In ZDM – Mathematics Education, https://doi.org/10.1007/s11858-022-01437-4

History of mathematics in the context of mathematics teachers' education: a dialogical/ethical perspective

David Guillemette[1] · Luis Radford[2]

Abstract
This paper deals with the use of original sources in mathematics education, with emphasis on preservice teachers' education and the exploration of historical texts. In this context, there is a real challenge, for both teachers and learners, to conduct both 'synchronic' and 'diachronic' reading. As reported extensively in research, learners seem to have a strong propensity to focus on and to 'translate' the texts into modern mathematics, which makes it difficult to deepen both their understanding of history and their own set of conceptualizations. Rather than propose practical solutions, in the paper we explore a theoretical positioning that may help us think differently about these difficulties and, ultimately, provide articulated and different avenues for interventions. Drawing on the philosophy of language of Mikhaïl Bakhtin and Valentin Voloshinov, the idea is to question the Saussurean perspective that underpins the notions of synchrony and diachrony. From this theoretical perspective, we propose to think about the challenges associated with the reading of historical texts in terms of the ethical stance of answerability and engagement in the context of preservice mathematics teachers' education, and we suggest envisioning a third possible reading, namely, that of the educator.

Keywords History of mathematics · Mathematics education · Readings of historical texts · Dialogism · Ethics

1 Introduction

There is a close relationship between the history of mathematics and mathematics education that has constantly been developed and implemented. In this context, numerous arguments supporting the presence of history in the teaching and learning of mathematics have been advanced, as well as many ways of putting these elements into practice (see Fauvel & van Maanen, 2002).
However, some major issues are still discussed and are keenly experienced. For instance, the development of theoretical or conceptual frameworks appropriated to the field of research is still sorely lacking (Barbin et al., 2020; Clark et al., 2016; Fried et al., 2016). It seems necessary to develop new concepts and perspectives for a more in-depth reflection on the foundations and the role of the history of mathematics in teaching and learning. In this contribution, we focus on the use of original sources in mathematics education, with emphasis on preservice teachers' education and the practice of exploring historical texts. We offer a theoretical and conceptual investigation that allow us to think about this precise educational practice. We clarify, first, the need for such theoretical investigation and then detail a specific theoretical position before highlighting some possible implications for practices and research.

David Guillemette guillemette.david@uqam.ca

Luis Radford lradford@laurentian.ca

[1] Département de Mathématiques, Université du Québec à Montréal, Montréal, Canada

[2] École d'éducation, Université Laurentienne, Sudbury, Canada

2 Research context

2.1 History and teaching–learning of mathematics

In the proceedings of the 2016 HPM[1] meeting, Clark et al. (2016) put forward some elements related to the needs and issues presently discussed in the field. In their concluding

[1] HPM is the International Study Group on the Relations between the History and Pedagogy of Mathematics, affiliated with the International Commission on Mathematical Instruction (ICMI) (see International Mathematical Union, 2022).

remarks (p. 175), the authors mentioned the following issues that are currently central for the researchers: (1) to put emphasis on pre- and in-service teachers' education, (2) to design, make available, and disseminate a variety of didactic source material, (3) to systematically and carefully perform applied empirical research in order to examine in detail and convincingly evaluate the effectiveness of HPM perspective, and (4) to acquire a deeper understanding of theoretical ideas put forward in the HPM domain to carefully develop them into coherent theoretical frameworks and methodological schemes.

These issues can be linked to those recently raised by Barbin et al. (2020) in the chapter *History of Mathematics in Education* in the *Encyclopedia of Mathematics Education*. In the section *Current Concerns and Emergent Questions in the Field*, the authors mention the need for (1) common ground between the history of mathematics and mathematics education, (2) effective theoretical and conceptual frame- works, (3) more in-depth empirical studies, and (4) a more refined reflection around the interdisciplinary role of history. We can recognize a convergent and recurrent point, namely, the need for the development of theoretical positionings and a conceptual basis in the field. Putting emphasis on preservice teachers, we discuss theoretically some of the difficulties encountered when engaging students in the reading of historical texts in this context.

2.2 The research problem

For some educators, the reading of historical texts is a privileged way for learners to encounter history, particularly in the context of mathematics teachers' education (e.g., Arcavi & Isoda, 2007; Barbin, 1997; Fried, 2007, 2008; Guillemette, 2017; Jahnke, 2014).

Several studies have emphasized the experience of "dissonance" or "alienation" (e.g., Arcavi & Isoda, 2007; Jahnke, 1994, 2014), which students can feel when reading historical texts. Others have suggested that these reading activities could help the prospective teachers learn something about their own mathematics knowledge and positions by experiencing and "reflecting on the *contrast* between modern concepts and their historical counterparts" (Fried et al., 2016, p. 218).

This said, the different ways to read and to encounter mathematics from the past in the context of mathematics education are still discussed. Fried (2007, 2008, 2018) has suggested that the historian's goal is to immerse oneself in the age of the mathematician, to perceive the latter's idiosyncrasies, and to situate the work in the continuum of the modern concepts the essential mathematical aspect of the author's words.

Drawing on Ferdinand de Saussure (1857–1913), Fried (2007, 2008, 2018) characterised the historian's reading as being both *synchronic*— considering the situation of the text within a given historical system of mathematical concepts— and *diachronic*—considering the historicity of the concepts and their evolution in time and space. For Fried, historians see the text as existing on a synchronic plane that is "relative to the community of 'mathematical speakers' out of which the text arose" and are "interested in synchronies of the past and their crystallization" (2007, p. 213). On their side, the mathematicians' reading focuses more on the relation between the actual modern synchronic plan and other synchronic plans from the past. Fried (2018) highlighted different "mathematician's ways" of relating to the past, such as "privileged observer," "mathematical critic" or "treasure hunter", for instance. This distinction between the historian's and the mathematician's reading can be linked with the one put forward by Grattan-Guinness (2004) between *History* and *Heritage*, the first being associated with the effort to understand the context of the emergence of the concepts, and the latter being associated more with the search for explanations of "How did we get there?" (Fried, 2018, p. 6).

Concerning mathematics education, empirical studies in the context of teacher education in mathematics confirm the preservice teachers' strong tendency to deploy a reading based on a modern synchronic plan when they engage with historical texts (see Fried, 2000; Guillemette, 2016, 2017). The results show that the encounter with the author perceived in his sociohistorical and mathematical context does not happen straight away despite the attention and efforts of the teacher educator.

This Saussurean perspective, focusing on interpretation and relying on the concepts of diachrony and synchrony, has been frequently discussed in HPM literature (e.g.,Bråting, 2019; Guillemette, 2017; Neuwirth, 2016; Rodriguez & Lopez Fernandez, 2010). What has been less discussed in reports of the research carried out within the HPM community is the conception of language offered by de Saussure—a conception that gives rise and meaning to the concepts of diachrony and synchrony. It seems to us that the lack of a sustained discussion around de Saussure's proposed concept of language has obscured the understanding of the empirically observed problems surrounding the readings of historical texts.

