

# **Teaching and learning (algebra or something else): Working together to make sense of similarities and differences between theories (and understanding oneself)<sup>1</sup>**

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*Abstract: My presentation is divided into two parts. In the first part I mention some of the reasons and questions that are at the origin of the theory of objectification (TO) and summarize some of its central concepts. In the second part I discuss some connections between the TO, the Anthropological Theory of the Didactic (ATD), and Commognition.*

## **Why another theory?**

I would say that the previous century ended with a kind of discomfort in mathematics education research. In the English-speaking mathematics education community constructivism had had a tremendous influence and had provided mathematics education researchers with a paradigm of learning. The learning paradigm was based on a Piagetian-inspired epistemology that led to understanding learning in a very subjectivist way. Although other avenues started to be explored in the 1980s (in particular with the work of D'Ambrosio (1985), Bishop (1988), and Bartolini Bussi (1991)), it was at the end of the last century that an increasing critique took form, leading to what Sfard (1999) suggestively called "The paradigms wars."

What was the problem? The problem was a scant attention to the role played by culture in the students' acts of knowing and learning. In light of the increasing critique, constructivism tried hard to incorporate a social dimension into its principles: "it is worth noting," Cobb and Bower (1999, p. 9) noted, "that our theoretical orientation when we first began working intensively in classrooms 12 years ago was primary individualistic" Constructivists sought to overcome the critique through the idea of *participatory reflective classroom practices* (Cobb, Boifi, McClain, & Whitenack, 1997). However, to remain coherent with its own principles, the social dimension was conceived of as playing a mere *support* role: participation in communal practices was "seen to support, enable, and constrain the ways in which students reorganize their individual acts of participating" (Cobb, 1998, p. 197). The constructivist move was coherent with Piaget's conception of the social. From a Vygotskian perspective, however, such a solution was considered to be totally insufficient (see, e.g., Lerman, 1992, 1996; Waschescio, 1998).

In the middle of the first decade of the new century, the TO emerged bit by bit with the intention to offer a non-individualistic account of teaching and learning from a cultural-historical perspective (Radford, 2006, 2007). The TO is a theory inspired by Vygotsky's cultural-historical school of thought and by a neo-Hegelian dialectical philosophy (e.g., Ilyenkov, 1977; Fischbach,

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2014). It has progressively developed in a continuous dialogue with other theories, mainly with Sfard's work, with constructivism, and the theory of didactical situations (TDS). The dialogue with Sfard's work is understandable given not only the sophisticated level of theorization that her work has reached, but because of the fact that the TO shares with it a common Vygotskian stance. The dialogue with constructivism derives from the fact that I live in Canada where constructivism has had a tremendous influence. Last but not least, my relationship with the TDS is to be found as a result of my immersion during the 1980s in the French intellectual tradition as a PhD student.

### **The TO's basic challenges**

The theory of objectification is embedded in an educational project that is not the same as the one adopted by constructivism or the one followed by the TDS. Constructivism identifies two general goals of mathematics education: "the construction of increasingly powerful conceptual structures and the development of intellectual autonomy" (Cobb, 1988, p. 100). The TDS, by contrast, is oriented towards the diffusion of mathematical knowledge. The TDS is embedded in a social project whose goal is to make the students acquire a constituted knowledge (Brousseau, 2003). As a result, in the TDS, the emphasis has generally been put on mathematical knowledge and the efficient management of the learning environment. In constructivism, the emphasis has generally been put on the knowing subject—more specifically, in the understanding of the idiosyncratic manners in which students build their own knowledge. In the first case, the underpinning theoretical orientation has been essentially epistemological. In the second case, the theoretical orientation has been psychological. The theory of objectification inscribes itself in a different educational project: it posits the goal of Mathematics Education as a political, societal, historical, and cultural endeavour aimed at the dialectical creation of reflexive and ethical subjects who critically position themselves in historically and culturally constituted mathematical practices, and who ponder new possibilities of action and thinking. As a result, the focus is not on the mathematical content alone; the focus is not only on knowing but also on becoming.

The theory rests on the fundamental idea that learning is both about knowing and becoming.

Two of the initial challenges of the TO were to formulate a concept of knowledge and a concept of the human from where to articulate a cultural-historical account of teaching and learning.

### **Knowledge in the TO**

In the TO, knowledge is defined as a historically and culturally constituted system of embodied, sensible, and material processes of action and reflection. Knowledge, as defined here, changes from culture to culture and in the course of time. It is produced in human activity and is more than a technology to do something. Knowledge, indeed, is considered to be highly aesthetical, ethical, symbolic, and political. For instance, in the Mazahua community in Mexico (de Hann,

1999), knowledge about planting corn seeds is tied to a cosmological view of the world where days and numbers acquire a specific symbolism that informs human action. Regardless of the culture in which it emerges, knowledge is always immersed in cultural super-symbolic structures. Another example, maybe closer to us, is the super-symbolic structure of the Occidental modern period that allows us to conceive of the natural and social world in a Galilean way: as something governed by laws that can be read through mathematics (e.g., the way your mortgage is calculated by your bank).

