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ON THEORIES IN MATHEMATICS EDUCATION AND THEIR CONCEPTUAL DIFFERENCES

Luis Radford

Abstract

In this article I discuss some theories in mathematics education research. My goal is to highlight some of their differences. How will I proceed? I could proceed by giving a definition, T, of the term *theory* and by choosing some differentiating criteria such as c1, c2, etc. Theories, then, could be distinguished in terms of whether or not they include the criteria c1, c2, etc. However, in this article I will take a different path. In the first part I will focus on a few well-known theories in Mathematics Education and discuss their differences in terms of their *theoretical stances*. In the last part of the article, I will comment on a sociocultural emergent trend.

Introduction

In order to make sense of problems around the teaching and learning of mathematics, mathematics educators have come up with different theories. Currently, there is a large number of theories in use. My goal is to highlight some of their differences. How will I proceed? I could proceed by giving a definition, T, of the term *theory* and by choosing some differentiating criteria such as c1, c2, etc. Theories, then, could be distinguished in terms of whether or not they include the criteria c1, c2, etc. see Radford [2008a, 2017a]. In this article, however I will take a different path. In the first part of the article, I will focus on a few well-known theories in Mathematics Education and discuss their differences in terms of their *theoretical stances*. In the last part of the article, I will comment on a sociocultural emergent trend.

My choice of theories has been guided by what may be termed their historical impact in the constitution of mathematics education as a research field. By historical impact I do not mean the number of results that a certain theory produced in a certain span of time. Although important, what I have in mind here is something related to the foundational principles of a theory. The foundational principles of a theory determine the research

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questions and how to tackle them within a certain research field, thereby helping to shape the form and determine the content of the research field itself.

To discuss the types of theories in our field is to discuss their differences and, more importantly, what accounts for these differences. My argument is that these differences are better understood in terms of *theoretical suppositions*. Sriraman and English [2005] argued that the variety of frameworks in mathematics education is directly related to differences in their epistemological perspectives. I suggest that, in addition to the underpinning corresponding epistemologies, differences can also be captured by taking into account the cognitive and ontological principles that theories in mathematics education adopt.

Obviously, I will neither be able to present a rich sample of theories in mathematics education nor will I be able to delve deeply into the intricacies of any of them. I hope, nonetheless, that by focusing on a few theories, and contrasting their theoretical suppositions, we may gain a sense of their distinctiveness and thereby better understand the notion and the types of theories in our field.

Because of space constraints, I will deal with three theories. Although other choices are certainly possible, I will deal with Constructivism, the Theory of Didactic Situations, and Socio-Cultural Theories.

1 Constructivism

1.1 The Theoretical Principles. During the 1980s and 1990s, Constructivists introduced their theory as based on two main principles:

> *p1*: knowledge is not passively received but built up by the cognizing subject; and
>
> *p2*: the function of cognition is adaptive and serves the organization of the experiential world, not the discovery of ontological reality. von Glasersfeld [see 1995, p. 18]

Principle *p1* stresses constructivism's opposition to teaching by transmission. Constructivism, indeed, emerged as an option against behaviourism and its pedagogy of direct teaching. It is in this context that Paul Cobb remarked some twenty years ago that

> An abundance of research indicates that students routinely use prescribed methods to solve particular sets of tasks on which they have received instruction without having developed the desired conceptual knowledge. Cobb [1988, p. 90]

However, although historically important, the true novelty of the constructivist perspective does not rest on the first principle. It rests, rather, as von Glaserfeld claims, on the epistemic and ontological attitudes conveyed by the second principle and its concomitant concept of *knowledge*. Without necessarily denying the existence of a pre-existent reality, and in a move consistent with Kant's theory of knowledge, constructivism does not claim that the knowledge constructed by the cognizing subject corresponds to such a reality; its epistemology rests precisely on the denial of the possibility of any certain knowledge of reality Ernest [1991].

In the beginning, constructivism envisioned the goals of mathematics instruction along the lines of Piaget's epistemology. At the end of the 1980s, Cobb argued that the goal of instruction is or should be to help students build [mental] structures that are more complex, powerful, and abstract than those that they possess when instruction commences Cobb [1988, p. 89]. The pedagogical problem was then to create the classroom conditions for the development of complex and powerful mental structures.

