a middle point between rationalism and empiricism. Thus, for Piaget, the body and sensations play an important role. This is the case with the sensorimotor stage in conceptual development. However, although the body plays a role in explaining how individuals arrive at knowledge, it is only as a transitional stage towards abstract thought. The sensorimotor stage for Piaget is only an ephemeral passage towards the stage of formal operations, which brings the Piagetian approach (as well as Kant's) to the side of rationalist approaches. One of Piaget's collaborators, Hermine

Sinclair, notes that Piagetian genetic epistemology remains 'closer to rationalism than to the empirical hypothesis' (Sinclair 1971, p. 121).

Piagetian genetic epistemology has had a significant influence on approaches that focus on the learning of mathematics; this is the case, in particular, of the so-called 'process-object' theories; that is, theories that conceive of thought as developing from action to the structures of operational knowledge. Two examples are the 'actions, processes, objects and schemas' theory (APOS) (Dubinsky and McDonald 2001; Dubinsky 2002) and the 'three worlds of mathematics' theory (Tall 2013).

However, there are new trends in research that offer a different approach to the understanding of human cognition. They consider our tactile and kinesthetic experience of the world, and our interaction with artifacts and signs, as being much more than forms of access to increasingly abstract cognitive operational configurations. One of these trends is offered by George Lakoff and Rafael Núñez (2000). Starting from a linguistic tradition, they focused on metaphors to try to show that the fundamental concepts of mathematics come from our sensory experience and how this is conveyed and expressed in language. Several researchers have followed this route to study the learning of mathematics (see, for example, (Edwards 2009)). But there are other approaches, such as one, inspired by phenomenology, which emphasizes the carnal nature of thought (Roth 2011; Thom and Roth 2011) and another that underlines the material dimension of thought from the perspective of a new materialism (de Freitas and Sinclair 2013, 2014).

The perspective on corporeality that we present here, 'sensible cognition', is inspired by dialectical materialism. From a dialectical materialist perspective, sensation, the senses, and matter are considered an important part of the foundation of cognition and of any psychic activity (affection, emotion, volitional activity, etc.). This position would seem to draw us towards empiricism. This is not entirely true. In empiricist perspectives, sensation, the senses and matter appear as *already given* entities. In other words, they are considered to be the starting point for the study of the mind and cognition. This is the case with David Hume, for example, who considered the human sensory apparatus to be already given, always the same and ready to receive impressions of the world. The human sense remains unproblematized. A significant number of contemporary approaches to embodiment follow Hume and British empiricism. This is the case especially for the philosopher Maxine Sheets-Johnstone (1990, 2009) who, by trying to overcome the limits of rationalism, emphasized not the senses, but corporeal movement. She argues for the epistemological role of the body through its movement. One of her main claims is that the formative source of our ability to do things (*agency*) is to be found in our spontaneous corporeal movements. In this approach, corporeal movement is also the source of our subjectivity and our sense of self (*selfhood*) (Sheets-Johnstone 2011, p. 119).

For Sheets-Johnstone, we are first of all 'animated organisms'. It is in movement that we find 'the start of cognition' (Sheets-Johnstone 2011, p. 118); 'our first cognitive steps are made by means of our own movement' (Sheets-Johnstone 2011, p. 118). She notes:

> 'It is in and through movement that the life of every creature... acquires reality... In the beginning, we are simply imbued with movement – not only with a propensity to move, but with the real thing [i.e., a primary animation – LR]). This primal animateness, this primary kinetic spontaneity that infuses our being and defines our aliveness, is our point of departure for living in the world and making sense of it. It is the epistemological foundation of our learning to move ourselves with respect to objects, and thus the foundation of a developing repertoire of "I cans" with respect to both the naturel and artefactual array of objects that happen to surround us.' (Sheets-Johnstone 2011, p. 117)

In brief, from this perspective, the discovery by an individual of what their body can do is not the result of a contemplative rationalist cogitation. It is thanks to the movement of the body that the individual discovers a 'real kingdom of kinetic "I cans": I can stretch, I can twist, I can reach, I can turn, etc.' and that the body discovers an 'open domain of possibilities' (Sheets-Johnstone 2011, p. 117). It is from movement that 'kinesthetic awareness' comes to life and we arrive at cognition.

Like other contemporary works on corporeality, the work of Sheets-Johnstone has the great merit of making us rethink the role of the body. The problem with such a theoretical position, which has been common to all subjectivist positions since Hume in the eighteenth century, is that it focuses on the individual alone. This position forgets that individuals come to create concepts within a historical-cultural world that, before the body moves, present it with possibilities and constraints. In fact, the movements that materialize the 'I cans', movements at the basis of our sense of ourselves as individuals, happen and always take effect within economic, political, social, conceptual, cultural and historical networks. And these structuring networks would have little importance if they did not shape our movements in the world and what we can and cannot do there. The point is that these networks profoundly affect the way in which we arrive at knowledge. In the best case, Sheets-Johnstone's theoretical position finishes by offering a partial explanation of cognition and the sense of self – an explanation that remains without overcoming the limits and difficulties of subjectivist approaches in general. Such a position would be adequate if the subject of the discourse was Adam or Eve. For human beings like ourselves, who have arrived in a world with its historical and cultural forms of thinking and action already formed (although always in a process of transformation), explaining cognition from the body and its movement is certainly insufficient. For dialectical materialism, by contrast, our forms of thinking about the world, our forms of moving and feeling this movement are conceived as being entangled, from the start, in forms of sensation (sensitivities) constituted culturally and historically.

Let us take the example of hearing. In his book, *Joseph Haydn: La mesure de son siècle*, (*Joseph Haydn: The Measure of his Century*) Marcel Marnat (1995) mentions the fact that an authentic Schantz pianoforte was recently restored to play one of Haydn's sonatas. The idea was to restore this piano to listen to the sonata in its original 1790s splendor. The musicologist Marcel Marnat agrees that the restored piano had 'an exquisite sound', but, he says, the piano was:

> 'so powerless to offer *our* ears a sound impact equivalent to what it achieved in 1790. […] We forget too quickly that we wrongly make comparison with what came *after*, whereas listeners of the 'period' appreciated it in comparison with what they had heard *before*.' (Marnat 1995, p. 78-79)

Hearing has undergone historical-cultural transformations so that we no longer hear like people of the eighteenth century. The same argument can be made for the other senses, too. Concerning perception, chess players such as Jacob Aagaard (2004) argue that the appearance of electronic chess programs have produced transformations in players' perceptions. In short, the senses are transformed historically through social practices. What we find before us at our birth is not an empty space that the body traverses in its movements. Nor is it a space occupied by objects and artifacts that would be cognitively neutral. On the contrary, before us stands a world hewn and chiseled by previous generations – a world populated by knowledge; that is, historical and cultural systems that allow us to think, to feel and to act in certain ways.

