Radford, L. (2018) A Plea for a Critical Transformative Philosophy of Mathematics Education. In: P. Ernest (ed.), *The Philosophy of Mathematics Education Today* (pp. 1-10).

A Plea for a Critical Transformative Philosophy of Mathematics Education

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A disciplinary educational research field cannot, I think, avoid tackling general questions about the educational aims it pursues, as well as more specific questions concerning the teaching and learning of its contents, the nature of these contents and its methodology and theoretical foundations. Mathematics education is not an exception. These questions and their possible answers define a specific area of inquiry that has been termed the *philosophy of mathematics education*. Ernest (1991a, b, 2009), whose work has been influential in shaping this area of inquiry, suggests that the philosophy of mathematics education revolves around two axes. On the one hand, the philosophy of mathematics deals with the philosophical aspects of *research* in mathematics education. On the other, it deals with the *aims* of mathematics education (Ernest, this volume). Taking both axes together, the philosophy of mathematics education tackles questions such as our understanding of, and the meaning we attribute to, mathematics and its nature. It also includes questions about the purposes of teaching and learning mathematics, the meaning of learning and teaching mathematics and the relationship between mathematics and society.

The answers that we can offer to the previous questions go beyond mathematics itself. In order to tackle those questions, we need, indeed, to go beyond mathematics and step into new territory. We need to immerse ourselves in a series of theoretical domains like history, politics, ontology, metaphysics, aesthetics, epistemology, anthropology, ethics and critical philosophy (Ernest, this volume).

Consider for instance, the question about the relationship between mathematics and society. Since ancient times, what we call today "schooling" has been related to societal needs. The education of the scribes in Mesopotamia is a case in point. Mesopotamian scribes were instrumental in the organization and administration of the City (Høyrup, 2007). The mathematics that they learned and practised was influential in the measuring and distribution of lands, the collection of taxes, the calculation of the amount of food to be distributed to the soldiers, etc. One of the three oldest known problems goes back to *ca*. third millennium BC. It was found in 1975 by an Italian Archeological Mission while excavating the site of the Royal archives of the city of Ebla. The problem, contained in text TM.75.G.1392, is about the amount of cereal that is required to be distributed among a large number of individuals. In Fribergs' (1986) reconstruction, the problem reads as follows:

> Given that you have to count with 1 gu-bar for 33 persons, how much do you count with for 260,000 persons? (Friberg, 1986, p. 19).

The mathematics that the Babylonian produced and that scribes learned in school (what they called the "House of Tablets") was not a disinterested endeavour. It was related to the way the Babylonian administrative and political body sought to respond to societal needs.[1] It does not mean, however, that all Babylonian mathematics was about solving *practical problems*. This is the case, for instance, of many geometric problems at the basis of what has been called "Babylonian algebra." The first problem of a tablet known as AO 8862 that goes back to *ca*. 1750 BC reads:

1. Length, width. Length and width I have made hold:
2. A surface have I built.
3. I turned around (it). As much as length over width
4. Went beyond,
5. To inside the surface I have appended:
6. 3` 3. I turned back. Length and width
7. I have accumulated: 27. Length, width, and surface w[h]at? (Høyrup, 2002, p. 164)[2]

Without being "applied problems," many geometric problems like this, formulated as a kind of riddle, evoke the sensible actions of walking around a field while measuring parts of the field and operating on those measures. More than being simply inspired by surveying practices, problems such as the above, and Babylonian mathematics at large, convey ideas, values, interests and needs of the society from which they emerged (Nemet-Nejat, 1993). The same could be said of the mathematics of other historical periods. For example, the mathematics produced by the masters of Abacus in the Renaissance responded to problems that arose with the emergence of Western capitalism (see Swetz, 1989).

A general formulation of what these examples offer is that the mathematics that is produced and imagined in a particular historical and cultural context is related to the ideas, values, interests and needs of mathematics' cultural-historical context. In other terms, mathematics always *refracts* ideas, values, interests and needs of the

[1]Kramer (1949) and Lucas (1979) present a portrait of the scribal education. A more recent account can be found in Robson and Stedall's 2009 book. For an overview of the Sumerian administration structure, see Diakonoff (1974).