The constructivist research was oriented to a great extent to the study of the development of the students' mental arithmetic and other mathematical structures and to the investigation of the students' difficulties in developing them. Particular attention was paid to the students' counting types and construction of arithmetic units see e.g. Cobb [1985], Steffe and von Glasersfeld [1983] and Steffe, von Glasersfeld, Richards, and Cobb [1983].

The creation of the classroom conditions for the development of mental structures led unavoidably to the question of the role of the teacher. Cobb said:

> The teacher's role is not merely to convey to students information about mathematics. One of the teacher's primary responsibilities is to facilitate profound cognitive restructuring and conceptual reorganizations. Cobb [1988, p. 89]

A close examination of the role of the constructivist teacher shows that the constructivist epistemic and ontological principles were underpinned by a general concept of the cognizing subject that framed the specific role of the student and the teacher. For constructivism, the epistemic and ontological principles *p1* and *p2* make sense only in the context of a self that is autonomously constructing her knowledge. If we remove the autonomy principle, constructivism becomes simply a variant of certain socio-cultural approaches. This third principle can be formulated as follows:

> *p*3: the cognizing subject not only constructs her own knowledge but she does so in an autonomous way.

Intellectual autonomy was in fact part of two of the general goals identified by constructivism from the outset:

> teaching by imposition is incompatible with two general goals of mathematics instruction that follow from constructivism, the construction of increasingly powerful conceptual structures and the development of intellectual autonomy. Cobb [ibid., p.100]

As I argued elsewhere Radford [2008c], the idea of the autonomous cognizing subject conveyed by constructivism was not a novelty in education. In fact, just such an idea is at the heart of the concept of the self of Western modernity–an idea that goes back to the very roots of Kant's theory of knowledge and its related epistemic subject. Kant's epistemic subject is not one that receives knowledge but one that produces it. It is a constructor that epitomizes the idea of man as *homo faber*. However, as we shall see later, although interesting from a historic viewpoint, this epistemic concept of the cognizing subject as an autonomous constructor of its own knowledge is considered too restrictive to account for the concrete processes of learning in the classroom and constitutes a point of divergence of theories in mathematics education.

1.2 The Ontology of Constructivism. The constructivist denial of the possibility of knowledge of reality is not mere fancy nor extravagant ontological position. It is, rather, one of the consequences of the remarkable subjectivism in which it was rooted from the start. The cognizing subject of modernity found itself in a world whose understanding was no longer assured by tradition and the interpretations offered by religion. The understanding of the world could only come from what the cognizing subject could accomplish through its sensing body and its intellect. Starting from the senses as the basic structure of knowledge, David Hume argued in the 18th century that the establishment of logical necessity was impossible to ascertain, for all that we can witness are particular associations occurring among events. Hume was perhaps the first thinker to express in the clearest way the finitude of the human condition that results from a subjectivism that started to arise from the Renaissance and that was clearly articulated by the philosophers of the Enlightenment. The long period that followed Kant's *Inaugural Dissertation*, published in 1770 (for a modern translation see Kant [1894]) and the first critique, that is the *Critique of Pure Reason*, published in 1781 (for a modern translation see Kant [2003]), the so-called silent decade, is explained by the intense cogitations in the course of what Kant sought for a solution to Hume's problem. This decade of intense cogitations led Kant to the development of his ontology Goldmann [1971], a neutral ontology, the main feature of which is, as von Glasersfeld noted, the abandonment of claims about the knowability of reality – i.e., an ontology that neither asserts that knowledge is about reality nor that it is not.

However, Kant's neutral ontology has an exception: the neutral ontology of Kant does not apply to mathematical knowledge. For Kant, mathematics was the paradigmatic example of certain knowledge. This is what Kant meant by the a priori status of mathematics, a status that put mathematical objects (in opposition to phenomenological objects such as chairs and dogs) within the realm of the truly knowable.

Kant's ontology rests on a form of *a priorism* that Piaget did not endorse. For Piaget, and for the ensuing constructivism in education, knowledge (mathematical or not) has to be constructed. Since there was no way to check the correspondence between subjective constructs produced by the cognizing subject and reality, von Glasersfeld suggested that knowledge is not about *certainty* but about *viability*. A piece of knowledge is kept by the cognizing subject as long as it seems to work. All knowledge is hypothetical.