The criticism of empiricism that we make here is not new. It is the criticism that Marx made of Feuerbach's materialism, a materialism of the eighteenth century 'which stops only at what catches the eye' (Marx 1982, p. 1078). Marx tells us:

> '[Feuerbach] does not see that the sensuous world around him is not a thing given immediately from all eternity, remaining ever the same, but the product of industry and of the state of society; and, indeed, [a product] in the sense that it is a historical product, the result of the activity of a whole succession of generations [. . .] Even the objects of the simplest ''sensuous certainty '' are only given him through social development, industry and commercial exchange. We know that the cherry tree, like almost all fruit trees, was introduced to our country by *commerce*, just a few centuries ago; and therefore only by this action of a definite society in a definite age it has become "sensuous certainty" for Feuerbach.' ((Marx 1982, p. 1078), italics from the original text)

According to Marx, the problem with this materialism is that it does not succeed 'ever in grasping the sensible world as the sum of the sensible activity of the individuals that form it' (Marx 1982, p. 1080) and in recognizing the fact that it is this sensible activity that produces and transforms our senses.

In the 'Parisian manuscripts', Marx writes:

> 'Only through the objectively unfolded richness of man's (sic) essential being is the richness of subjective *human sensibility* (a musical ear, an eye for beauty of form, in short, *senses* capable of human gratifications, senses confirming themselves as essential powers of man) either cultivated or brought into being. For not only the five senses but also the so-called mental senses – the practical senses (will, love, etc.) – in a word, *human* sense – the humanness of the senses – comes to be by virtue of its object, by virtue of *humanized nature*.' ((Marx 2007, p. 151), the italics are in the original text)

It is in the context of this theoretical position on the senses that one can, as has previously been suggested (Radford 2014), understand human cognition as *sensuous cognition*: a culturally and historically constituted multimodal sentient form of creatively thinking, acting, imagining, feeling, transforming and giving meaning to the world. This conception of cognition conveys the idea that our thoughts, our feelings, our acts and, in fact, all our relationships with the world (through taste, smell, hearing, touch, sight, etc.), are an intertwinement of our body and our material and ideational culture. And what makes this intertwinement possible is human *praxis*; that is, the sensuous, concrete activity of the individuals (Radford 2021).

In the following section, we will briefly consider the emergence of algebraic symbolism and the way in which it is accompanied by a reorganization of the senses, in particular hearing and sight.

7.3. The body and the historical emergence of algebraic symbolism

In the history of the West, it is known that the Renaissance was a particularly important moment in the search for an appropriate system of symbols to deal with algebraic problems. Before the formation of an appropriate semiotic algebraic system, mathematicians expressed everything in natural language. Then, they incorporated a series of abbreviations of words and mathematical operators, giving rise to what is called *syncopated algebra*. Jens Høyrup (2008) shows that our contemporary alphanumerical algebraic system was preceded by attempts that sought to conceptualize and express correctly and succinctly the successive powers of the unknown and to create diagrams of operations on algebraic expressions that remained operationally cumbersome to carry out using natural languages. The search for a proper system, freed from natural language, was not a direct process; it passed through a series of attempts, giving rise to what Høyrup (2008) calls 'the tortuous ways toward a new understanding of algebra'.

In this section, we would like to spend some time on the transition in mathematics practices that had an important impact on our senses: the transition from a practice of algebra based on orality to a visual practice based on a written symbolic system. To do this, we shall look at excerpts from two texts: one from the *Trattato d'Abaco*, a fifteenth century Italian manuscript by Piero della Francesca, and the other from *L'Algebra*, a book by Rafael Bombelli published in the sixteenth century.

The *Trattato d'Abaco* belongs to a genre of manuscripts developed by teachers of abbacus schools, which were frequented by the sons of artisans and merchants 'including those belonging to the highest mercantile patriciate' (Høyrup 2018, p. 2). The manuscripts produced by teachers at abbacus schools were often teachers' notes. In general, they contain a series of problems tackling subjects relating to commerce. The algebra taught there was not part of the curriculum; it was intended for amateurs in mathematics and for those who wished to become teachers in abbacus schools. Concerning teaching methods in these schools, Raffaella Franci (1988) noted that these were based on repetition; students had to do numerous exercises, written or oral, ranging from the simplest to the most complex. Beyond the many exercises carried out at school, homework was also given. Franci cites a fifteenth-century manuscript, in which we read:

> 'And note this general rule: each evening, students are given problems matching their abilities, which they should complete and return the following morning [. . .] And note that when [the next day] is a holiday, [the number of] problems mentioned above must be doubled.' (Franci 1988, p. 185)

Figure 7.3 includes the first excerpt we would like to discuss here. It contains the statement of a problem from a manuscript by Piero della Francesca and its solution.

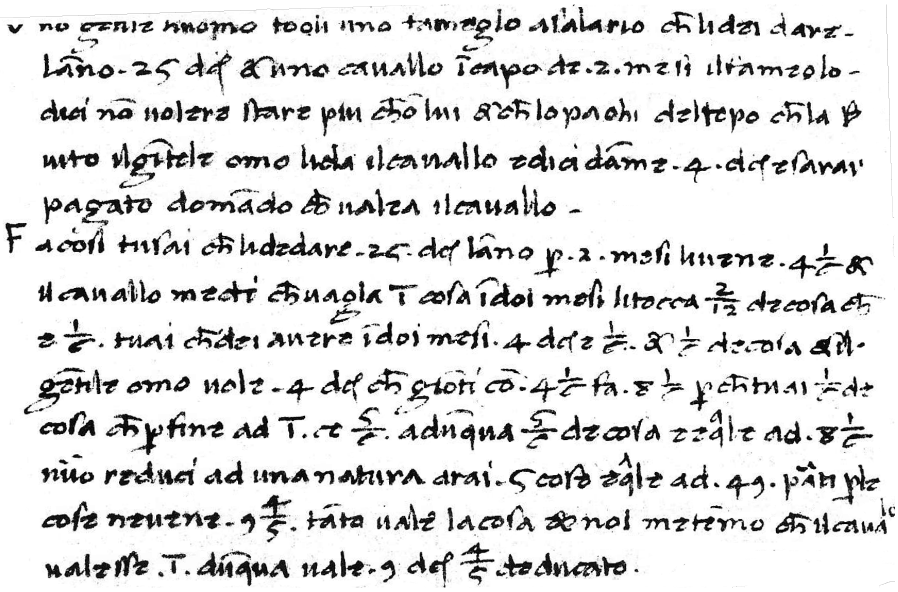


Figure 7.3. *Excerpt from Trattato d'Abaco (della Francesca 1460)*

Box 7.1 includes an English translation based on Arrighi's (1970) Italian edition of the manuscript.