[2]Rephrased, the text talks about a (rectangular) surface built out of a length L and a width W to which the difference (L − W) is added. The result is (in the Babylonian sexagesimal system) 3`3. The text also tells us that L + W = 27 (see Høyrup, 2010, p. 25).

society from which it emerges. It is in this sense that mathematics can be said to always be *ideological* (that is, not as something that conveys a false portrait of culture's reality, but as something that *embodies* the ideals and tensions of its own sociocultural context). It is in this sense that mathematics in general is not, as I claimed above when discussing Babylonian mathematics, a disinterested endeavour. It has never been so—not even in Plato's Academy, where mathematics, as opposed to the sensuous and kinesthetic Babylonian mathematics, was conceived as unrelated to practical matters. To conceive of mathematics as unrelated to practical matters is already the result of an ideological posture.

Plato's ideas about philosophy in general and mathematics in particular arose and evolved during the turmoil of the Peloponnesian War between Athens and Sparta and the post-war oligarchic Athenian regime established by Sparta. Just before the war Athens experienced a population growth. Athenians of the time saw the rise of commerce, and the emergence of new social classes, leading to a social restructuring where the old values of the aristocratic elite were shaken. The concept of the "good," related to manliness and good birth, which had been progressively elaborated since Homer's time, was challenged by the new context shaped by the arrival of "[r]ootless foreigners in their origins; skeptical, nominalistic, subjectivistic, and relativistic thinkers . . . [who] had no axioms, no epistemic certainties, no fixed axes of value, no ancestral pieties" (Levi, 1974, p. 61). "We can be certain," states Beavers (n.d.) in his biography of Plato, that the Peloponnesian War, "the establishment of a government by Sparta (after the chaos of Athens' final defeat in 404), and the events that followed, dramatically affected the direction of [Plato's] thinking."

Plato grew up in an aristocratic family. His "father's lineage went back to the first kings of Athens" (Levi, 1974, p. 58). Because of his aristocratic ancestry, he was destined to become a member of Athens' ruling class. His path, however, was interrupted by the Peloponnesian War and the subsequent course of events, which led to the decline of the Athenian empire and the Athenian aristocracy's loss of political power. Greatly affected by the execution of Socrates, Plato turned to other endeavours and travelled for several years, seeking comfort in philosophy. It is in this historical and political context that Plato fought for the restoration of the Greek world ruled by a "cultured elite" (Levi, p. 58) and that, during his return to Athens in *ca*. 387, he founded the Academia—"to instruct a new generation to become the legislators and the aristocratic statesmen of a future world" (p. 60). It is against this historical-political backdrop and the aristocratic outlook that opposed epistemological relativism and despised social and political change, practical labour, commerce and all mundane activities, that Plato came to formulate his philosophy of permanent Forms and the ensuing idea of truth as something immutable, perfect and timeless. Truth was conceived of as something that was accessible not through practical labour with artefacts but through "kóco1" (*logos*), the reasoned discursive activity of cultured citizens whose aim was to rise to higher levels of knowledge. In Plato's view, mathematics was not about calculations or using mechanical instruments (Radford, 2003, 2008). Plutarch reports how Plato got offended when he learned that Eudoxus and Architas were resorting to mechanical instruments in their geometric inquiries. Plutarch says:

> But Plato took offense and contended with them that they were destroying and corrupting the good of geometry, so that it was slipping away from incorporeal and intelligible things towards perceptible ones and beyond this was using bodies requiring much wearisome manufacture. (Plutarch, Lives: Marcellus, xiv; quoted by Knorr, 1986, p. 3)

In Plato's conception, the forms of mathematics (the mathematical objects) have delimiting boundaries that make it possible to clearly distinguish one form from another (e.g. a triangle can be distinguished from a square with certainty; by contrast the boundaries separating courage from cowardice are not necessarily clear). In addition to this boundary feature of its objects, in mathematics, through *reason*, "we gain access to [a] purely intelligible, formal stable entity" (Roochnik, 1994, p. 559). This is why, in Plato's view, mathematics offers a paradigmatic model of clear and authoritative knowledge, where one can "shift one's sights, away from the sensible towards the noetic" (p. 559), and that mathematics becomes invested with moral value: "The study of mathematics is good for turning [away from the sensible world] the souls of the future philosopher-kings" (p. 560).