In the following section, we present an example coming from an empirical study about engaging mathematics pre- service teachers in the reading of historical texts.[2] We try development of mathematics. On the contrary, the mathematician tries, for his part, to decode obsolete symbols, to restore them to modern language, and to grasp through

[2] More details on this Saussurean analyses of the prospective teachers' tendency to deploy a reading based on a modern synchronic plan can be found in the report by Guillemette (2016). These analyses are part of a larger empirical study concerning the description of the pro-

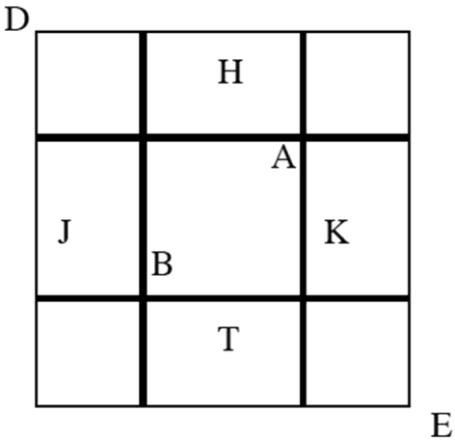

Fig. 1 Al-Khwārizmī's rhetorical and geometric explanation of how to solve quadratic equations (from Djebbar, 2005)

to show that the Saussurean diachronic/synchronic dichotomy—on which the ideas of the historian's reading and the mathematician's reading are based—might not be enough, and that there is a need to envision a third possible reading: that of the *educator*. Exploring the Saussurean concept of language and some of its intrinsic problems will help us to better articulate this third possible reading.

2.3 Reading al-Khwārizmī

To set the scene, the focus of the topic reported upon here is solving quadratic equations using the methods of al-Khwārizmī that can be found in the *The Compendious Book on Calculation by Completion and Balancing*. The text used can be found in the publication by Djebbar (2005). It consists of a French translation of al-Khwārizmī's rhetorical and geometric explanation of how to solve quadratic equations. The text was read by six preservice teachers taking part in an undergraduate program in secondary mathematics teachers' education. With the participants, the focus was on an excerpt presenting the case in which a square (*bien* in the French translation) and 10 roots are equal to 39 units (dirhams). This specific passage of al-Khwārizmī's was chosen because it was easier to introduce in a series of reading activities, conducted through a semester in a history of mathematics course, that presents a certain plot (intrigue) that unified the presented texts, with the aim of facilitating a reading that focuses more on the synchronic plan from which the text arose and the historicity of the concepts involved or their evolution in time and space (diachronic reading). Here is our translation of the text (Fig. 1):

> As for the justification of "one square and ten roots equal thirty-nine dirhams," its figure is a square sur- face of unknown sides, and it is the square that you want to know and whose root you want to know. This is the surface (AB), and each of its sides is the root. Each of its sides, if you multiply it by a number among the numbers, whatever the numbers, will be numbers of roots, each root being like the root of this surface. As it has been said that with the square there are ten of its roots, we take the quarter of ten—and that is two and a half—and we transform each of its quarters [into a segment] with one of the sides from the surface. There will thus be, with the first surface, which is the surface (AB), four equal surfaces, the length of each of them being like the root of the surface (AB) and its width two and a half, and these are the surfaces (H), (T), (K), (J). The result [is] an equal-sided surface, also unknown, and deficient in its four corners, each corner being deficient by two and a half by two and a half. So, what is needed as an addition so that the surface is square, will be two and a half by itself, four times; and the value of all this is twenty-five. Now we have learned that the first surface, which is the surface of the square, and the four surfaces which are around it and which are ten roots, are [equal to] thirty-nine in number. If we add to them the twenty-five which are the four squares which are in the corners of the surface (AB), the quadrature of the largest surface, and which is (DE), will then be completed. Now we know that all this is sixty-four, and one of its sides is its root, and that is eight. If we subtract from eight the equivalent of two times a quarter of ten—and that is five—, at the extremities of the side of the largest surface which is the surface (DE), there remains its side three, and that is the root of this square.

After a 75-min period of study of the historical and mathematical context of the text, as well as a quick presentation about the mathematician, the participants were invited to begin reading in small groups. This presentation was done by an historian of mathematics, an experienced professor in this context of teachers' training. We (one of the authors) took charge of the reading activities, with the explicit objective, following Fried, of highlighting both the synchronic (exploring the synchronic plan from which the text arose) and diachronic perspective (situating the text in a historical continuum). Here, we follow Martha, Alyosha, and Ninotchka.

Martha turned her desk around to face Alyosha and Ninotchka. They started reading individually and silently for 5 min (see Fig. 2).

Footnote 2 (continued)

spective teachers' lived experiences when engaging in the reading of historical texts (see Guillemette, 2017).

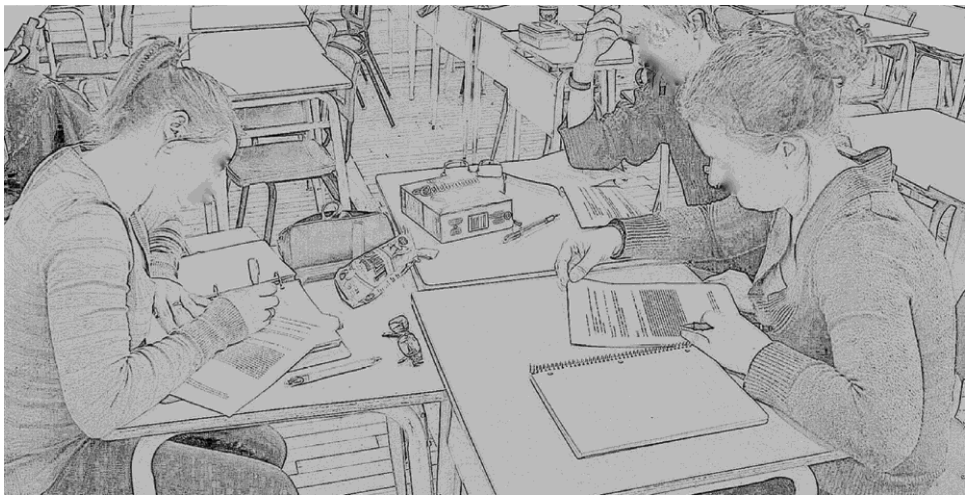

Fig. 2 Martha, Alyosha, and Ninotchka start reading al-Khwārizmī

Fig. 3 Martha, Alyosha, and Ninotchka making sense of Al-Khwārizmī's rhetorical and geometric explanation

Martha questions Ninotchka about what the figure drawn by the author represents. Together, they take up the meaning of the different elements of the figure and quickly return to their reading.

Martha asks her teammates why al-Khwārizmī takes a quarter of the value of the term in x. Alyosha explains that "since the author's initial figure is a square, he must then add four rectangles around it, which are associated with the x term". The latter must therefore be divided into four. Alyosha points to the four rectangles on the figure drawn by Martha.

We can already notice after a few minutes a translation into modern algebraic language. Participants discuss an "*x* term" without justifying or explaining themselves. Of course, the use of letters in the enunciation of explicit algebraic reasoning is not found anywhere in this excerpt.