A gentleman hires a servant with a salary; he must pay him 25 ducats and a horse per year. At the end of two months, the servant says that he no longer wishes to remain with him and that he wishes to be paid for the time he has served. The gentleman gives him the horse and says to him: give me 4 ducats and you will have been paid. I ask you: how much was the horse worth?

Do this. You know that you should give him 25 ducats per year, for 2 months this makes 4 I/6; and you say that the horse is equal to $\overline{I}$ thing, for 2 months he earns 2/12 of the thing that is I/6 (*sic*: it should be $\overline{I}/6$ and not I/6 as shown in the manuscript that Arrighi faithfully transcribed). You will have in 2 months 4 ducats and I/6 and I/6 of the thing. And the gentleman wants 4 ducats which added to 4 I/6 make 8 I/6. Now, you have 1/6 of the thing, [and] up to $\overline{I}$ there is 5/6 of the thing; so 5/6 of the thing is equal to 8 I/6 numbers. Reduce to a nature [i.e. to a whole number], you will have 5 things equal to 49; divide by the things, there results 9 4/5: this is the value of the thing and we have said that the horse is worth $\overline{I}$, so it is worth 9 ducats and 4/5 of a ducat.

Box 7.1. *English translation based on the edition made by (Arrighi 1970, p. 107)*

As we can see, the abbacus manuscript keeps the entire cadence of oral language. By discussing how to solve the problem with his students, the abbacus school master very probably left written traces (on a tablet[1] or another medium) when he was in the process of explaining the solution in class. Indeed, 'pupils did not own their own a[b]bacus treatises; the a[b]bacus syllabus was imparted directly by teachers without pupil-owned textbooks' (Black 2007, p. 162). It was the voice above all, and sight and memory (and probably the teacher's gestures) on which the practice of mathematics relied.

Like other abbacist manuscripts of the period, Piero della Francesca's manuscript expresses the solution of the problem in the syncopated form mentioned above. In other words, Piero della Francesca expresses the solution in natural language enriched by the abbreviations of some words. But in reality, our author goes a little further: at the *writing* level, we see the appearance of a rudimentary mathematical symbolism. To represent unknown quantities, in some parts of the text, following the abbacist tradition, Piero della Francesca uses the term 'thing' (*cosa*). However, in other parts, he uses a small dash placed above some numbers. The small dash is not yet an arbitrary sign. The arbitrary character of signs will appear later in, for instance, Descartes' work (1637). In Piero della Francesca's work the small dash signifies the *side of a square,* and from the geometric shape of the side it acquires its material form (a dash or line). In other problems where the square of the unknown number was needed for calculations, Piero della Francesca uses a small square above the number of squares of the unknown and he uses, as in the previous excerpt, a small dash to signify the quantity of the unknown number to be considered.

Historically speaking, the very rudimentary symbolism that Piero della Francesca uses was one of the first algebraic symbolic systems of the Renaissance. The main interest it holds for us here is that it allows us to retrace a transformation that was underway in mathematical practice. The abbacist manuscripts (for an inventory, see (van Egmond 1980)) are written notes that served as memory aids to direct and organize oral practice. We see this orality projected into the very structure of the abbacist text. But, at the same time, we begin to see the appearance of small technical symbols that go beyond abbreviation of the spoken word. Indeed, on the first line of the solution, Piero writes 'p.2 mesi' (*per* 2 mesi, *for* two months), the 'p' then being an abbreviation of the word 'per'. By contrast, the small dash is no longer an abbreviation of the word 'cosa' (the thing, the unknown). In reality, the abbacists had an abbreviation for the word cosa: 'co.', which was widely used. The small dash, like the small square, is a *technical* sign, although they both remain attached to the geometric imaginary (side, area of the square).

---

[1] In a fifteenth century manuscript kept in the *Bibliothèque nationale de France*, there is a reference to a wooden tablet covered in black wax being used during a geometry lesson. See http://classes.bnf.fr/ema/grands/373.htm. Black (2007, p. 163) cites a document that mentions a 'tavoletta del gesso, ch'è buona per fare ragioni'; i.e., a chalk tablet that is useful for solving problems.

Although we cannot attribute the appearance of emergent algebraic symbolism to a move from oral to written practice alone (Radford 2006; Høyrup 2008), it nevertheless remains that this algebraic symbolism we saw appear in Piero della Francesca or, later, in Viète (1630) and Descartes (1637), came from *writing* and not from the oral use of language. How would one orally express Piero's written text 'and say that the horse is worth $\overline{I}$ things'? Would one say 'and say that the horse is worth a dash thing'?

Whatever the case, art historians frequently emphasize this characteristic trait of the Late Middle Ages and the Renaissance, which distinguishes these historic periods from those that preceded them; that is, their relationship with visual culture, with a pictorial representation of the world.

Let us return to Biernoff's analyses that show how, over the course of the Late Middle Ages, there began a long process of cultural transformation. The process began with people considering the perceptible world as a simple starting point from which to look at a higher and invisible reality beyond. Then, people came to understand that it was to this world, to nature, to everyday life, that they must look.

Walter Isaacson (2017, p. 173) cites a passage from Leonardo da Vinci's notebooks: 'My intention', Leonardo says, 'is to consult experience first, then, by reasoning, to show how this experience is called to act in this way.' The famous fourteenth-century Italian humanist, Leon Battista Alberti, author of *De Pictura* and *Ludi rerum mathematicarum* (modern translations in (Alberti 2011, 2002) respectively), made constant reference 'to the visual experiences of life' (Belting 2011, p. 172). From this attention that turns its glance to nature and to this entirely new experience that the individual makes of the world, the body and the senses will become the structuring elements of new ways of knowing and representing the world. Particular attention is paid in paintings to the 'naturalness' of the human subjects – their bodies, the hands, the emotion, movement. Isaacson contrasts the tableau of *Tobias and the Angel* by Antonio Pollaiuolo with the one produced in the workshop of Andrea del Verrocchio (a tableau to which da Vinci contributed)[2]. 'One difference is that Pollaiuolo's version is stiff, while Verrocchio's conveys movement' (Isaacson 2017, p. 50).

In Verrocchio's *Tobias and the Angel* tableau, the two individuals in the painting

> 'turn to each other naturally. Even the way they hold hands is more dynamic. Whereas Pollaiuolo's faces seem vacuous, the body motions in

[2] For Pollaiolo's version, see: https://commons.wikimedia.org/wiki/File:Antonio_del_Pollaiolo_-_Tobias_and_the_Angel_-_WGA18047.jpg. For Verrocchio's version, see: https://www.nationalgallery.org.uk/paintings/workshop-of-andrea-del-verrocchio-tobias-and-the-angel.