This concept of knowledge has some interesting corollaries. One of them is that since everyone constructs his or her own knowledge, we can never be sure that we are talking about the same things. We can just assume or pretend that we are perhaps sharing something. For constructivists, we take knowledge and meanings as *taken-as-shared*. Naturally, one question that has been raised in this regard is whether or not the subjectivist idea of knowledge and meaning conveyed by constructivism is a form of solipsism. Constructivists answer negatively, stressing the role of social interaction in the cognizing subject's construction of viable knowledge.

1.3 Social Knowledge in Constructivism. Although some mathematics educators were intrigued by the extreme relativism of the Kantian constructivist neutral epistemology see e.g. Goldin [1990], ontological questions seemed to recede into the background as constructivist teachers and researchers were preoccupied with the understanding of good practices to ensure the students' development of mental structures. Naturally, the search for solutions was framed by constructivism's principles. In particular, the question was to devise pedagogical actions coherent with the idea of avoiding teaching the answers and influencing the student's reasoning. In short, the question was how to teach without trespassing into the domains of the student's self-determination. The solution was sought in the idea of the classroom as a space of *negotiation* of meanings.

Later on, this idea was developed further, perhaps as a result of the dialogue between constructivists and the German interactionists Bauersfeld [1980], Voigt [1985], etc. Thus, in the early 1990s, constructivism was formulating the learning-teaching process as a process that is interactive in nature and involves the implicit and explicit negotiation of mathematical meanings. In the course of these negotiations, the teacher and students elaborate the *taken-as-shared* mathematical reality that constitutes the basis for their ongoing communication Cobb, Yackel, and Wood [1992, p. 10].

Through the insertion of the idea of mathematics as a social practice and the classroom as aspace of negotiation of meanings, constructivism moved into a new direction. In an article published in 1994, Cobb described two different constructivist research lines. The first remained centred around the investigation of the students' development of mental structures. The second focused rather on the evolution of meanings in the course of the students' interaction in the classroom Cobb [1994].

One of the challenges for this second line of research was to make the idea of interaction operational within the constraints imposed by their three basic principles. The operationalization was made through a clear distinction between: (1) the students' psychological processes, on the one hand, and (2) the social processes of the classroom, on the other. While the investigation of students' psychological processes went along the lines of Piaget's concept of reflective abstraction, the social processes were related to the idea of *collective classroom reflection* Cobb, Boifi, McClain, and Whitenack [1997].

Certainly, developing the new research line was not an easy move. It had to take into account social interaction in a context where, as a result of the theoretical principles, constructivism found itself with not too much room left. Indeed, interaction had to be devised in such a way that the inclusion of the Other in the cognizing subject's act of knowing left no room for interference with the autonomous constructivist cognizing subject. From the outset, there was a vivid tension between the students' mathematical meanings and those of the teacher: "The teachers' role in initiating and guiding mathematical negotiations is a highly complex activity that includes … implicitly legitimizing selected aspects of contributions" Cobb, Wood, Yackel, Nicholls, Wheatley, Trigatti, and Perlwitz [1991, p. 7]. To explicitly legitimize selected students' contributions would jeopardize, indeed, the constructivist project and its principle that knowledge construction is a personal and self-determining matter.

The dichotomy that constructivism erects between its culturally detached autonomous cognizing subject and the socio-cultural historical traditions in which this cognizing subject thinks and acts, turns out to be, as many find, an unsatisfactory solution. Thus, given the theoretical principles adopted by constructivism, Waschescio [1998] argues that a link between the individual and the cultural realm is certainly missing. Actually, as Lerman claims, such a link is simply impossible to find Lerman [1996].

To sum up, constructivism is a student-centred theory. Its influence in education has been very impressive, not only in North America but all over the world. The detailed analyses of classroom interaction and the sophisticated methodologies designed to scrutinize the negotiation of meanings underpinning the students' conceptual growth have helped the community of mathematics educators become aware of the variety of meanings that the students mobilize in tackling mathematical problems. Constructivism has certainly helped us to better understand the complexities surrounding the students' processes of learning and provides us with an alternative to direct teaching.

2 The Theory of Didactic Situations

The Theory of Didactical Situations (TDS) seeks to offer a model, inspired by the mathematical theory of games, to investigate, in a scientific way, the problems related to the teaching of mathematics and the means to enhance it.

In the beginning, the term *situation* referred to the student's environment as handled by the teacher for whom it appears as a tool in the process of teaching. Later, the situation was enlarged in order to include the teacher herself and even the educational system as a whole Brousseau [1997a].