Martha then asks: "The length 10, how do we know it?" Alyosha expresses his incomprehension. She explains: "In my diagram, how do I know how much it measures 10/4, I give myself a reference unit?" Alyosha points out that the author "speaks of the 'dirham,' a currency which can here be considered as the unit" (see Fig. 3).

Once again, the participants' focus here is to translate the author's vocabulary into their own words.

They resume their reading individually. Alyosha now concentrates on the figure while Ninotchka and Martha reread the excerpt word for word, developing their drawing with new elements. Martha again highlights passages from the excerpt, while Alyosha tries to solve the problem algebraically.

We can observe that the participants refer to an algebraic solution strategy using geometric representations (algebraic tiles), a tool acquired during their earlier didactic training. Alyosha begins a classic modern algebraic approach.

Alyosha stands up and asks the teacher educator for help. The latter considers Alyosha's strategy of thinking about a modern algebraic solution (see Fig. 4). *Alyosha replies that "the quadratic formula should be applied." Ninotchka proposes the completion of a square as a solution strategy and shows her steps to Alyosha.*

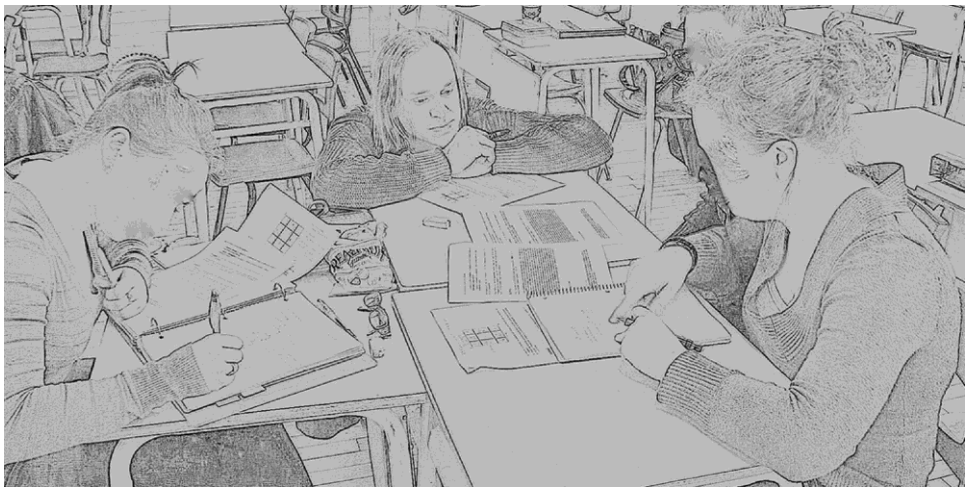

Fig. 4 Martha, Alyosha, and Ninotchka developing a modern solution

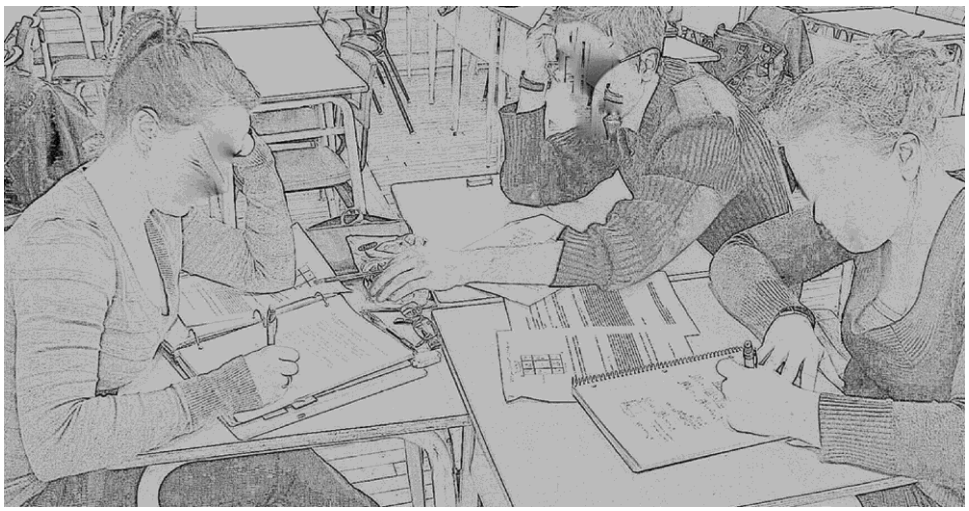

Fig. 5 Martha, Alyosha, and Ninotchka trying to reconcile their modern solution with al-Khwārizmī's approach

The teacher educator suggests that the next step could be to draw a parallel with al-Khwārizmī's approach to bet- ter understand the latter. The group then tries to reconcile Ninotchka's approach with that of the author. The teacher educator then leaves the group and leaves the participants to their reflections. They each resume their work individually. We can observe the teacher educator's effort to guide the learners in such a way that they will not exclusively translate and solve the problem on their own terms, but will focus on the elucidation of al-Khwārizmī's perspective.

Ninotchka then remembers that the completion of a square is usually associated with a geometric figure. She shows her drawing to the group and asks Alyosha's opin- ion on her approach. She points out that in this case, it is necessary to divide by two and not by four as al-Khwārizmī did. Martha explains that al-Khwārizmī adds four rectangles around his square rather than two, as is usually done when completing a square.

Returning to the square completion method, which appears closer to that of al-Khwārizmī, the participants try to reconcile their ways of doing things with those of the author. The focus is now on the geometrical representation of algebraic quantities, as al-Khwārizmī does, and as they are used to doing with algebraic tiles.

Alyosha therefore explains that if we want to follow the author's method, "we must then divide the term in x by four". Martha goes on to explain that "the rectangles will cover the same area, but will be arranged differently". Alyo- sha agrees and adds that the associated algebraic approach will then be different. They again turn back to the modern algebraic exploration of the author's approach.

Advancing in the reconciliation between the square com- pletion method and that of al-Khwārizmī, the participants propose in parallel to provide a modern algebraic approach. *Martha attempts to generalize the case treated by the author using an algebraic expression. She explains how she got her expression to Alyosha* (see Fig. 5). *The latter*