> the Verrocchio version connect to emotional expressions, conveying mental as well as physical movements.' (Isaacson 2017, p. 50)

And so, following the cultural transformations of the Middle Ages and the Renaissance, for the first time in the history of the West, the devotional image draws out the sacred in the human sphere:

> 'Where ocular desire had previously been denigrated or redirected 'up' the ladder of perception; where the visible creation had occasioned contemplation of an invisible creator; now God could be seen in the flesh.' (Biernoff 2002, p. 163)

In the visual arts, *perspective* is the expression of this new way of representing the sacred and the world in which 'the shaping of the world, the organizational structure of the representation' occurs 'under the principle of the imitation of nature' (Arasse 2004, p. 46). This involves:

> 'a perfectly arbitrary system of representation, that was invented by a whole society over almost a century […] What is the function of this such arbitrary system of representation? A perspective […] supposes an immobile spectator, fixed at some distance from that at which he looks, and looking at it with a single eye.' (Arasse 2004, p. 50-51)

Daniel Arasse emphasizes the arbitrariness of the perspective that we know, and which, historically speaking, came into competition with other forms of perspective (for example, convex perspective). The perspective that the Renaissance retained rested on the idea of a fixed, immobile eye that 'has nothing to do with the way in which we perceive, [since] our eyes do not stop moving' (Arasse 2004, p. 51). Aside from the ease of drawing according to the principles of Alberti's and Masaccio's perspective, the Renaissance retained this type of perspective. On one hand it was for reasons linked to the politics of power and on the other, by the privileged position 'that it gives to the individual, to man in the world' (Arasse 2004, p. 57). This perspective affirmed the new form of subjectivity that appeared during the Renaissance. It led the individual to imagine themselves as *homo faber* (Arendt 1958); that is, as the producer and manufacturer of things in the social and economic network of emerging artisanal capitalism. It finished by placing 'the center of gravity of [the individuals'] existence in ordinary life', that of work and family (Taylor 1994, p. 185).

We would like to suggest that it is the same *episteme* (Foucault 1966) which, during the Renaissance, informed the visual world of representation in the arts as much as the symbolic world of algebra. In other words, a phenomenon similar to that of visual representation influenced mathematical texts within which algebraic symbolism was refined. Under the effect of this episteme, algebraic text was detached little by little from speech, which was its organizing component among the abbacists. The role of the spoken word never entirely disappeared. But the algebraic text gained a *spatiality* it had never before enjoyed in the West. Not only did new, specific signs appear, but algebraic discursivity was profoundly altered.

Let us look at an excerpt from the sixteenth-century book, Bombelli's *Algebra*. Unlike *Trattato d'abaco* by Piero de la Francesca, Bombelli's *Algebra* is a printed book, published in 1572, although the manuscript version was produced between 1557 and 1560. As Jayawardene notes, mathematics itself was not Bombelli's vocation. 'He was neither a teacher of the subject nor was he a gentleman of leisure' (Jayawardene 1965, p. 298). Bombelli was an architect. In writing this book:

> 'Bombelli thought that only Cardano among his predecessors had explored the subject algebra in depth. However, he felt that Cardano had not been clear in his exposition [...] So he decided to write a treatise which would enable a beginner to master the subject without the aid of any other book.' (Jayawardene 1973, p. 513)

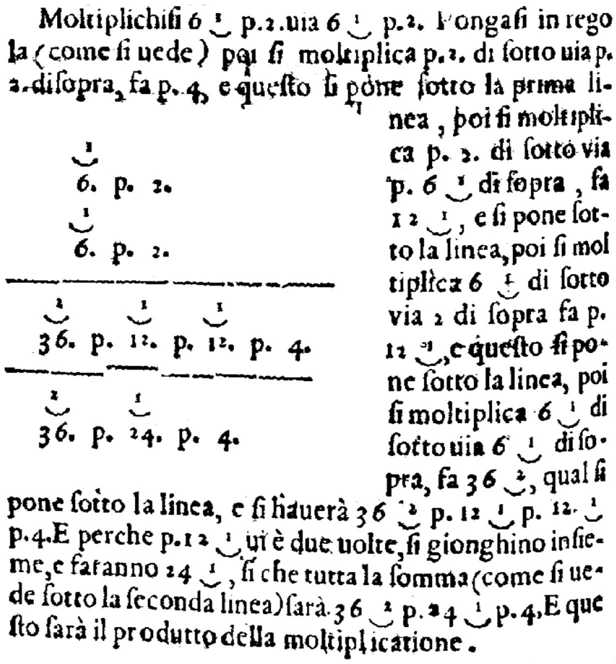
Bombelli's book was influenced by the veneration shown by the humanist tradition of the Renaissance for the thinkers of Antiquity. Inspired by Diophantus's *The Arithmetic*, Bombelli wrote his book to break with the tradition of the abbacus masters who posed their problems under 'the guise of human actions or affairs' (Bombelli 1572, p. 414), as was the case with buying and selling and other commercial problems.

The first excerpt that we would like to discuss is about the multiplication of binomials; that is, algebraic expressions comprising two terms. We note first of all that Bombelli introduced small curved dashes above which he added a number; the curved dash indicates that we are dealing with a power of the unknown and the small number placed above this dash indicates the power of the unknown. The curved dash and the number above are placed beside a number (sometimes above this number), which would correspond to what we call, in modern terminology, the *coefficient* of the unknown. So $6\overset{1}{\smile}$ means $6x$. What is interesting in Bombelli's algebraic symbolism is that the unknown is not represented. It is *implicit*. Following the abbacist tradition, Bombelli used abbreviations to designate operations (such as 'p' for 'plus' and 'm' for minus). Let us look at a problem that consists of multiplying $6x + 2$ by $6x + 2$.

The explanation is a bit similar to that of Piero della Francesca; it includes a verbal description and technical signs (see figure 7.4). But instead of the personal pronoun 'tu' (you) (*you* make, *you* add, for example) or the associated imperative formula 'do this, multiply', etc.), Bombelli uses an impersonal command: 'Molti-plichisi $6\overset{1}{\smile}$ p. 2 *via* 6 $\overset{1}{\smile}$ p. 2'. The command is no longer addressed to someone in *particular*. The subject who carries out the calculations is no longer the concrete subject – the student – before the abbacus master, but an *abstract* subject. And it is this abstract subject who now *looks* at a table on the left, a table that no longer contains any words. It contains only symbols.