As any theory, the TDS works on a set of principles, among them the following epistemic ones:

> *p1*: knowledge results as the —optimal solution to a certain situation or problem.
>
> *p2*: learning is –in accordance to Piaget's genetic epistemology— a form of cognitive adaptation.

As in the case of constructivism, these principles are supplemented by a conception of the roles that teacher and students have to play in the classroom:

2.1 The Role of the Teacher. An essential part of the teacher's role is not to show the students how to solve the problems, but rather to let the students deal with them, for doing mathematics does not consist only of receiving, learning, and sending correct, relevant (appropriate) mathematical messages Brousseau [1997b, p. 15]. Like Constructivism, the TDS is opposed to direct teaching. The teacher's role is rather to identify the problems or situations that will be given to the students and that will provoke the expected learning.

2.2 The Role of the Student. The student which the TDS talks about is an epistemic subject, a sort of ideal model of the individual, conceived of as behaving (or having to behave) in a rational manner, in a way close to the behaviour of the mathematician. Her role is to engage in mathematical problems in a way that is coherent with the professional scientific practice. In the course of a faithful reproduction of scientific activities, the student is required to produce, formulate, prove, and construct models, languages, concepts and theories. Brousseau [ibid., p.22].

The roles of the teacher and the student are explained in the following passage:

> The modern conception of teaching … requires the teacher to provoke the expected adaptation in her students by a judicious choice of problems that she puts before them. These problems, chosen in such a way that the students can accept them, must make the students act, speak, think, and evolve by their own motivation. Brousseau [ibid., p. 30]

The judicious choice of problems is, of course, a delicate part of the teaching process. Its concrete possibility rests on the following epistemological assumption:

> *p3*: for every piece of mathematical knowledge there is a family of situations to give it an appropriate meaning.

This family is called a *fundamental situation*. For Brousseau [1997b, p. 24], the search for fundamental situations and their insertion into the more general classroom project of teaching and learning requires at least two elements: a good epistemological theory, which would reveal the deepness of mathematical knowledge and positively inform the teaching process, and a good didactic engineering, which would be oriented to the design of situations and problems to be solved by the students.

A fourth principle specifies further the concept of learning in the TDS. The general epistemic principle *p2* tells us that learning is of an adaptive nature; it consists of the students' adaptations to a milieu, but it does not say anything about the socio-interactional conditions to be fulfilled for it to occur. Principle four fills the gap and gives an impeccable theoretical consistency to the TDS –although, as we will see, some paradoxes will appear later on:

> *p4*: the student's autonomy is a necessary condition for the genuine learning of mathematics.

Thus, if the process of learning was not accomplished autonomously vis-à-vis the teacher, learning could not have happened. For "if the student produces her answer without having had herself to make the choices which characterize suitable knowledge and which differentiate this knowledge from insufficient knowledge, the evidence [of learning] becomes misleading" Brousseau [ibid., p. 41]. In other words, "if the teacher teaches her [the student] the result, she does not establish it herself and therefore does not learn mathematics" Brousseau [ibid., pp. 41-42].

The student is hence expected to engage with a fundamental situation in a particular type of game that gives rise to another situation, called *adidactic* Brousseau [ibid., p. 30], characterized by the student's autonomy vis-à-vis the teacher. What makes the *adidactic* situation different is the fact that it is partially freed from the teacher's direct interventions Brousseau [2003, p. 2]. This is why, referring to the *adidactic* situations –the only one through which true knowledge acquisition can be said to happen (knowledge by adaptation)– Brousseau asserts that "Between the moment the student accepts the problem as if it were her own and the moment when she produces her answer, the teacher refrains from interfering and suggesting the knowledge that she wants to see appear" Brousseau [1997b, p. 30].

Within this context, the teacher's mission is not only to ensure the successful devolution of the fundamental situation to the student in the *adidactic* situation, but also to maintain a fruitful interaction with the milieu (i.e., the antagonist system of the actors) in an encompassing context called the *didactic* situation. As Brousseau puts it,

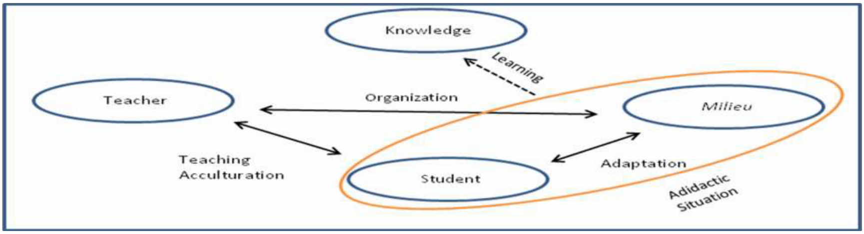

Figure 1: The four-pole (simplified) diagram shows the basic components of a Didactic Situation.