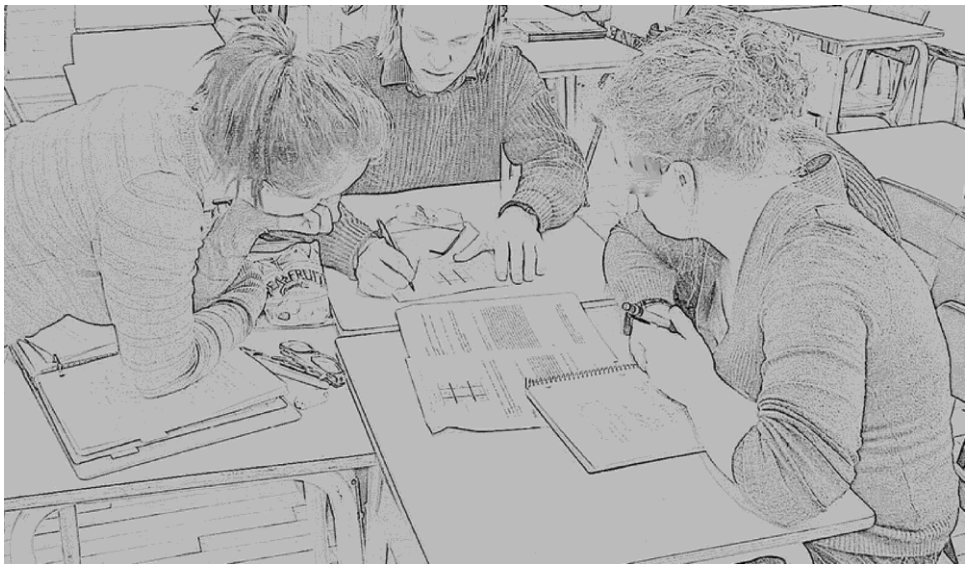

Fig. 6 Martha, Alyosha, and Ninotchka proposing to the teacher trainer to reconcile the quadratic formula with al-Khwārizmī

further generalizes Martha's expression and accompanies her in the refinement of her approach. After some advances, Alyosha concludes, however, that Martha's generalization does not help them in the understanding and validation of the author's approach.

Alyosha then continues the process on his sheet and announces that he believes he is approaching the quadratic formula. The group then accompanies Alyosha on this search. Alyosha concludes that he succeeded in reconciling the author's approach with the quadratic formula, except for the sign of one of the terms of his equation. He gets up and then asks the teacher educator for help, in order to explain this difference and complete his process. With the group, the teacher educator then goes over in more detail the reasoning associated with the application of the quadratic formula (see Fig. 6).

It is then that Ninotchka suddenly understands that the sign is reversed in the application of the quadratic formula from the general case, which explains the problem previously raised by Alyosha. The group then goes back to work individually. Alyosha concludes that the use of geometry brings the approach closer to that of the mathematicians of the Hellenistic period. He also points out that al-Khwārizmī proceeds from a particular example to discuss a general result. The reading session is suspended by the teacher educator.

At the end, we can see that Alyosha offers some thoughts about the author's approach and tries to make connections between his remarks and the mathematical context, but also to situate the text within the historical evolution of mathematics. These reflections occur only at the very end of the reading activity. We can see that the participants began quickly to restore the text into modern language, to grasp the essentially mathematical aspect of the author's words (modern synchronic plan). Despite some efforts of the teacher educator, the focus is clearly not on al-Khwārizmī's idiosyncrasies, style, or representativeness of his era (the synchronic plan from which the text arose), and the work is hardly situated in the continuum of the development of mathematics (diachronic reading). Moreover, we observe the tendency to make links with didactic tools (such as algebraic tiles) and mathematics classroom culture.

In this paper, we attempt a different interpretation. The objective is to discuss the limits of the diachronic/synchronic binary perspective and to sketch a new theoretical positioning. This leads us to think differently about the reading of historical texts in the context of teachers' education and the associated difficulties, but, most of all, to emphasize a third possible reading besides that of the mathematician and the historian, that of the *educator*. We argue that the educator seems to read the text differently, focusing on its estrangement potential in a particular way. Ultimately, we attempt to provide some articulated new avenues of reflection and intervention.

3 Theoretical exploration

We start this section by dwelling on some ideas in de Saussure's work. Then, we move to a brief presentation of Voloshinov's critique of de Saussure and to Bakhtin's idea of language and dialogicality with the aim of rethinking the question of diachronic and synchronic readings of historical texts.

3.1 On the Saussurean perspective

De Saussure's major work is the *Cours de Linguistique Générale* (*Course in General Linguistics*) (1916/1959). It deals with linguistics, as the title suggests, but de Saussure describes it as "general linguistics." Indeed, he studied language as such, which brings him to be situated in a larger framework, that of semiology. He states that "*a science that studies the life of signs within society* is conceivable; it would be a part of social psychology and consequently of general psychology; I shall call it *semiology*" (de Saussure, 1916/1959, p. 16, emphasis in the original).

De Saussure brings up the notions of "signifier" and "signified" which, isolated in a closed system, would form a sign. The "signifier" corresponds to the acoustic image (the sound or the word or the set of sounds or the intonation, etc.) and the "signified" is the object that is targeted by the speaker, the concept itself. However, the most important aspect in de Saussure semiology remains the essentially arbitrary character of the sign. This theoretical assumption accounts, first of all, for the diversity of languages and, secondly, for the paradoxical aspect of de Saussure's thought, seeking a "general linguistics." Indeed, he emphasizes that signs evolve over time and under the influence of the *masse parlante* (community of speakers). This leads him to say that

> Language is no longer free, for time will allow the social forces at work on it to carry out their effects. This brings us back to the principle of continuity, which cancels freedom. But continuity necessarily implies change, varying degrees of shifts in the relationship between the signified and the signifier. (de Saussure, 1916/1959, p. 78)

At this point, again, there seems to be a paradox between the evolutionary and social aspects of the sign. How can a sign be both variable and invariable? Regarding this point, de Saussure mentions that "if we considered language in time, without the community of speakers […] we probably would notice no change; time would not influence language. Conversely, if we considered the community of speakers without considering time, we would not see the effect of social forces" (de Saussure, 1916/1959, p. 78). We can distinguish two perspectives here namely, an ahistorical perspective as if we observe, like a photo, the effective signs in the community of speakers, and a historical perspective where the signs are in perpetual change (see Fried, 2008). From then on, it appears that "the stream of language flows without interruption" (de Saussure, 1916/1959, p. 140). It is here that de Saussure introduces the terms "synchronic" and "diachronic" in order to distinguish respectively the ahistorical and the historical perspective: "Everything that relates to the static side of our science is synchronic; everything that has to do with evolution is diachronic. Similarly, synchrony and diachrony designate respectively a language-state and an evolutionary phase" (1916/1959, p. 81).

3.2 Voloshinov's critique

One of the main admirers of de Saussure, and, at the same time, one of his main critics, was the Russian linguist Valentin Voloshinov (1895–1936). In *Marxism and the Philosophy of Language*, Voloshinov (1929/1973) considered advantages and disadvantages of structuralism that arose in the posterity of de Saussure. Unlike de Saussure, who in his works focused on the synchronic and formal aspects of language, Voloshinov puts an emphasis on the socially and historically situated ideological dimension of signs and language.

Three of the main objections of Voloshinov are the following. First, de Saussure presents a rationalist and formal approach to language "that regards history as an irrational force distorting the [assumed] logical purity of the language system" (1929/1973, p. 61).[3] The diachronic view of language can hence be understood only formally, as the mere effect of the community of speakers, abstracted from the materiality of the world, and the ideas and social practices that shape language. Second, de Saussure continues the longstanding philosophical tradition that posits the social and the individual in antagonist terms. "[de] Saussure's main thesis: *language stands in opposition to utterance in the same way as does that which is social to that which is individual*" (1929/1973, p. 60, emphasis in the original). The emphasis on the formal aspects of language impoverishes the synchronic view of language, where *parole* (speaking) finds itself removed from scientific investigation as does the subjective and expressive dimension of communication. Third, and related to the previous point, there is this problematic axiom established by de Saussure, which consists in the idea that linguistic communication results from the fact that categories by which the recipient knows the identity of a sign are necessarily those by which the issuer has produced it. For Voloshinov, accepting the Saussurean model and its presuppositions leads to an impoverishment of the linguistic experience and to treating the individual as a code user. It also brings us to see the social world as a universe of symbolic exchanges where our linguistic actions become acts of communication that are intended to be deciphered by the means of a code.