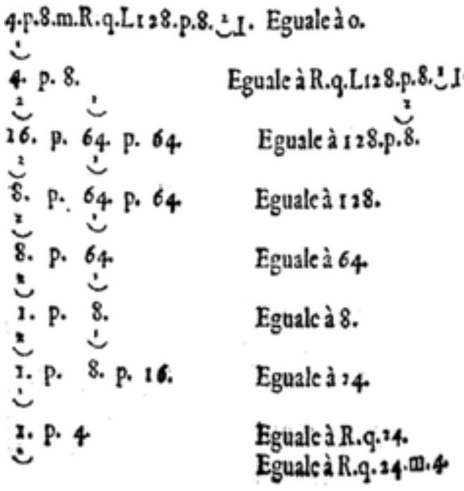
Let us now look at a second extract from the same book (figure 7.5). This extract is about solving the equation that we would write in the alphanumeric semiotic system as $4x + 8 - \sqrt{128 + 8x^2} = 0$ (to the right we have added a translation into modern symbols). In the text, this problem was preceded by a similar problem, the solution of which is explained by Bombelli with recourse to impersonal commands like what we see in figure 7.4.

In addition to abbreviations of operations (such as ‘p’ for ‘plus’ and ‘m’ for minus), there were new signs, such as ‘L’ and an inverted ‘L’: they enclose the expression affected by the square root (‘R.q.’). Bombelli does not use any specific sign for the expression ‘equal to’. The equal sign is a late invention (Heeffer 2009).


Moltiplichisi 6 ⌣1 p.2. via 6 ⌣1 p.2. Pongasi in rego la (come si uede) poi si moltiplica p.2. di sotto uia p. 2. di sopra, fa p. 4, e questo si pone sotto la prima linea, poi si moltiplica p. 2. di sotto via p. 6 ⌣1 di sopra, fa 12 ⌣1, e si pone sotto la linea, poi si moltiplica 6 ⌣1 di sotto via 2 di sopra fa p. 12 ⌣1, e questo si pone sotto la linea, poi si moltiplica 6 ⌣1 di sotto uia 6 ⌣1 di sopra, fa 36 ⌣2, qual si pone sotto la linea, e si hauerà 36 ⌣2 p. 12 ⌣1 p. 12 ⌣1 p. 4. E perche p. 12 ⌣1 ui è due uolte, si gionghino insieme, e faranno 24 ⌣1, si che tutta la somma (come si uede sotto la seconda linea) sarà 36 ⌣2 p. 24 ⌣1 p. 4. E questo sarà il produtto della moltiplicatione.

6. p. 2.
6. p. 2.
36. p. 12. p. 12. p. 4.
36. p. 24. p. 4.

Figure 7.4. *Excerpt from Algebra by (Bombelli 1572, p. 214)*


4.p.8.m.R.q.L128.p.8.1. Eguale à o.
4. p. 8. Eguale à R.q.L128.p.8.1.
16. p. 64. p. 64. Eguale à 128.p.8.
8. p. 64. p. 64. Eguale à 128.
8. p. 64. Eguale à 64.
1. p. 8. Eguale à 8.
1. p. 8. p. 16. Eguale à 24.
1. p. 4 Eguale à R.q.24.
Eguale à R.q.24.m.4

$$4x + 8 - \sqrt{128 + 8x^2} = 0$$

$$4x + 8 = \sqrt{128 + 8x^2}$$

$$16x^2 + 64x + 64 = 128 + 8x^2$$

$$8x^2 + 64x + 64 = 128$$

$$8x^2 + 64 = 64$$

$$x^2 + 8 = 8$$

$$x^2 + 8 + 16 = 24$$

$$x + 4 = \sqrt{24}$$

$$x = \sqrt{24} - 4$$

Figure 7.5. *Excerpt from Algebra by (Bombelli 1572, p. 250), with translation into modern symbols on the right*

What is striking about Bombelli's text is that, now, everything happens on a kind of table – we may say a *tableau*. Some years ago, Brian Rotman noted that the Renaissance perspective includes a semiotic system with its own rules. The perspective is, in a sense, a system: a system that serves to produce scenes that do not necessarily exist (such as the exact scene of *Tobias and the Angel*, which we cited above).

> 'The system becomes both the source of reality, it articulates what is real, and provides the means of 'describing' this reality as if it were some domain external and prior to itself; as if, that is, there were a timeless, 'objective' difference, a transcendental opposition, between presentation and representation.' (Rotman 1987, p. 28)

Rotman made the parallel with the number zero that, in his analysis, came to play in some way the same semiotic role as the vanishing point in perspective: articulated around the notion of absence, zero is a sign – the sign of the cardinal of the empty set, $\emptyset$. In the semiotic system of arithmetic, this sign, which is written $\emptyset$ *or* 0 (Bourbaki 1970, p. 30), *produces* other signs $\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ ... that are the natural numbers 1, 2, 3, etc. But what Rotman saw for zero is in fact valid for Bombelli's algebraic semiotic system. This semiotic system (with its 'p', 'm', R.c., the sign for powers of the unknown, etc.) now makes it possible to *generate* (as zero does for arithmetic) algebraic *scenes* at will (polynomial calculations, equations, etc.). And these algebraic *scenes* are more and more detached from orality.

The spoken word, in fact, gives way to *visual* activity. In the wake of problems/scenes that can now be generated at will with the symbols that Bombelli gave us, the visual relationship allowed for bodily participation in algebra the same way that 'the visual relationship – more than any other sensory interaction – allowed [in art] for bodily participation in the divine' (Biernoff 2002, p. 134).

In the same way that the observing subject of a tableau does not figure explicitly and stands at a distance to look perspectively at whatever there is to look at, the subject of the new algebraic symbolism becomes an observing subject as well. This subject begins to no longer figure explicitly in the text. Just as in the case of perspective, there is a displacement of the subject: whereas the subject was at the center of Piero della Francesca's text in the middle of doing this or that, the subject of Bombelli's text is displaced to the margins of the text, at a distance from what is seen and said. The eye can now examine the world from the new locus of subjectivity.

But, unlike the traditional interpretation that conceives the Renaissance eye as a neutral, objective, disembodied entity detached from the perceived, we would like to suggest, on the contrary, that from the Late Middle Ages and the Renaissance, the eye is not simply a receptive instrument, but an active and emotionally affected eye. In his work on perception, Roger Bacon already noted that 'vision is active and passive' (Bacon 1962, p. 470). The eye affects perception and perception affects the eye. This was an entirely new bodily esthetic experience. The eye traverses the forms and symbols as if it touched from a distance, as if palpating an object. And by palpating the object, the eye, like the hand, is in turn *affected*. This is a 'perceptual *relationship* [that] is distinct from a unidirectional act of perception' (Biernoff 2002, p. 86). The eye's grasp of the perceived object is as much participative as creative. The eye adds *meaning* to what is presented to it – a scene, a text, etc. Thus, in figure 7.5, to move from one equation to the next, the mathematical eye must *add* what is implicit or lacking so that one can move deductively. The eye should learn to see what is not there and add it to the picture. Of course, this does not mean the total disappearance of language. Natural language does not disappear; it has not even disappeared from the austere work of Bourbaki (1970). As Gérard Vergnaud notes, 'no diagram, no non-language symbolism, no algebra can fulfil its function without the accompaniment of language, even internally (Vergnaud 2001, p. 14). In Bombelli's *Algebra*, language is displaced and replaced. It works as a meta-language.