> This situation or problem chosen by the teacher is an essential part of the broader situation in which the teacher seeks to devolve to the student an adidactical situation which provides her with the most independent and most fruitful interaction possible. For this purpose, according to the case, the teacher either communicates or refrains from communicating information, questions, teaching methods, heuristics, etc. She is thus involved in a game with the system of interaction of the student with the problem she gives her. This game, or broader situation, is the *didactical situation*. Brousseau [ibid., pp. 30–31]

Figure 1 (which is a simplified and modified version of Perrin-Glorian and Hersant [2003] diagram) conveys the complexity of a didactic situation.

The didactic situation is in the end a model that can be better conceptualized as a game see Brousseau [1988]. The situation models the interaction of a subject with a milieu by a game (e.g. a problem to solve) where players have to take decisions: some states of the game are more favourable than others to win; thus the situation defines a piece of knowledge as a means for the subject to reach or maintain a favourable state (for the game) in this milieu Perrin-Glorian [1994]

In practice, however, the game does not necessarily proceed smoothly. The student may fail to solve the problem or simply may avoid it. A negotiation takes place:

> Then a relationship is formed which determines – explicitly to some extent, but mainly implicitly – what each partner, the teacher and the student, will have the responsibility for managing and, in some way or other, be responsible to the other person for. This system of reciprocal obligation resembles a contract. What interests us here is the *didactical contract*, that is to say, the part of this contract which is specific to the "content", the target mathematical knowledge. Brousseau [1997b, pp. 31-32]

Brousseau acknowledges that this system of reciprocal obligations is not exactly a contract in so far as it is not fully explicit. It is rather something like a flexible, ongoing negotiation. However, this is not a negotiation in the sense of constructivism, for what is being negotiated in the TDS is neither the mathematical meanings constructed in the classroom by the students and the teachers nor the mathematical forms of proving, arguing, etc. For the TDS, in opposition to constructivism, mathematical meanings and the mathematical forms of proving are not negotiable: they are part of the target knowledge, the cultural knowledge of reference. Negotiation is about the fluctuating borders of a teacher-student division of labour that seeks to ensure that the teacher's devolution of the fundamental situation is accepted by the student; that is to say, that the student takes responsibility for the solution of the problem and enters into an adidactic situation.

Because of its own nature, the unavoidable fuzzy didactic contract is haunted by some paradoxes. Let me dwell briefly on this point.

2.3 The Paradoxes of Learning. Teachers have the social obligation to make sure that learning is happening in the classroom.

What to do, then, if the student fails to learn? The student will ask the teacher to be taught. But

> the more the teacher gives in to her demands and reveals whatever the student wants, and the more she tells her precisely what she must do, the more she risks losing her chance of obtaining the learning which she is in fact aiming for. Brousseau [1997b, p. 41]

Brousseau does not consider this paradox as a contradiction. The paradox reveals the tricky situation that the teacher will be often called upon to live in the classroom. If the teacher gives up, knowledge attainment will be compromised:

> everything that she [the teacher] undertakes in order to make the student produce the behaviours that she expects tends to deprive this student of the necessary conditions for the understanding and the learning of the target notion; if the teacher says what it is that she wants, she can no longer obtain it. Brousseau [ibid., p. 41]

Another paradox may arise when it is not possible to find a fundamental situation that would fit the students' intellectual possibilities at a certain point of their development. In this case, the teacher

> gives up teaching by adaptation; she teaches knowledge directly in accordance with scientific requirements. But this hypothesis implies that she must give up providing a meaning to this knowledge and obtaining it as an answer

> to situations of adaptation because then the students will colour it with false meanings. Brousseau [ibid., p. 42]

According to Brousseau, the student is also put in a paradoxical situation: "she must understand AND learn; but in order to learn she must to some extent give up understanding and, in order to understand, she must take the risk of not learning" Brousseau [ibid., p. 43].

For the TDS, these (and other paradoxes) are an intrinsic part of didactic situations. They are part of the teaching of mathematics and knowledge acquisition. However, these paradoxes can also be seen as the result of a tension in the TDS' account of teaching and learning –a tension that results from a particular conception of learning, an epistemological and ontological rationalist view of mathematics and its adherence to a classical concept of the cognizing subject.