### **Learning in the TO**

Student-centred pedagogies define learning as that which results from the students' own actions. In other words, the students are considered to *construct* their *own* knowledge. In this context, to *construct* a concept is equated to *learning* such concept. As I mentioned before, the intention behind the TO was to move beyond this individualistic stance. Before we go further and explain the concept of learning in the TO, let me note, *en passant*, that the metaphor behind the idea of knowledge as construction arose gradually during the 16<sup>th</sup> and 17<sup>th</sup> centuries, when the manufacture and commercial production of objects replaced the feudal economy of subsistence and barter and became the main form of human production in Europe. It is within the general vision of a world of manufactures that knowledge was also conceived for the first time as a form of manufacturing. A very clear exposition of this point of view appeared at the end of the 18th century in Kant's *Critique of Pure Reason* where mathematics is pictured as proceeding through a "construction" of concepts (Kant , 2003, p. 590 [A 734 / B 762]).

This formidable shift in the epistemological arena went hand in hand with another formidable shift in the ontological realm: it was not only *knowledge* and *knowing* that shifted out of the economic and political transformations that the feudal world underwent as the Occident entered into the early forms of capitalism: the concept of *self* was also completely transformed (Arendt, 1958). The individual stopped being a simple receiver of knowledge along the web of social hierarchies and institutions and became the *origin* of knowledge, meaning, and intentionality. It is only against this background that the idea of a subject as constructor of her own knowledge started making sense and that the 20<sup>th</sup> century student-centred pedagogy could be imagined and materialized. I do not have space here to dwell further on this important point; I limit myself to noting that the interest of this short incursion into the historical background of knowledge as construction is that it allows us to see how the fundamental concepts of education, namely the concept of knowledge, the concept of learning, and the concept of the knowing subject, are tied to the political and economic context from where they emerged.

Coming back to our point, if, from a cultural-historical educational perspective, learning cannot be suitably defined in terms of students' own constructions, how can it be defined? Sociocultural theories have resorted to a series of concepts, such as enculturation (mainly formulated in anthropological research) and internalization (borrowed from Vygotsky's work). I

have argued elsewhere (Radford, 2010, 2013a) that both concepts are insufficient to come up with an operational definition of learning from an educational perspective. To put it in a nutshell, the concept of enculturation seems to excessively emphasize the idea of social practice, to leave it uncritically touched, and to de-emphasize the individuals—even if they are considered as active participants. Rogoff, for instance, conceives of learning as apprenticeship in a context of guide participation. She says: “The concept of guided participation attempts to keep the roles of the individual and the sociocultural context in focus” (Rogoff, 1990, p. 18). She goes on to say that she uses the analogy of apprenticeship “to focus on how the development of skill involves active learners observing and participating in organized cultural activity with the guidance and challenge of other people” (Rogoff, 1990, p. 19). Learning, however, remains in the end a process whose goal is to *adapt* oneself to social practices. There is little room to investigate the individuals as agentic entities, such as the manners in which the individuals come to position themselves and be positioned in those practices. There is little room to investigate the tensions that arise from the normative dimension of cultures (what Bakhtin (1981) called a centripetal force) and the agentic movements of the individuals (the centrifugal force in Bakhtin’s terminology). The same critique holds for Vygotsky’s concept of internalization. It is rarely noticed in mathematics education research that Vygotsky’s concept of internalization (Вращивание – vraschivanie) appears as a theoretical construct intimately related to Vygotsky’s conception of the development of the mind, a conception that the Russian psychologist enunciates in terms of “genetic laws of cultural development.” The content of internalization is not learning, but the higher psychological functions (such as memory and perception) and the problem this concept seeks to explain is how the psychological functions evolve. The second of the three laws of cultural development that Vygotsky formulated states that

Every [psychological] function in the cultural development of the child appears on the stage twice, in two forms—at first as social, then as psychological; at first as a form of cooperation between people, as a group, an intermental category, then as a means of individual behavior, as an intramental category. This is the general law for the construction of all higher mental functions. (Vygotsky, 1998, p. 169)

Vygotsky then states the third law of the cultural development of higher mental functions as follows: “The third law, connected with the second, may be formulated as the law of transition of a [psychological] function from outside inward” (Vygotsky, 1998, p.170).

From the short overview of the concept of internalization we can pinpoint a few problems with using it as the basis of a definition of learning. First, internalization is a psychological concept—not a pedagogical one. We could try to expand it, and to move its content from the higher psychological functions to the learning of school disciplinary contents, although the task is not evident in itself. And even if we succeed, there are still two problems to be solved.