Reading and understanding Bombelli's *Algebra* requires a reorganization of the senses in which perception now comes to play a fundamental role. And from this bodily reorganization, a dynamic relationship is established between body and matter. At the foundation of this relationship there is an *effect* of the perceived that affects the eye: the perceived, by affecting the eye, becomes an object that, on being exposed, by giving in to the eye, invites us to follow and pursue it. To use Biernoff's (2002) expression, it is now a vision that could be called *carnal*, to the extent that it is an *affection* of the being and of the object. Carnal vision is, from this point of view, *movement* – movement between that which perceives and that which is perceived.

Before moving on to the next section, let us recap what we have just seen and heard. In this section, we have proposed an analysis of short passages from two algebraic texts, one from the fifteenth and the other from the sixteenth century. In the first text, by Piero della Francesca, algebraic practice is organized through orality. Through language, the teacher explains to the student the calculations that need to be made. Although we see the appearance of a rudimentary symbolic language, the production of mathematical meaning (the signification of symbols, the perception of these, the calculations) is articulated around language. In the second text (Bombelli's), language remains important. But we are witnessing the reorganization of the *sensorium* or the sensorial complex. In the first excerpt from Bombelli's text, a symbolic schema of the calculation of binomials is still explained through language. But little by little, there is an increasing focus on symbols. These are organized in the form of a table where the whole is given *at once*. The eye is called to traverse the symbols, as it traverses figures in a *tableau* of visual arts. This phenomenon becomes clearer still in the case of the equation and its solution seen in the second excerpt from Bombelli's text. Here, the word stops and the eye looks. There is a transformation in the production of mathematical meaning. The verbal thought that was placed at the center of Piero della Francesca's text gives way to another type of mathematical thought: symbolic thought.

We would like to emphasize the idea that this transformation should be understood in light of the importance that the visual dimension acquired during the Renaissance. There was a secularization of the eye that began with the natural philosophers and was part of a historic movement that questioned the relationship between the body and the world. The ascetic relationship between the body and the world that was bequeathed by the Middle Ages is replaced by a relationship where the eye and the seen interact and mutually form one another.

7.4. Sight, touch, orality and symbol

In this section, we discuss the example of an equation in a Grade 6 class (students of 11–12 years of age in primary school). The example comes from the conclusion of a three-year longitudinal study on a class of students that we began to follow when the students were in Grade 4 (ages 9–10). Our interest was in understanding the way in which pupils' algebraic thinking emerges and is transformed over the course of teaching and learning activity when we move from a non-symbolic algebra to a symbolic algebra. So, in Grade 4, the students worked on equations modeled through the idea of a balance. The balance was drawn on the page; on the plates, there were plastic objects: small red, green or blue squares, and green cylinders. The squares all had the same mass and this mass was known. The cylinders all had the same mass and this mass was unknown (see figure 7.6a). The students also worked on equations modeled with the help of a known number of physical cards and envelopes containing a same but unknown number of cards (see figure 7.6b; for more details, see also (Radford *et al.* 2009)). Here, the students' work on concrete objects was based on touch, action, perception and the spoken word. In Grade 5, a gradual introduction to equations written with the help of letters and numbers took place (i.e., a writing of equations in alphanumeric notation); this was done from concrete modeling on the context of the cards and envelopes seen in Grade 4. In Grade 6, equations were given directly in the alphanumeric semiotic system.

The excerpt that we discuss below was preceded by work carried out the day before. During this work, students working in small groups had tackled the following equations:

a) $4 \times n + 2 = 2 \times n + 18$

b) $3n - 8 = n + 8$

c) $4n = 20 - n$

Equation $x + 2 = 4$ — a)

Equation $2x + 4 = x + 7$ — b)

Figure 7.6. *a) Students explaining to the teacher (second right to left) the solution of the equation* $x + 2 = 4$ *modeled with the help of small squares and a cylinder (which plays the role of* x*). On the left-hand side of the balance, we see two squares and one cylinder; on the right-hand side of the balance, we see four squares. b) The students have just written the equation* $2x + 4 = x + 7$ *modeled with the help of cards and envelopes (that play the role of* x*) and are about to solve it.*

The students are learning to move from the writing $a \times n$ to an. They are also encountering equations including subtractions (which cannot be modeled with the help of the two previous contexts: the balance, the cylinders and the squares on one hand; the cards and the envelopes, on the other).

After a general discussion, students were invited to solve the following problem, which is the one we are going to discuss in the remainder of this chapter:

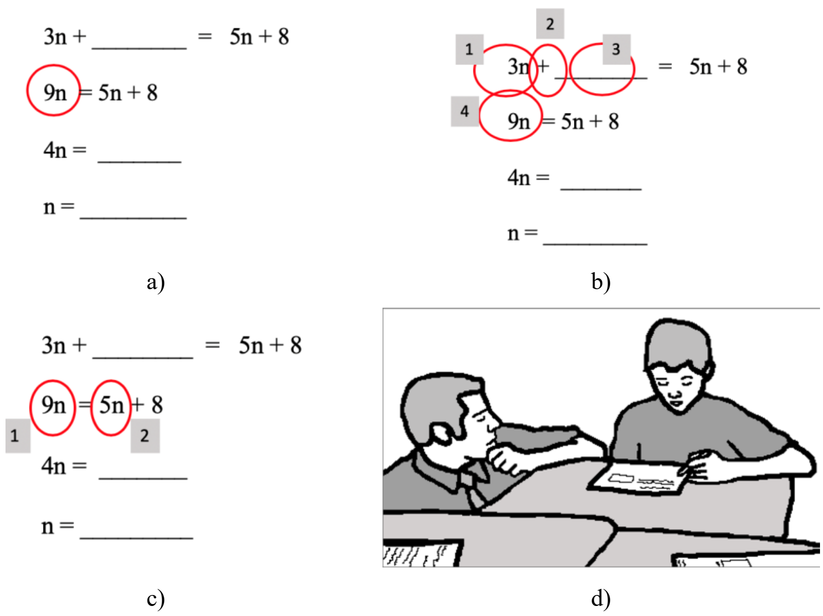
> 'Digging around her grandmother's attic, Julie found an algebra book. She opened a page and found an equation with its solution. Unfortunately, the book was damaged, and some parts were no longer legible. Below, Julie has copied the equation and its solution. Lines indicate the parts that are no longer legible. Can you help her to retrieve the missing parts?'
>
> 'Write the missing parts on the lines below:
>
> $3n + \_\_\_ = 5n + 8$
>
> $9n = 5n + 8$
>
> $4n = \_\_\_$
>
> $n = \_\_\_$'

As usual, the class was divided into groups of two or three students. We discuss the conversation held within one of these groups, with the help of figure 7.7. The teacher arrives when the students are about to start the task.