2.4 The Idea of Learning. As seen previously, for the TDS, genuine learning can only arise from the individual's own deeds and reflections. It is this theoretical stance on learning that gives sense to the ideas of adidactic situation and devolution. Although the TDS involves social interaction at different levels Kidron, Lenfant, Bikner-Ahsbahs, Artigue, and Dreyfus [2008], knowledge acquisition is, in the end, the result of the student's personal relationship with the object of knowledge. There is no difference in this respect between constructivism and the TDS. Differences arise in terms of, for example, the epistemic role of the problem: while in the former, a problem may lead to diverse, equally genuine *viable* pieces of knowledge, in the latter, in contrast, the design of the didactic situation should lead to the target cultural knowledge.

As we will see in the next section, the road taken by Vygotskian Sociocultural contemporary approaches to the problem of teaching and learning is different in important ways.

The TDS has had a significant influence in France and French-speaking communities around the world. It has also had an important influence in Spain and Central and South America. The detailed epistemic analyses of fundamental situations, their engineering and control in the classroom by the teacher, have helped mathematics educators understand the key role of suitable problems in the development of students' mathematical thinking.

3 Sociocultural Approaches

We have seen that for both constructivism and the TDS, the autonomy of the cognizing subject vis-à-vis the teacher, is a prerequisite for knowledge acquisition. For sociocultural approaches, autonomy is not the prerequisite of knowledge acquisition. Autonomy is, in fact, its result. This is one of the central ideas of Vygotsky's concept of *zone of proximal* *development*. Therefore, it is easy to imagine that, within sociocultural approaches of Vygotskian ascent, the roles of the teacher and the students are theorized along very different lines from what is found in other theories.This important difference will become clearer as I present a summary of the main principles of sociocultural approaches.

3.1 The Ontological and Epistemological Principles. The *ontological* position of a theory consists of specifying the sense in which the theory approaches the the nature of conceptual objects (in our case, the nature of mathematical objects, their forms of existence, etc.). The *epistemological* position consists of specifying the way in which, according to the theory, these objects can (or cannot) end up being known.

One of the most popular ontologies is Realism. Realists consider that the existence of mathematical objects precedes and is independent from the activity of individuals and that they exist independently of time and culture. Contemporary sociocultural approaches take a different route:

> *p1*: knowledge is historically generated during the course of the mathematical activity of individuals.

The principles of the TDS and constructivism seem to be in agreement with this ontological stance. If there is not a discrepancy in the "mode of being" of mathematical knowledge, there might be nonetheless some discrepancies in terms of its "modes of production." As seen earlier, the TDS and constructivism consider knowledge as the result of the adaptive actions of the cognizing subject. For socioculturalists, however, adaptation is insufficient to account for the production of knowledge. One of the reasons is that socioculturalists consider cognition as a cultural and historically constituted form of reflection and action embedded in social praxes and mediated by language, interaction, signs, and artifacts. As a result, knowledge is produced by cognizing subjects who are, in their productive endeavours, subsumed in historically constituted traditions of thinking. The cognizing subject of sociocultural theories is a subject that thinks within a cultural background and that, in so doing, goes beyond the necessities of mere ahistorical adaptive urges. In other terms, the "will to knowledge" (to borrow Foucault's term) and the way knowledge comes into being are neither driven nor shaped by adaptive needs or impulses to produce "viable" hypotheses or "optimal" results. The "will to knowledge" and knowledge itself are rather mediated by cultural forms of thinking and values (scientific, aesthetic, ethic, etc.) that orient (without imposing) the growth of knowledge into certain new directions. Within sociocultural contexts, viability cannot be understood as a mere subjective game of hypothesis generation by a cognizing subject in its attempt at getting around its environment. Much in the same way, *optimality* cannot be understood in terms of some universal, intrinsic mechanisms of mathematical knowledge. Mathematical thinking and mathematical responses are always framed by the particular rationality of the culture where they take place; within these cultures optimality can have different meanings and may not be the main drive to move mathematical thinking to new levels of development Radford [1997b], Radford [2008a].

For instance, the ways of dealing with the prediction of future events or the understanding of past events in early 20th century Azande culture was not at all moved by questions of optimality. The Azande reasoning was inscribed in a different worldview from the Ho versus Ha view of hypotheses testing of Western mathematics. And yet, like the latter, the Azande's ceremonial procedures were clear processes of understanding and making sense of their reality Evans-Pritchard [1937], Feyerabend [1987], and Radford [2017b].