To sum up, in Voloshinov we find a critic of de Saussure's conception of language, and the notion of

[3] As de Saussure contends: "In separating language from speaking we are at the same time separating: (1) what is social from what is individual; and (2) what is essential from what is accessory and more or less accidental" (1916/1959, p. 14).

synchronicity—perceived as an abstraction that does not correspond to any actuality—and of diachrony—perceived as a historical evolution abstracted from real life.

Let us now turn to a different approach where language is seen both as an empowering dimension of subjective individual expression and thinking, and the continuous and evolving link between the social and the individual. As we suggest, this position invites us to surmount de Saussure's formal view of the synchronic and the diachronic by considering our cultural-historical situated and embodied nature and the ensuing implications for ethics.

3.3 Dialogicality and ethics

Recognizing the insurmountable historicity and social anchoring of language, the philosopher and literary critic Mikhail Bakhtin (1895–1975) developed a sociolinguistic theory that can be qualified as "interpretative" instead of the "logicogrammatical" one of de Saussure (Vauthier, 2002).[4] Indeed, for Bakhtin, when we speak, we invest ourselves subjectively. This is why "no word can be spoken without an evaluative accent, without an attitude adopted towards that of which it speaks" (Dentith, 2005, p. 36). The same is true of the listener. When the listener hears us, "he [or she] simultaneously takes an active, responsive attitude toward it [what has been said]. He [or she] either agrees or disagrees (completely or partially), augments it, applies it, prepares for its execution, and so on" (Bakhtin, 1986, p. 68). Bakhtin adds: "[The speaker] does not expect passive understanding that, so to speak, only duplicates his own idea in someone else's mind. Rather, he expects response, agreement, sympathy, objection, execution, and so forth" (1986, p. 69). The result is that what is spoken is always done in a language that unites and divides, a language that is a whirlpool of divergent and convergent forces, for language is, as Bakhtin (1981) noted, heteroglossic (i.e., contextually and historically dependent, affording and conveying thereby differences) and monoglossic (i.e., subjected at the same time to the centripetal or unifying effects of culture).

This is where the ethical aspects of Bakhtin's perspective are crucial. Indeed, considering the heteroglossic dimension of language, the relation I-Other is necessarily a leitmotif in the Bakhtinian corpus, as he insisted on the situated and embodied nature of lived existence and experience (*perezhinanie*) and its implications for ethics (Bakhtin, 1981, 1986). Precisely, he put great emphasis on what transcends or what is located beyond our immediate subjective existence and our cognitive activity (the objective unity of a domain of culture), which necessarily participates in what can be called "otherness." Bakhtin highlighted the idea that our Being-in-the-world is necessarily dialogical, that we are co-participants in a world where other subjects exist. Furthermore, our very identity is constituted of these relationships, of an incarnate dialogism. The Other has, for Bakhtin, a "surplus of vision" on us, and vice versa, making possible a "reversal of perspectives" which is capital here in the human intersubjective experience and the formation of a proper ethical consciousness (Bakhtin, 1986/1993).

Bakhtin's idea of language as something intrinsically dialogical—language as a dynamic organ that bears in itself the contradictions of a heteroglot historical and cultural world that brings Self and Other to each other, as subjectivities in the making, as consciousness producing themselves mutually—opens a door not to dismissing diachrony and synchrony, but to conceiving of them as mutually constitutive, as concepts in relation. Bakhtin's idea of language leads us also to recognize the dialogical and ethical dimension of language and invites us to think differently—or at least to bring important nuances—about what it means to read a historical text in the context of mathematics teachers' education.

4 Investigating a dialogical/ethical perspective

Let's get back to Martha, Alyosha, and Ninotchka trying to cope with al-Khwārizmī's rhetorical and geometric explanation of how to solve quadratic equations. As we have seen, the participants have a strong tendency to translate the text into modern language, and to grasp the essentially mathematical aspect of the author's words. Despite some efforts of the teacher educator, the focus is clearly not properly on al-Khwārizmī's idiosyncrasies, style, or representativeness of his era, and the work is hardly situated in the continuum of the development of mathematics. There is, we may say, a mathematician's reading taking place, accentuating mathematical correctness, terminology adequacy, robustness of the argument, etc. (reading from a contemporary synchronic plane). Indeed, we saw Alyosha approaching the problem with the quadratic formula and considering al-Khwārizmī's explanations in the light of modern algebraic perspective (see Fig. 5). We can hardly see a form of conciliation or a form of appreciation of al-Khwārizmī's work on his own terms (reading from the synchronic plane from which the text arose) and an appreciation of the text within the historical evolution of mathematics (diachronic reading), which would be the historian's reading of the text. However, there

[4] On the one hand, it is important to be aware that Bakhtin and his close collaborators do not propose a unified and precise theory, but an articulated epistemological position which is applied in quite het- eroclite domains. On the other hand, it is also important to be aware of the important polemic around the authorship of this corpus. The origin of the various texts that appeared under the name of Bakhtin and the existence of a single and homogeneous Bakhtinian corpus are now discussed among literary scholars (see Bronckart & Bota, 2011).

is a proposition by Ninotchka, followed by her two team- mates, to articulate the explanation of al-Khwārizmī's with the didactic algebraic tiles (see Fig. 4), having in mind and caring about the potential moments of teaching and learning involving these objects. Bakhtin's dialogical approach would allow us to witness the appearance of a form of student agency (a reflective stance) that does not correspond to the mathematician's or the historian's approaches, as the readers (i.e., the students) are here embedded in their teacher training cultural-historical context and are obviously more inclined to put forward their pedagogical or didactic concerns. In other words, what the students read is already imbued with their responsive evaluative accents.

In this sense, there seems to be a third possible reading of the episode in the context of mathematics teachers' training, which is not properly the reading of the mathematician, nor properly the reading of the historian, but the reading of the *educator*. Having in mind the dialogical/ethical perspective developed above, we would like to argue that the educator seems to read the text in a *different* manner, which is not driven by translating into modern ways, nor by situating the text in a historical continuum. It is a dialogical reading that brings with it its potential estrangement. Furthermore, this dialogical educational reading questions the self in its theoretical assumptions and offers vicarious aspects around the ways of *being* in mathematics and ways of *doing* mathematics. What happens here with the tension between the mathematician's and the historian's reading of the historical text? In the theoretical perspective that we are outlining here, this dialectical tension is not articulated in a synthesis of a positive reason that would cause it to be overcome. On the contrary, we want to argue that there is space for the dialogical/ethical reading of the educator, that is, an open-ended critical reading that, resisting a synthesis, remains the effect of what Hegel calls "negation"; that is, the affirmative force of *difference* (1807/1977). The focus here is not principally on mathematical correctness, nor historical correctness, but on the ethical correctness, that is to say on the recognition of Otherness and the active expectation of a constructive dialogue in terms of ways to engage in mathematics. Let us develop this idea further.