> '1. Professor: Are there any clues?
>
> 2. Paul: Yes!
>
> 3. Professor: What are these clues?
>
> 4. Paul: The $9n$ here (*on the second equation on his sheet, Paul points to* $9n$ *with his crayon*; see figure 7.7a).
>
> Well for me, that says that $3n$ (*on the first equation*, *he points to* $3n$; see [1] figure 7.7b) plus (*he points to the sign '+',* see [2] same figure) $6n$ (*he points to the line*; see [3] same figure) is equal to $9n$ (*on the second equation, he points to* $9n$; see [4] same figure) [. . .]
>
> Then, there, after, that would be $9n - 5$ [he means 'minus $5n$'] (*he points to* $9n$; see [1] figure 7.7c); that gives $4n$; then, that would be minus 5 on the other side [he means 'minus $5n$']. That (*he points to* $5n$ *in the second equation*; see [2] same figure) would be 0.
>
> So, $4n$ (*he points to* $4n$ *on the third equation*; see figure 7.7d) equals 8 (*he points to the line*; see figure 7.7e).
>
> Then, if we divide that (*i.e.,* $4n$) by 2, that gives $2n$; (*a little annoyed, he continues*), but there is just $1n$ there (*he points to* n *on the last equation*; see figure 7.7f).

5. Albert: (*coming to his aid, says*), but, if you divide . . . OK, if you divide that (*i.e.,* $4n$) by 4, that gives $1n$.’

Paul has succeeded in solving the equation without writing. He has still not filled in the lines. As with Bombelli’s text, the equation and the subsequent equations appear as a table. It is a table that the eye crosses sensuously. But here, unlike in Bombelli’s text, the didactic device puts the reader-subject (or, better, the perceiving subject) in a situation where they are asked to fill in some parts of the table that have been removed intentionally. The eye is aided by indexical gestures and words. We might return here to Piero della Francesca’s text to better appreciate the disruption created: In Piero’s text, language and the cadence of its flow mark the rhythm of thinking, leaving written traces here and there; the problem-solving procedure is revealed through *seeing* and *hearing* at the *same time*. In Paul’s approach, the rhythm of thinking is subject to the way in which the semiotic table is perceptually traversed.



e) f)

Figure 7.7. *Paul's perceptual activity on the alphanumerical algebraic text*

To find what must be placed in the first line, Paul moves down from the first equation to the second. It is there, in this *perceptual* movement, that he will look for clues. 'The $9n$ here', he says, referring to the first monomial of the second equation, 'that tells me that $3n$ (see [1] figure 7.7b) then $6n$ (see [3] figure 7.7b) is equal to $9n$'. The perceptual coordination of multiple elements of the table allows him to find what needs to be placed on the first line, and the other lines as well.

In the algebra activities of Grade 4 and 5, language still organized the students' work on cultural artifacts (such as the squares and cylinders placed on the drawing of the balance or the cards and the envelopes), with the help of perception and touch (much as in the case of della Francesca). In Grade 6, touch is still here in the hand movements that Paul makes throughout his solution of the problem. But it has been transformed completely. It has become more 'theoretical'. The same is true of perception. Now, symbolic thinking (i.e., the perceptual thinking that moves across the algebraic signs) surpasses and goes beyond verbal thinking. This surpassing of verbal thinking by symbolic thinking is one of the greatest accomplishments of the cultural transformation in the students' mathematical development, a transformation that was still not entirely accomplished in Grades 4 and 5.

But Albert is not entirely in agreement with Paul's procedure. Although he thinks it is logical, he does not agree with the idea of placing 6n on the line.

'6. Albert: Yes, but here (*he points to the line*), why would you put $6n$, if $3n$ is already there? I think that here (*he points to the line*), that should be a number!

7. Teacher: A number? Why?

8. Albert: Because you don't just write $3n + 6n$; why would you do that?

9. Paul: Because, well, here (*he points to the first monomial of the second equation*), that says $9n$ . . .

10. Teacher: (*addressing Albert*) Do you see that?

11. Albert: Yes, but I think that I can find a number for that (*i.e., to fill in the line*).

12. Teacher: Can you find a number . . .?

13. Albert: Yes . . .

14. Teacher: That still makes the equation true?

15. Albert: That will be 12! (*the teacher leaves for a discussion with another group*).

16. Paul: Yes, but where does the 12 go?

17. Albert: 12, because you should add $6n$ then you will make 2×6.

18. Paul: Yes, (*in a frustrated tone*), but I don't understand; where does the 12 go?

19. Albert: Here (*Albert uses his pen to indicate placing the 12 on Paul's sheet*)'

As we see in this excerpt, Albert has in mind an equation of the form $ax + b = cx + d$. On line 7, he expresses doubts about the writing '$3n + 6n$'. As Paul is not entirely convinced, the discussion goes on. After some exchanges, Albert agrees that his procedure may pose a problem:

'20. Albert: Yes, but how do we know that 12 is equal to 6n?

21. Paul: I told you so!

22. Albert: Yes, but here (*to fill in the line*) you should know the value of n before, to make . . .

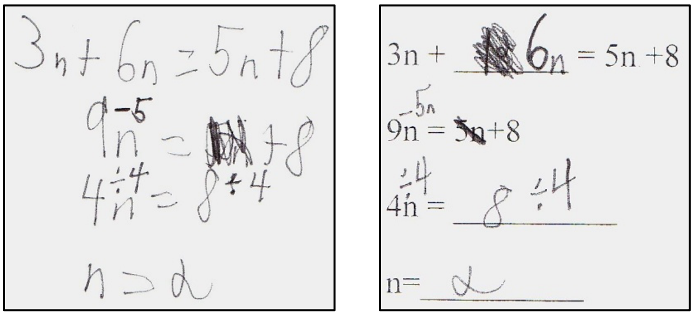
23. Paul: Yes, exactly! For me, I think my way is easier, because . . . look, it's really as if 3 plus 6, $3n$ plus $6n$.