It is in this discursive society, torn by the distinction between appearance (doxa) and truth, with its scorn of the material and the sensible, that speech and its social use took on an epistemological dimension that remained unthinkable to the Babylonians, the Mayas, the Inuits, the Azande, the Maori, etc.

As an expression of its society, mathematics appears as the refraction of the manner in which knowledge is ideologically expressed and power is exerted. However, the manners in which mathematics in general and the mathematics that we teach at school embody such an ideological refraction need to be spelled out in detail. It is here that I find the promises of a philosophy of mathematics education most welcome. In my view, a philosophy of mathematics education should not appear merely as another field of inquiry, but as an urgent endeavour. For if there has always been a relationship between mathematics and society, this relationship has taken a very particular turn during the period in which we are living. Our historical period can sadly be characterized as the unprecedented historical age of the most radical assault on schools and educational systems at large by the economic forces of society. No school system before has ever been engulfed in such a virulent manner by one of society's components. The school of today appears, indeed, as an appendix to political economy, defined by global capitalism. And it is against this background that curricular contents are determined and that expectations about students and teachers are set.

Referring to public education in the USA, Lavallee notes that

> "public" schools have not only had their educational practices and curriculum taken over by edu-businesses, but schools' hidden curricula have also been likewise infiltrated by capitalism . . . Like a colonial occupying force, the for-profit publishers, test makers, test-prep profiteers, tutoring companies, curriculum designers, and so on are determining what our children learn and how their futures (economic, ideologic, etc.) will be shaped – not the community and parents, not the teachers, and least of all not the students themselves

> (who should actually have the greatest say). One cannot deem an occupied territory a "public" space. (Lavallee, 2014, pp. 6–7)

The assault on education that Lavallee talks about is happening farther north too. A central document that defines the goals of education in Ontario is *Achieving Excellence: A Renewed Vision of Education*. In this ministerial document, which is the reference *par excellence* in our province and frames all the initiatives of our Ministry of Education, achievement is explained as "raising expectations for valuable, higher-order skills like critical thinking, communication, innovation, creativity, collaboration and entrepreneurship" (Ontario Ministry of Education, 2014, p. 3). Then, candidly, the document continues: "These are the attributes that employers have already told us they seek out among graduates" (p. 3). The term entrepreneur/entrepreneurship appears 10 times in this document of 19 pages—a very worrisome frequency! In the opening lines of the document, we are told that we have one of the best educational systems in the world. What is the evidence? It comes from "respected international organizations such as the Organisation for Economic Co-operation and Development (OECD), McKinsey & Company, and the National Center on Education and the Economy in the United States." They "have all applauded Ontario, our programs and our results" (Ontario Ministry of Education, 2014, p. 2). We are on the right track. We are developing the taskforce that capitalism requires to keep the machinery going on—the same machinery that produces as many commodities as inequalities.

A philosophy of mathematics education should, I think, denounce the current political trend that defines human existence in mere economic terms and that reduces education to the development of actions (competencies) that are necessary to maintain, expand and refine the current capitalist forms of production.