We can summarize this discussion in the following principle:

> *p2*: the production of knowledge does not respond to an adaptive drive but is embedded in cultural forms of thinking entangled with a symbolic and material reality that provides the basis for interpreting, understanding, and transforming the world of the individuals and the concepts and ideas they form about it.

3.2 Learning. In the previous section it was argued that socioculturalists claim that from a phylogenetic point of view, conceptual objects are generated in the course of human activity. From an ontogenetic point of view, the central problem is to explain how acquisition of the knowledge deposited in a culture can be achieved: this is a fundamental problem of mathematics education in particular and of learning in general.

The metaphor of knowledge construction seems to convey very well the idea that knowledge is not something transcendental to the human sphere and that knowledge is rather something made by human beings. Constructivism, the TDS, and sociocultural perspectives agree on this point.

However, from a sociocultural perspective, the extrapolation of this metaphor to the ontogenetic dimension leads to a series of important irresolvable problems. Instead of talking about students constructing knowledge, some socioculturalists prefer to talk about students making sense of, and becoming fluent with, historically constituted modes of thinking. One of the advantages in putting the problem of learning in this way is that the student's knowledge is not seen as something coming from *within* (a kind of private or subjective construction endlessly seeking to reach a culturally-objective piece of knowledge) but from *without*. Principle 3 summarizes this idea:

> *p3*: learning is the reaching of a culturally-objective piece of knowledge that the students attain through a social process of *objectification* mediated by signs, language, artifacts, and social interaction as the students engage in cultural forms of reflecting and acting.

The idea of learning as the reaching of cultural knowledge should not be interpreted as if the students reach knowledge in a passive way. Unfortunately, we have become used to making a dichotomy and to thinking that either students construct their own knowledge *or* knowledge is imposed upon them. This is a too easy and misleading oversimplification –what Lerman has termed the absolutist view about learning Lerman [1996]. Learning, from a sociocultural perspective, is the result of an active engagement and self-critical, reflexive, attitude towards what is being learned. Learning is also a process of transformation of existing knowledge. And perhaps more importantly, learning is a process of the formation of subjectivities, a process of agency and the constitution of the self.

Sociocultural approaches resist indeed the idea that learning is just the uncritical appropriation of existing knowledge absorbed by a passive student-spectator. Knowledge has a transformative power: it transforms the object of knowledge and, in the course of knowing and learning, the subject is itself transformed. There is a dialectical relationship between subject and object that can be better understood by saying that learning is a process of objectification (*knowing*) and subjectification (or *agency*), that is a process of *being* Radford [2008c].

3.3 The Role of the Teacher and the Students. The role of the teacher is not, as it can be imagined from what we just said, to dispense knowledge. Since sociocultural approaches argue that knowledge cannot be injected into the students' mind[1], in order to get the students to know (in the sociocultural transformative sense) objects and products of cultural development, one of the roles of the teacher is to offer students rich classroom activities featuring, in a suitable manner, the encounter with the various layers of generality of historical cultural objects and the encounter with other voices and forms of understanding.

The configuration of these activities (both in terms of the mathematical content and its social- interactive dimension) is framed by the ultimate socioculturalists' idea of how learning occurs. As already mentioned, for socioculturalists, learning will not necessarily or uniquely occur as the result of the student's autonomous cogitations in her attempt to create viable hypotheses or to give optimal solutions to a problem. Learning, in fact, very often starts when the student is no longer able to continue by herself and requires the active participation of the teacher (this is one of the ideas of Vygotsky's *zone of proximal development*). This participation may become apparent in terms of questions and clues to redirect the student's attention to certain unattended features of the problem under consideration and that are vital to the attainment of a certain form of mathematical thinking. But it also can result from actively and critically interacting with the teacher while both

[1] Knowledge does not spring up in the individual as a result of a direct projection on his brain of the ideas and concepts worked out by preceding generations Leont'ev [1978, p. 19].

teacher and students solve the problem *together*. Of course, such a way of doing cannot be accounted for as an instance of learning in other theories, where the intellectual autonomy of the student plays the role of a prerequisite for learning. For sociocultural theories, however, autonomy is not a prerequisite, but, as already mentioned, its result.