24. Albert: Yes, but 12 makes sense!

25. Paul: Er, well yeah, but . . .'

Following the idea of putting 6n on the first line, Paul starts to write the equation on his sheet. At the same time that he writes the terms of the equation, he quickly states them out loud. Once he has finished writing and reading the equation '$3n + 6n = 5n + 8$', he writes below and announces the second equation (see figure 7.8a). Then, he says quietly, as if talking to himself: ‘Because that (*he points to* $3n + 6n$) is equal to that (*he points to* 9n *on the second equation*)’. He begins to subtract $5n$ from the first monomial of the second equation. He writes a small 5 on the second line, but he is interrupted by Albert:

> ‘26. Albert: Wait! I think I’ve found something . . . We’ll solve that (*the second equation*) and if that (*he points to n*) is equal to 2, then, we got it right. And if that is not equal to 2, then we got it wrong.
>
> 27. Paul: That, that means we would have two ways of finding . . .’



a) b)

Figure 7.8. *a) Writing an equation and its solution as proposed by Paul. b) Paul fills in the dashes*

Paul returns to his equation and subtracts $5n$ from the right-hand side of the second equation barring the term ‘$5n$’. Then, he continues to write and at the same time to say each of the terms of the third and fourth equations. When he has finished, he exclaims, satisfied: ‘Yes! OK, I’ve got it (*we see his body relax*).’ Then, Paul fills in the lines (see figure 7.8b).

During the general discussion, Paul went to the table to show his solution. Then Albert showed his solution. He explained that starting from the second equation, he found that *n* is equal to 2. He then replaced *n* by 2 in 6*n* to obtain 12.

He had some objections:

> ‘28. Alexandre: (*speaking to Albert*) But we would need the first line to be able to find the value of *n*.
>
> 29. Albert: That’s why I didn’t know if I was right.

30. Teacher: So, you [Alexandre], you seem to be saying that if we solve the equations we always begin with the first line and we move downward . . .

31. Alexandre: Yes . . . Because the second [equation] says $9n$ . . .

32. Teacher: (*repeating for the whole class, to emphasize the idea*) Because there is $9n$. Can you explain that to us?

33. Alexandre: Yes, because it doesn't give the value of *n*. [The value of n is] all the way down, and that says $9n$ (*pointing from his desk to the first term of the second equation*); then, how would we know that $3n + 12$ equals $9n$?'

The teacher encouraged the class to take a position on the ideas proposed. Afterwards, the class tackled, in small groups, the next problem in the day's lesson. Solving these problems and a general discussion of them in class allowed pupils to better understand the subtleties behind writing and solving linear equations.

We see in the actions and discussions of pupils the reorganization of the body and the senses that underlie this conceptual journey proposed by the teacher through the chosen problems. The production of meaning now changes and occurs around symbolism and a meta-language provided by natural language.

7.5. Conclusion

In this chapter, we were interested in the problem of embodiment in the formation of meaning in mathematics. Our interest was focused, in particular, on the formation of mathematical meaning in its relationship with the body, matter and signs. In the first section, we posed the question of the importance of investigating the embodied aspects involved in the students' multimodal semiotic activity. The question is entirely relevant since the multimodal semiotic dimension that brings to the fore the question of the body has not really been included – or, in any case, not decisively – in mathematics education. It took almost twenty years (since the pioneering work of (Lakoff and Núñez 2000)) for reflection to begin on the importance of the body, gestures, perception, rhythm and some phonic phenomena of natural language in learning. We would need to look perhaps for reasons behind this 'delay' in the more rationalist tradition that has influenced our conceptions of teaching and learning mathematics and other disciplines. The rationalism we inherited from Descartes, Leibniz and other philosophers stands openly against the role of the body and the senses in our acts of knowing.

The second section of this chapter made a small incursion from the historical side to show the rationalist epistemology that preaches, in practice, an ascetic exercise of the body. Historically speaking, this epistemology goes back to Plato, then to Christianity, as practiced in the Middle Ages. But the return to the body that we see more and more in contemporary discussions in philosophy (Massumi 2002), in the cognitive sciences (Johnson and Rohrer 2007) and in anthropology (Geurts 2002) among others – 'a return' to the body since we are *returning* in one way or another to the precepts of eighteen-century empiricist philosophy – poses great difficulties. It is a question in fact of rethinking our understanding of thinking and its relations with the body, the flesh and the senses. It is a question that is not self-evident. The return to the body is often presented as a relatively simple opportunity to *reposition* the subject in the epistemic scene; that is, the return to the body is seen as a possibility for the subject to try to find an effective place in the production of knowledge. There is nothing, indeed, more intimate or more personal to the subject than their own body. But this position results, as we have tried to demonstrate by taking the example of the theoretical position of Maxine Sheets-Johnstone, in a subjectivist form of radical empiricism that Marx had already refuted in the nineteenth century. From the perspective of dialectical materialism developed by Vygotski (1987, 2014), Leontiev (1976, 2005), Luria (1984) and others, sensation, the senses and matter are considered an important part of the foundations of cognition and of any physic activity (affective activity, volition, etc.). However, in this perspective, unlike the empiricist ones, the senses, the body and the flesh are not conceived of as *already given*; they are not considered to be the starting point of cognition; they are rather considered as entities that are *produced* and *transformed in social practice* (Radford 2021). In this context, human cognition is conceived of as *sensual cognition*: a sensible and multimodal cultural and historical form of thinking, acting, imagining, feeling, transforming and giving meaning to the world.

The last two sections of this chapter (sections 7.3 and 7.4) tackled the transformation of the senses from the phylogenetic and ontogenetic angles respectively. We suggested that the appearance of algebraic symbolism during the Renaissance went hand in hand with a whole transformation of the senses that can be seen in the work of other domains such as the visual arts. This transformation should be linked with changes in the production of material and spiritual life in light of the nascent artisanal capitalism and the period's new procedures or techniques for subjectivation. In the same way that the perceiving subject is removed from *tableaux* in the perspective of the Renaissance and is placed in front of these tableaux and the scenes they represent, the subject of algebraic practice is also placed in front of the algebraic text/tableau. In either case, the subject *looks*. Orality gives way to a *visual* activity; perception (transformed into *symbolic perception*) becomes the organizing principle of the production of meaning.

Section 7.4 should not be read as suggesting that there is a parallel (or recapitulation) of phylogenesis in ontogenesis (Radford and Puig 2007). Although ontogenesis (i.e., the subject's cognitive development) seems to reproduce phylogenesis (i.e., the historical development of knowledge), the reason for this apparent reproduction of historical processes is not to be found in a *natural* movement of thought. On the contrary, it is the effect of a whole series of didactic devices that give a direction to cognitive development.

In short, this seemingly developmental reproduction is not a biological law of development, but the effect of educational cultural choices.

Multimodal semiotic analyses can help us to better understand the cognitive transformative journey that we offer our students at school and allow flexibility in the organization of learning activities. One way to shed light on the understanding of cognitive transformation – the one we followed in this chapter – is through an articulation of phylogenetic and ontogenetic analyses. Such analyses can help us better understand the historical presuppositions on the body, the senses and matter in historical mathematical practices – which remains a very scarcely explored domain in didactics – and allows us, at the same time, to better engage in critical stances in relation to our own theoretical and practical presuppositions.

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