In my view, a philosophy of mathematics education is the space to investigate and to denounce what Ferreira de Oliveira calls "the ideology of the market;" that is, the "transformation of things, inanimate or alive, in passive elements of commercialization" (Ferreira de Oliveira, in Freire, 2016, p. 113). The ideology of the market, with its emphasis on competitiveness, reduces the human to a means; it reduces the student to human capital: an atom that is trained to jump later in the inclement machinery of supply and demand to produce, consume and reproduce. It perverts the basis of true human relations, leading to a model of alienated society that schools repeat again and again. Within this context,

> Nature, water, the air, the earth, the world, the planet, the universe, the human beings, and all other beings, their minds, their organs, their feelings, their sexuality, their beauty, their workforce, their knowledge, their existence, their homes and their lives, are considered as merchandise. (Ferreira de Oliveira, in Freire, 2016, p. 113)

The articles in this volume ask different questions and try to answer them through different perspectives. Some chapters move around philosophical matters about language, pedagogy and conceptions of mathematics. Other chapters interrogate our often taken for granted assumptions about teaching and learning, about the nature of mathematics, and the role that mathematics plays in society and in the shaping of teachers and students.

Platonism is featured in several papers. In his contribution to this volume, Skovsmose reminds us of the influential role played by Platonism in referential theories of meaning. Platonist referential theory of meaning, Skovsmose notes,

> provides the basis for logicism and for many attempts to construct mathematics on a foundation of logic. It also provided a basis for the whole New Math movement, establishing set theory not only as the logical but also as an educational foundation of mathematics. (Skovsmose, this volume)

Otte (this volume) distinguishes various forms of Platonism. In the discussion, he refers to the distinction between the object and its representations and the role of representations in our knowing of the object. There is an often-quoted passage in *The Republic* where Plato deals with this problem:

> And you will also be aware that they [the geometers] summon up the assistance of visible forms, and refer their discussion to them, although they're not thinking about these, but about the things these are images of. So their reasoning has in view the square itself, and the diagonal itself, not the diagonal they have drawn. And the same with other examples. (Plato, 2000, 510d, p. 217)

Otte (this volume) mentions an acquaintance of his for whom ideal objects (mathematical, musical, etc.) appear in "the classical sense of a universe of eternal ideas," a conception that has been largely considered as "a foundational conception of pure mathematics." Otte writes:

> Once we had a colleague at our mathematics department at the University of Bonn, who would not listen to music, but would read it from the partiture [score]. He did not visit music performances because he thought music becomes distorted by playing it.

In this Platonic view about ideal objects, a human intervention would ruin the purity of the object. Kant held a similar, although not exactly equal, position: since all knowledge starts with our senses, or as Kant puts it, in our capacity to be affected by the representation of the objects (Kant 1781/2003, p. 93; A51/75), what we come to know of the ideal object is not the object itself but what results from the mediation of our senses (Radford, Arzarello, Edwards, & Sabena, 2017). As a result, we cannot know the object *itself*, but only its *appearance*. Consider the drops of rain that you feel when it suddenly starts raining and you hurry to find some shelter. These drops are appearances, objects of the phenomenological experience you are undergoing, not drops of rain as ideal objects. What we get to know is precisely *that*: the drops of rain that we feel over our body, not the transcendental object. Kant says:

> We then realise that not only are the drops of rain mere appearances, but that even their round shape, nay even the space in which they fall, are nothing in themselves, but merely modifications or fundamental forms of our sensible intuition, and that the transcendental object remains unknown to us. (Kant 1781/2003, p. 85; A46/63)

This is in a nutshell the argument behind Kant's epistemological relativism. In the case of Otte's acquaintance, the problem is not the impossibility of the human accessibility to the ideal object, but the fact that its representation (here the musical performance of the orchestra) seems to end up representing something else—a distorted version of the musical work.

Without a doubt, Platonism has had a privileged seat at the table of the mathematicians. The mathematicians' ontological position that attributes to the ideal objects an existence independent of human labour certainly has consequences in the manner in which research is conducted. It is not the same to assume that you create something as to assume that you are discovering it. The French 2010 Fields medallist Cédric Villani put this question as follows:

> Of course, philosophical thinking can influence the way in which research is done in mathematics, in the sense that if one is persuaded that there is something intrinsic to discover, one will not look in the same way as if one is persuaded that it is a human movement of construction. We will not have the same reflexes, not the same tension. (Villani, in Cartier, Dhombres, Heinzmann, & Villani, 2012, p. 60)