As discussed in Sect. 3, for Bakhtin, understanding others is not a bipolar activity where self and others remain separated. Understanding others entails the immersion of self into a genuine dialogical relation that brings self and others *together*. Rather than being a mere message, "Any utterance is a link" (Bakhtin, 1986, p. 69). Let us say that, from the dialogical perspective outlined here, the objective is the same, namely, to understand a historical text made by a mathematician from the past. But the ways of its realization and the very notion of understanding are quite different. Indeed, for Bakhtin, "First, the task is to understand the work as the author himself understood it, without exceeding the limits of his understanding. […] Second, the task is to take advantage of one's temporal and cultural outsideness. Inclusion in our (others's for the author) context" (Bakhtin, 1986, p. 144).

From a dialogical/ethical perspective, there are always two subjects that are participating in the act of meaning. The focus is more on the dialogical and historical dimension of the phenomenon. In this sense, the reading of a text is thus perceived as "the meeting of two texts—of the ready-made and of the reactive text being created—and, consequently, the meeting of the two subjects and two authors" (Bakhtin, 1986, p. 107). The focal point here, based on a dialogical/ ethical perspective, is the emergent dialogue, not really at the level of concrete exchange, but rather on the intrinsically tied levels of ideas, ways-to-engage-in-the-world or forms of consummation.

4.1 Engagement and answerability

On this basis, to understand something would mean to enter responsively in a dialogue; that is to say, to bring engagement and answerability within the encounter with the Other. Indeed, the appearance of otherness throws into relief elements that are indispensable to any genuine dialogue. It is not the positive or constructive exchange of goods, approval, or power that is crucial here, but rather a recognition of the 'Otherness' of 'the Other'.

The possibility of the Other, the experience of Other- ness, seems to be the very condition that enables the learner to engage with something new—a key experience that is characterized by a fundamental *vulnerability* (from Latin *vulnerabilis*, "who could be wounded"); that is to say, an *exposure to a distance* (in our case an historical or cultural one), which may be received either "as a terrifying abyss or as a 'dialogic span' that motivates the desire to know the other, as well as the self, and fosters movement and change" (Gurevitch, 1988, p. 1180). In this sense, there is a fundamental ethical aspect of the act of understanding, as every act of understanding is responsive and dialogical, seeking a counter diction—which means the possibility of showing answerability and engagement toward the Other. Only an active/responsive/engaging comprehension can grasp the theme of the meaning of the speech.

If we turn back to our classroom episode, we see Alyosha, Martha, and Ninotchka *engaging* with al-Khwārizmī's text. They engage (etymologically speaking, they *bind* themselves) with the text in a way that brings different nuances and evaluative accents. While Alyosha and Ninotchka seek connections through a quadratic formula and the completion of a square, respectively, Martha focuses on areas and their spatial positions. The students' polyphonic engagement encounters in the instructor's reaction an invitation to approach the text from its own perspective, highlighting the

need to attend to the Other in its Otherness. The centripetal and centrigual forces that derive from engagement shape how students *answer* to the text. Answerability, the concrete answer to al-Khwārizmī's text, is the attending way the subject's subjectivity shows itself in the reception (*l'accueil*) of the other in its unavoidable estrangement (the *bien*, the roots, the dirhams) that marks the Other *as* Other. Martha (re-voicing al-Khwārizmī): *"The length 10, how do we know it?"*. In other words, answerability is the coming to the Other within the possibilities of the self —a coming that trans- forms being and is, hence, properly speaking, a be-*coming*. Answerability is not simply driven by the epistemic requirements of precision. Answerability is also driven by the ontological-ethical demands of being. As Bakhtin puts it "the limit of precision in the natural sciences is the identity ($a = a$). In the human sciences, precision is surmounting the otherness of the other without transforming him into purely own's own (any kind of substitution, modernization, non- recognition of the other, and so forth)" (1986, p. 169). The criterion of answerability in the dialogical/ethical perspective outlined here is not the correctness of knowledge such as in natural sciences, nor the adequation between the world and the self who is thinking it, but the depth of penetration in the text and the engagement in the emergent dialogue.

Let us turn to some implications for both teachers education and research within this context, which this way of thinking about the reading of historical texts could entail.

4.2 Implications for mathematics teachers' education

For the prospective teachers engaged in the reading of historical texts, the focus would be more directed to the engagement/answerability in the interpretation of the text by emphasizing the voices manifesting themselves (voices from the past, like al-Khwārizmī in the example, voices from the classroom, and voices from a larger sphere of communication such as those active in the actual sociopolitical environment).

Within this Bakhtinian perspective, the main difficulty here is not associated with the objective of knowing the mathematics of the text both synchronously and diachronically in an articulated way. The main difficulty here could be precisely to, phenomenologically—related to the inner- most part of the lived experience—face the adversity that resides in the experience of otherness (due to historical and cultural distance) in order to engage in an active/responsive/engaging relation with the text. As mentioned above, in the act of understanding, there is a need to welcome the alterity of what is *Other* without transforming it into some- thing which is a *mine*; that is, without assimilating it to our own conceptual categories—a modernized or reified Other, instead of an authentic interlocutor. Thus, the latter points not to the difficulty of understanding the Other, but rather to the difficulty of beholding the Other as Other and avoiding alienating it.

On the side of the teacher educator, there is a need to engage, support, and accompany the students with inclusive gestures in their interpretative enterprise, and also to offer, highlight, challenge, and expound interpretations, as someone who has developed (at least potentially) a certain sensibility regarding the manifestations of different voices involved in this special encounter. Indeed, from a historicocultural perspective in mathematics education, the classroom is not a neutral space because the modes of activity (objectives, interactions, operations) taking place are mediated by the surrounding culture. Mathematical activity is thus imbued with scientific and ethical values that are consubstantial with learning (Radford, 2021a), which can be under- stood as a social process of critical awareness of culturally and historically constituted knowledge. This understanding implies, inevitably, the recognition of the social and ethical dimensions of mathematical activity (Radford, 2021b).

We can also find similar insights in the work of Barbin, who thinks of both history itself and the work of historians in terms of a Bakhtinian perspective (see Barbin, 2014), and also the interaction between voices from the history of mathematics and the mathematics classroom (see Barbin, 2011). Her emphasis lies on the rejoinder role of utterances and the ensuing configuration of speech genres. Our emphasis, by contrast, moves to the onto-ethical dimension of dialogicality. It brings to the fore subjectivities of the present and subjectivities of the past in an encounter of self and Other in the answerability; that is, the polyphonic active/responsive/engaging relations that the readings of historical text convoke.