The nuance is in fact subtler, for the idea of autonomy is not taken by sociocultural perspectives as something that develops from within the individual, or as something latent that the subject manages to expand: autonomy is not seen as my capability to do things without the help of others: autonomy is a social relation that I acquire as I engage in social praxes, and as such, is always a commitment to others Radford [2008c, 2012].

Sociocultural approaches to teaching and learning are younger than the other two approaches discussed in this paper. They were introduced in the early 1990s into mathematics education by mathematics educators, such as Ubiratan D'Ambrosio, Alan Bishop, Steve Lerman, and Mariolina Bartolini Bussi. The sociocultural approaches have gained some impetus in the past few years and shed some light on the problem of the cultural nature of mathematics D'Ambrosio [2006] and Bishop [1991], classroom interaction and discourse M. G. Bartolini Bussi [1998] and Lerman [1996, 2001], classroom conceptualization Radford [2000, 2008d] and Radford, Bardini, and Sabena [2007], semiotic mediation Arzarello and Robutti [2004], M. G. Bartolini Bussi and Mariotti [1999], M. Bartolini Bussi and Mariotti [2008], and Radford [2005], and the question of culture and cognition Radford [1997a, 2008b,e].

4 A New Trend

In this last section, I want to briefly mention a new trend as observed in the Fifth Congress of the European Society for Research in Mathematics Education (CERME-5, February 22-26, 2007). The European Society for Research in Mathematics Education organizes biannual conferences that are designed to encourage an exchange of ideas through thematic working groups. A few plenary activities take place, yielding most of the space to group work. One of the recurring CERME working groups is the one devoted to theories in mathematics education. For instance, in the CERME-5 conference held in the city of Larnaca, Cyprus, the working group 11 *Different Theoretical Perspectives / Approaches in Research in Mathematics Education* was one of the most popular, which attests to the interest in understanding that which makes theories different. However, the goal of this working group was not just to understand differences, but to seek new forms of linking and connecting current theories. More specifically, the idea was to discuss and investigate theoretical and practical forms of *networking* theories. Most of the papers presented at the meetings of working group 11 will appeared in an issue of the journal *ZDM - The International Journal on Mathematics Education*. As I mention in the commentary paper written for this ZDM issue Radford [2008a], this new trend consisting of investigating ways of connecting theories is explained to a large extent by the rapid contemporary growth of forms of communication, increasing international scientific cooperation, and the attenuation of political and economic barriers in some parts of the world, a clear example being, of course, the European Community.

This new trend is leading to an enquiry about the possibilities and limits of using several theories and approaches in mathematics education in a meaningful way. The papers presented at the conference provided an interesting array of possibilities.

Depending on the goal, connections may take several forms. Prediger, Bikner-Ahsbahs, and Arzarello [2008] identify some of them, like comparing and contrasting, and define them as follows. In comparing, the goal is to find similarities and differences between theories, while in contrasting theories, the goal is to stress big differences. Cerulli, Georget, Maracci, Psycharis, and Trgalova [2008] is an example of comparing theories, while Rodríguez, Bosch, and Gascón [2008] is an example of contrasting theories. These forms of connectivity are distinguished from others like coordinating and combining. In coordinating theories, elements from different theories are chosen and put together in a more or less harmonious way to investigate a certain research problem. Halverscheid's article (2008) is a clear example of an attempt at coordinating theories, in that, the goal is to study a particular educational problem (the problem of modelling a physical situation) through the use of elements from two different theories (a modelling theory and a cognitive one). In combining theories, the chosen elements do not necessarily show the coherence that can be observed in coordinating connections. It is rather a juxtaposition of theories (see Prediger et al.'s paper, (2008)). Maracci [2008] and Bergsten [2008] furnish examples of combining theories.

At least in principle, comparing and contrasting theories are always possible: given two mathematics education theories, it is possible to seek out their similarities and/or differences. In contrast, to coordinate or to integrate theories, which is another possible form of connection Prediger, Bikner-Ahsbahs, and Arzarello [2008] paper, seems to be a more delicate task.

Connecting theories can, in sum, be accomplished at different levels (principles, methodology, research questions), with different levels of intensity. Sometimes the connection can be strong, sometimes weak. It is still too early to predict how this new trend will evolve. What is clear, in contrast, is that the investigation of integration of theories and their differentiation is likely to lead to a better understanding of theories and richer solutions to practical and theoretical problems surrounding the teaching and learning of mathematics.

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