And as many mathematicians (Bernays, 1935), Villani recognizes himself as one of those that adopt a pre-existing harmony that is already waiting to be discovered:

> I am one of those who believe that there is a pre-existing harmony and that, on a given problem, will seek the nugget, persuaded that it exists. I am one of those who seek the miracle, not of those who will create it or seek something very clever in their own resources. (Villani, in Cartier et al., 2012, p. 60)

A number of papers in this volume deal with another range of questions identified by Ernest in his overview of the philosophy of mathematics education. These questions have more to do with the relationship between mathematics and society. Andrade-Molina, Valero and Ravn's contribution examines the role of mathematics education in producing children of a certain kind: rational and logical children. Their inquiry features Euclidean Geometry as a model of inquiry that, historically speaking, grew up entangled with a worldview that provides explanations about the natural world. Mathematics loses here its innocence. Rather than being beyond the vicissitudes of cultures, mathematics, as well as its teaching and learning, unavoidably refract a conceptual view of the human world that is political through and through. As Walshaw notes in her chapter, "Objectively derived and propositionally formulated, it [i.e., school mathematics, although this is even truer of mathematics itself—LR] is constructed from the experiences of a privileged group of people." What is specific to our contemporary world that incessantly produces and reproduces inequalities through its own economic machinery, is the fact that, theoretically, it aspires to erase the same inequalities that it produces through its own individualist conception of democracy. Hunted by its own contradictions, global capitalist societies (and those that without being such are affected by them) imagine that the solution to the riddle of inequality is to be found in the achievement of an impossible equity and the dream that the mathematics that has been constructed from the privileged groups are "paradigmatic for all" (Walshaw). A critical philosophical attitude helps us understand that the uncertain solutions offered by neoliberal political benevolent and naïve discourses that tackle the question of diversity through the conflation of equity with equality are doomed to fail. Such a conflation is

> based on the understanding that full opportunities to learn within the classroom and respectful exchanges of ideas about mathematics between a teacher and her students' outcomes, yield a comprehensive picture of equitable mathematical access for students, irrespective of any social determinations. (Walshaw; this volume)

Underpinned by a utilitarian logic, this conflation of equity with equality assumes that it is possible to erase the social, cultural and historical pillars of human existence through an equalitarian repartition of positions and possibilities in the social web of a competitive market.

The ideological substrate of mathematics and its teaching and learning—one of its features being the one discussed by Walshaw—is a topic that appears in various chapters (see, e.g. Schürmann's contribution to this volume). One of the questions that surfaces in this regard is the one concerning the conditions for the emergence of genuine critical thinking (e.g. Barwell, this volume). Another question revolves around the possibility to move beyond the oppressing and alienating framework circumscribing most of the current practice of mathematics teaching and learning. Seeking some alternatives, Walshaw (this volume) turns to Foucault's idea of *governmentality*. Through this concept, she sees a possibility for us to come up with "an interpretation of individual experiences in which domination and resistance are no longer conceived of as ontologically different but as opposing effects of the same power relations."

The previous brief overview of some of the problems that haunt mathematics education and mathematics education research makes clear, it seems to me, the need for an urgent space of critical reflection that can be filled by a philosophy of mathematics education. Ernest formulates a possible role for such a philosophy as the endeavour directed "to analyse, question, challenge, and critique the claims of mathematics education practice, policy and research" (Ernest, this volume). Yet, I would like to contend that we must go one step further and *act*, *take action*, so that our analyses, questions and critiques come to make, through concerted movements, a *transformation* of mathematics education as it is practised today. This is why a philosophy of mathematics education today appears to me as the space in whose interior an encompassing struggle against the reduction of education in general, and mathematics education in particular, to a technical consumerist view can be organized and deployed. It is in this sense that a philosophy of mathematics education appears as a land of hope—the hope to understand, criticize and *transform* the aims of mathematics education and its concrete practice. This is why I would like to submit that what we need is a *critical and transformative* philosophy of mathematics education.

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