This said, a question arises about the implications of this perspective for teachers' education. Indeed, there is, on the one hand, the need to recognize the sensibilities that a practice of answerability might require—the sensibilities of engaging in an active/responsive relation with the historical text. On the other hand, there is the question of ensuring that the students confront the text through an authentic (that is, always emerging, contested, polyphonic, and conflicting) experience of Otherness. The response to this twofold question could result in antinomian training practices. Indeed, on the one hand, there is a need to prepare the students for this encounter. As reported extensively in research (Jankvist, 2009), the reading of an historical text cannot be done with- out a minimum of introductory instructions or explorations around the historical, cultural, or mathematical context of the mathematician. On the other hand, there is also the need to prepare pedagogically for the experience of Otherness and what has been called *dépaysement* (disorientation, sometimes translated as reorientation) (Barbin, 1997; Jahnke et al., 2002). Indeed, there is a need to *feel* a distance

in order to figure out the mediation of mathematics by the surrounding culture and the social and ethical dimensions of mathematical activity. On the one side, if the encounter between the students and the text is pushed too hard on a previously established path, the experience of Otherness could be potentially impoverished, as the students would be confronted with an already determined entity and the reading activity would be reduced to a recognition or high- lighting game. On the other side, if the encounter is not pre- pared enough, there is a risk of missing the encounter itself because of a much too large semantic distance between the students and the text, making it impossible to cope with dis- tant voices. The irony of the encounter is that it cannot really be fully prepared: its mode of being is that it always unfolds in unpredictable ways, moving dynamically according to the active/responsive/engaging voices that the encounter brings into dialogue.

4.3 Implications for research

From a dialogical/ethical perspective, these encounters with the history of mathematics arise in the classroom within an already ongoing dialogue on mathematics and on mathematics education. In the context of a research study, this dialogue could include the teacher or the teacher educator, but also the researcher. Bakhtinian concepts (such as dialogism and polyphony) can here provide the necessary means to think about the elaboration of a description of these encounters that includes voices of the participants (prospective teachers, teacher educators, etc.) and of the researcher, and also voices from the past (see Barbin, 2011; Guillemette, 2019).

At the end, the task then would be to restore the acts of meanings that have emerged in the context of the reading activities, but, above all, to restore them *in their dialogical interaction*. There is room here for casting a reflective and critical stance towards how the historical readings have challenged the prospective teachers', teacher educators', and researchers' own ideas and expectations, and how this reading has transformed their voices. For example, one task could be collectively to assess critically the depth of the engagement and to investigate the experience of the acts of the participants in the reading (students or teacher trainers educators, or researchers if we think of the participative approach in research), which is not exclusively about the exactitude, the correctness, or the congruity of the acts of meaning, but their active/responsive/engaging relations regarding the mathematicians and their work.

5 Conclusion

Concerning the reading of historical texts in the context of teachers education, one of the principal goals that have been put forward is to explore, with prospective teachers, their own mathematics knowledge and epistemological assumptions by experiencing and reflecting on the contrast between modern concepts and their historical counterparts (see Fried et al., 2016). One of the main difficulties, in this context, is to be able to welcome the mathematicians from the past 'on their own terms' in order to make this contrast apparent (see Radford & Santi, 2022). We suggested above that the lack of a sustained discussion around de Saussure's proposed concept of language has obscured the understanding of the empirically observed problems surrounding synchronic and diachronic readings of historical texts, and that there is a need in HPM literature for a discussion of these issues.

In this paper, we developed a different theoretical point of view. Taking into consideration Voloshinov's critique of de Saussure and seeking to overcome a formal view of language and its synchronic and diachronic aspects, drawing on Bakhtin, we articulated a dialogical/ethical perspective in the context of teachers education. Emphasizing the fact that dialogues always occur in specific forms of activity, with their own object-oriented specificities, this perspective leads us to highlight how preservice teachers seem to read historical texts in a manner that is neither the mathematician's (looking at the past from a modern synchronic plan) nor the historian's (both synchronously by considering a given historical system of mathematical concepts and diachronically by considering concepts' evolution in time and space). In this dialogical/ethical perspective, the classroom episode featured an engagement with the historical text where the students reacted to it in terms of an agentic culturally and historically situated positioning in which the subjectivity of the students shows itself polyphonically in the reception of the text and its unavoidable estrangement. Drawing on Bakhtin and Voloshinov, we called this dialogical, engaging, polyphonic reaction *answerability*. Then, within this perspective, we stressed some elements necessary for the preservice teachers to enter into a dialogue with mathematics from the past, such as the responsive/engaging relations regarding the mathematicians and the setting of a 'dialogical window' in the experience of Otherness.

This focus on the onto-ethical dimension of dialogicality is not a threat to mathematical and historical reading and analyses of historical text in the context of teachers education. We would like to argue that this theoretical position has the advantage of opening the spectrum of interpretation and investigation of these special educational practices by avoiding a polarization between a mathematician's and a historian's reading, and by manifesting other ways to

relate to historical text and to history. Before interpreting as an historian would do, there is, first, an Other that has to appear and, second, there is a dialogical window, in which the mathematician's way of reading or of engaging is only one possible part. In this dialogical/ethical perspective, for the educator who is in search of the prospective teachers learning something about their own mathematics knowledge and positions by experiencing and reflecting on the contrast between modern concepts and their historical counterparts, there is the necessity to be attentive and to enter into the students' polyphonic engagement. This seems an important path to help the emerging, cultivating and fostering of the feeling of a distance, the dimension of duration in mathematics, and the accompanying estrangement.

Possible implications for practice and research on the history of mathematics in mathematics education have also been suggested. Particularly for teachers education, we high- light the need to engage, support, accompany, offer, and challenge the students' emerging interpretations with inclusive gestures in their interpretative endeavour and to help emphasize the manifestations of different voices involved in the readings. Concerning research, we draw attention to the need to restore, in their dialogical interaction, the acts of meaning that emerge in this context and to assess, collectively and critically, the depth of the engagement of the preservice teachers.

Finally, we would like to emphasize again the importance of these educational practices, alongside investigation of the theoretical developments on the subject and the implementation of these practices. The reason is that the latter seems to have the potential to highlight, within a particular experience of Otherness that is quite difficult to imagine in another educational context, the very fundamental situated aspect of the mathematical activity, as well as the social, cultural, and political perspective on mathematics and mathematics education—a reflection that, we believe, is crucial for the reinterpretation and development of the meaning and the orientation of mathematics education itself.

Acknowledgements We would like to acknowledge the support of the Fond de Recherche du Québec—Société et Culture and the Conseil de Recherches en Sciences Humaines du Canada/Social Sciences and Humanities Research Council of Canada.

References

Arcavi, A., & Isoda, M. (2007). Learning to listen: From historical sources to classroom practice. *Educational Studies in Mathematics, 66*(1), 111–129. https://doi.org/10.1007/s10649-006-9075-8

Bakhtin, M. M. (1981). The dialogic imagination: Four essays. *University of Texas Press*. https://doi.org/10.2307/1772435

Bakhtin, M. M. (1986). Speech genres and other late essays. *University of Texas Press*. https://doi.org/10.2307/3684926

Bakhtin, M. M. (1993). *Toward a philosophy of the act*. University of Texas Press. https://doi.org/10.7560/765344-004. Originally published in 1986.

Barbin, É. (2011). Dialogism in mathematical writing: Historical, philosophical and pedagogical issues. In V. Katz, & C. Tzanakis (Eds.), *MAA notes*, (Vol. 78, pp. 9–16). MAA.