First, Veresov (1999) —one of the most knowledgeable contemporary Vygotkian scholars—notes the difficulty that the concept of internalization has to escape a dualist external-internal dichotomy that I find crucial to overcome in a redefinition of learning.

Second, much as in the case of learning as apprenticeship, the individual is, in the end, no more than a *replica* of her culture. The agentive dimension that I mentioned before, that dimension where we could see the cultural-historical, political production of the child in all its tensions, is not taken into account. It does not appear in the statement of Vygotsky's three laws of cultural development of the child.<sup>2</sup>

Let us come back to our previous point, then.

If, from a cultural-historical perspective, the construction and internalization of knowledge do not seem to offer suitable routes to theorize learning from an educational perspective, how can we theorize it? The answer has to be coherent with the adopted concept of knowledge. In the following section I provide the route adopted in the TO.

### **Processes of Objectification**

As suggested earlier, we can conceive of knowledge as culturally and historically constituted systems of thinking and action. Knowledge appears as a cultural-historical generative *capacity* (a latent capacity to do things and to think in certain ways, e.g., to plant corn seeds, to calculate mortgages, to solve linear equations, etc.). In this view, knowledge as a generative capacity cannot be something like a thing that we can “own” or that we can “possess.” Knowledge is neither a kind of merchandise nor a psychological entity. Rather, it is something that exists in our culture (in the form of knowing how to plant corn seeds, knowing how to calculate mortgages, etc.) that we may (or not) *encounter* in the course of our life (depending on the cultural-historical-political webs of knowledge access ubiquitously operating in our society). Our encounter with culturally and historically constituted systems of thought (e.g., mathematical, scientific, aesthetic, legal, etc.) is what we call *objectification*.

The noun “objectification” tries to convey the idea that the culturally and historically constituted systems of thought that, at our birth, are already there, in our culture, but we have not

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<sup>2</sup> Naturally, Vygotsky did not have in mind a kind of mechanical transition from the social to the individual. Thus, in *Tool and sign in the development of the child*, Vygotsky argues that

The transfer of social methods of behavior to the interior of the system of individual forms of adapting is not at all a purely mechanical transfer; it is not done automatically but is connected with a change in structure and function of the whole operation and is a special stage in the development of higher forms of behavior. Complex forms of cooperation transferred into the sphere of individual behavior begin to function according to laws of the primitive whole, of which they now make up an organic part. (Vygotsky, 1999, p. 53)

At the end of his life, in order to better understand the anchoring point between the individual and the social context, he came up with the concept of *perezhivanie* “a concept which allows us to study the role and influence of environment on the psychological development of children in the analysis of the laws of development” (Vygotsky 1994, p. 343). See also Veresov (2017). Yet, there is not much room, in the formulation of the laws of genetic development, for agency and subjectivity.

encountered yet, at first *object* us (that is, they resist or oppose us). They appear as something that is *not* us, as a form of *alterity*. *Object*-ification is our *encounter* with them. And because such an encounter is not something that happens suddenly, instead of objectification *tout court*, we prefer to talk about *processes of objectification*.

More precisely, processes of objectification are those social, collective processes of becoming, progressively and critically, conscious of a culturally and historically constituted system of thought and action—a system that we gradually notice and at the same time endow with meaning. Processes of objectification are those processes of noticing something culturally significant, something that is revealed to the consciousness not passively but by means of corporeal, sensible, affective, emotional, artefactual, and semiotic activity.

In this context, learning is defined as the outcome of processes of objectification. And since systems of thought (mathematical, etc.) are always revealed partially, these processes are always endless—and hence so is learning.

### **Processes of Subjectification**

Now, learning, as defined previously, includes an emotional and affective component. While *a priori* no educational theory would contend otherwise—or so I would like to believe—the emotional and affective components are not necessarily organic parts of most educational theories' account of learning. This is not the case of the TO. In the TO, emotions, for instance, are not seen as an additional component of thinking. On the contrary, following Vygotsky's insight, we consider emotions as an omnipresent part of thinking. In this line of thought, emotions are not relics of our phylogenetic past to be mastered in order to think properly. Emotions are ontological constituents of us, humans, as part of nature. Affect, that is, the capacity of being affected by things in our surroundings, on the other hand, is also part of our human makeup. The educational implication is that instead of being a purely mental endeavour, learning mathematics involves emotions and affects in manners that affect us profoundly as human beings. This is why classrooms do not produce knowledge only; they produce subjectivities as well.

In order to investigate the production of subjectivities in the classroom we resort to the construct of *processes of subjectification*: the processes where, co-producing themselves against the backdrop of culture and history, teachers and students *come into presence*.