Barbin, É., Guillemette, D., & Tzanakis, C. (2020). History of mathematics and education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (2nd ed., online). Springer. https://doi.org/10. 1007/978-3-030-15789-0_69

Barbin, É. (1997). Histoire et enseignement des mathématiques: Pour- quoi ? Comment ? *Bulletin De L'association Mathématique Du Québec, 37*(1), 20–25.

Barbin, É. (2014). Une approche bakhtinienne des textes d'histoire des sciences. In A. L. Rey (Ed.), *Méthode et histoire : Quelle histoire font les historiens des sciences et des techniques?* (pp. 217–232). Garnier. https://doi.org/10.15122/isbn.978-2-8124-1421-3.p.0217

Bråting, K. (2019). Development of school algebra: A comparison between the 1980 and 2011 Swedish mathematics curricula. In E. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *Proceedings of ESU 8* (pp. 711–726). Metropolitan University.

Bronckart, J.-P., & Bota, C. (2011). *Bakhtin démasqué*. Librairie Droz. Clark, K., Kjeldsen, T. H., Schorcht, S., Tzanakis, C., & Wang, X. (2016). History of mathematics in mathematics education: Recent developments. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 HPM meeting* (pp. 135–179). IREM de Montpellier.

de Saussure, F. (1959). *Course in general linguistics*. Philosophical Library. Originally published in 1916.

Dentith, S. (2005). *Bakhtinian thought*. Routledge.

Djebbar, A. (2005). *L'algèbre arabe : La genèse d'un art*. Vuibert.

Fauvel, J., & van Maanen, J. (Eds.). (2002). *History in mathematics education: The ICMI study*. Kluwer Academic Publishers. https:// doi.org/10.1007/0-306-47220-1

Fried, M. N. (2000). Some difficulties in incorporating history of mathematics in teaching-training. In A. Ahmed, J. M. Kraemer, & H. Williams (Eds.), *Proceedings of the CIEAEM 51* (pp. 73–78). *Cultural diversity in mathematics education:* Horwood Publish- ing Limited.

Fried, M. N. (2007). Didactics and history of mathematics: Knowledge and self-knowledge. *Educational Studies in Mathematics, 66*(2), 203–223. https://doi.org/10.1007/s10649-006-9025-5

Fried, M. N. (2008). History of mathematics in mathematics education: A Saussurean perspective. *The Montana Mathematics Enthusiast, 5*(2), 185–198. https://doi.org/10.54870/1551-3440.1100

Fried, M. N. (2018). Ways of relating to the mathematics of the past. *Journal of Humanistic Mathematics, 8*(1), 3–23. https://doi.org/ 10.5642/jhummath.201801.03

Fried, M. N., Guillemette, D., & Jahnke, H. N. (2016). Theoretical and/ or conceptual frameworks for integrating history in mathematics education. In L. Radford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 HPM meeting* (pp. 211–230). IREM de Montpellier.

Grattan-Guinness, I. (2004). History or heritage? An important distinction in mathematics and for mathematics education. *The Ameri- can Mathematical Monthly, 111*(1), 1–12. https://doi.org/10.2307/4145010

Guillemette, D. (2016). Quelques difficultés rencontrées dans la formation des enseignants du secondaire à l'aide de l'histoire des mathématiques : Une réflexion sur les modalités de lecture de textes historiques. *MENON: Journal of Educational Research*, [Special issue], 94–111.

Guillemette, D. (2017). History of mathematics in secondary school teachers' training: Towards a nonviolent mathematics education. *Educational Studies in Mathematics, 96*(3), 349–365. https://doi. org/10.1007/s10649-017-9774-3

Guillemette, D. (2019). Being in research and doing research on history and mathematics education in a Bakhtinnian dialogical perspective. In E. Barbin, U. T. Jankvist, & T. H. Kjeldsen (Eds.), *Proceedings of ESU 8* (pp. 91–106). Metropolitan University.

Gurevitch, Z. D. (1988). The other side of dialogue: On making the other strange and the experience of otherness. *American Journal of Sociology, 93*(5), 1179–1199. https://doi.org/10.1086/228868

Hegel, G. W. F. (1977). *Phenomenology of spirit*. Oxford University Press. Originally published in 1807.

International Mathematical Union. (2022). Organization—Affiliated Organizations—HPM. Retrieved April 8, 2022, from https://www.mathunion.org/icmi/organization/affiliated-organizations/hpm. Accessed 8 Apr 2022

Jahnke H. N. (1994). The historical dimension of mathematical under- standing: Objectifying the subjective. In J. P. da Ponte, & J. F. Mato (Eds.), *Proceedings of PME-18* (vol. 1, pp. 139–156). University of Lisbon.

Jahnke, H. N. (2014). History in mathematics education: A hermeneutic approach. In M. N. Fried & T. Dreyfus (Eds.), *Mathematics & mathematics education: Searching for common ground* (pp. 75–88). Springer. https://doi.org/10.1007/978-94-007-7473-5_6

Jahnke, H. N., Arcavi, A., Barbin, E., Bekken, O., Furinghetti, F., El Idrissi, A., Silva da Silva, C. M., & Weeks, C. (2002). The use of original sources in the mathematics classroom. In J. Fauvel & J. van Maanen (Eds.), *History in mathematics education: The ICMI study* (pp. 291–328). Kluwer Academic Publishers. https://doi. org/10.1007/0-306-47220-1_9

Jankvist, U. T. (2009). A categorization of the "whys" and "hows" of using history in mathematics education. *Educational Studies in Mathematics, 71*(3), 235–261. https://doi.org/10.1007/ s10649-008-9174-9

Neuwirth, S. (2016). Un cours musique et mathématiques. In L. Rad- ford, F. Furinghetti, & T. Hausberger (Eds.), *Proceedings of the 2016 HPM meeting* (pp. 715–725). IREM de Montpellier.

Radford, L. (2021a). *The theory of objectification A Vygotskian perspective on knowing and becoming in mathematics teaching and learning*. Brill/Sense. https://doi.org/10.1163/9789004459663

Radford, L. (2021b). Mathematics teaching and learning as an ethical event. *La Matematica e La Sua Didattica, 29*(2), 185–198.

Radford, L., & Santi, G. (2022). Learning as a critical encounter with the other: Prospective teachers conversing with the history of mathematics. *ZDM Mathematics Education*. https://doi.org/10. 1007/s11858-022-01393-z

Rodriguez, O. H., & Lopez Fernandez, J. M. (2010). A semiotic reflection on the didactics of the Chain rule. *The Mathematics Enthusiast, 7*(2), Article 10. Retrieved April 8, 2022, from https://scholarworks.umt.edu/tme/vol7/iss2/10. Accessed 8 Apr 2022 Vauthier, B. (2002). Bakhtine et/ou Saussure ? » ou de l'histoire du malentendu des « malentendus saussuriens. *Cahiers Ferdinand De Saussure, 55*, 241–266.

Voloshinov, V. N. (1973). *Marxism and the philosophy of language*. Seminar Press. Originally published in 1929.