To come into presence refers to the idea of the student as someone who, through classroom activity, comes to *occupy a space* in the social world and to *be a perspective* in it. To come into presence is a *dialectical* movement between culture and the individual. The dialectical nature of this movement brings us to conceive of the individuals as entities in flux—entities who are continuously co-producing themselves against the background of history and find in their culture the raw material of their own existence. Both the individual and culture are coterminous entities in perpetual change, one continuously becoming the other and the other the one. In this

dialectical movement, students as well as teachers are considered as subjectivities in the making, openness towards the world. Teachers and students are conceptualized as unfinished and continuously evolving projects of life, in search of themselves, engaged together in the same endeavour where they suffer, struggle, and find enjoyment and fulfillment together.

### **Two examples**

As a first example, let us consider the case of traditional teaching and its pedagogy of knowledge transmission. In traditional teaching the students engage in processes of objectification. Even if the critical stance mentioned in the definition of a process of objectification is almost inexistent, the students of traditional teaching pedagogy encounter systems of thinking that they had not encountered before (e.g., translating a story into an equation, solving the equation through algebraic methods, interpreting the results in the context of the story, etc.).

Along with these processes of objectification, they engage also in processes of subjectification: they come to occupy a space in the social world and take a perspective in it. The social space the students come to occupy may not be the most interesting for many of us, nor the perspective that the students adopt. Yet, processes of objectification and subjectification occur. What is wrong here? What is it that leads the students to occupy a space where they have little room to participate and engage in vivid debates and that leads them to adopt a perspective of obedience? The answer is very simple: the classroom activity: the *teaching-learning activity*.

Indeed, the teaching-learning activity is so poor that the students do not have room to express themselves. As a result, the activity *alienates* them from their own product — the knowledge that was produced in the classroom. The students spend energy trying to understand what is going on, doing repetitive exercises, and memorizing things that the teacher may say are important to remember. Yet, the students do not recognize themselves in the result of their efforts. The teaching-learning activity is alienating and the subjectivities that are produced are alienated.

As a second example, let us consider the case of constructivist learning and its student-centred pedagogy of knowledge construction. In this case, the student is involved, acts, spends her energy and expresses herself. However, that expression remains confined to the subjective sphere of the self. What is wrong here? There is no possibility for a genuine teaching-learning activity where processes of objectification may occur. Indeed, as knowledge is understood as that which is produced by the action of the subject, there is no conversation with humanity (in the sense of Oakeshott, 1993). There is a mere monological conversation of the subject with the subject itself. The process of subjectification is alienating. The subject is alienated from the historical-cultural world (Radford, 2014, 2016) and is confined to live in a “taken-as-shared” universe.

### **Activity as Tätigkeit/deyatel’nost’**

In the TO, what makes learning possible is human, sensuous, practical *activity*. Processes of objectification and subjectification occur in sensuous, practical activity. But the activity to which I am referring here has a definite sense that is very different from the usual conceptions that reduce activity to a series of actions that an individual performs in the attainment of his or her goal. The latter line of thinking reduces activity to a functional and *technical* conception: activity amounts to the deeds and doings of the individuals. Activity in the theory of objectification does not merely mean to do something. Activity (Tätigkeit in German and deyatel'nost' in Russian) refers to a *dynamic system* geared to the satisfaction of collective needs. This is why activity as Tätigkeit/deyatel'nost' should not be confounded with activity as Aktivität/aktivnost'; that is, as being simply busy with something (Roth and Radford 2011). Activity as Tätigkeit/deyatel'nost' is a social form of joint endeavour through which individuals produce their means of subsistence while producing themselves as humans. It “comprises notions of self-expression, rational development, and aesthetic enjoyment” (Donham 1999, p. 55). More precisely, Activity as Tätigkeit/deyatel'nost' is a *form of life*. To avoid confusion with other meanings, in the theory of objectification, activity as Tätigkeit/deyatel'nost' is termed *joint labour*.

### **Joint labour**

Joint labour is the chief ontological category of the theory of objectification. Its central role derives from a dialectical materialist anthropological conception of the human. Following Spinoza (1989), humans are considered to be *part of nature*: they are *natural beings*. Like all other natural living beings, humans are *beings of need* who find their satisfaction in objects *outside* of themselves.

To meet their needs (needs of survival and also artistic, spiritual, and other needs created by/in society), humans engage themselves actively in the world. They *produce*. What they produce to fulfill their needs occurs in a *social* process that is, at the same time, the process of the individuals' inscription in the social world and the production of their own existence. The name of this process is what in the previous section I have termed *joint labour*. Sensuous, material joint labour is considered the ultimate field of aesthetic experience, subjectivity, and cognition. Joint labour as the central category of the TO asserts the fundamental ontological and epistemological role of matter, body, movement, action, rhythm, passion, and sensation in what it is to be human.

One important consequence of this theoretical stance is that the individual cannot be conceived of as a substantial entity, produced from within, as articulated by both the rationalist and the empiricist humanist trends of the Enlightenment that informs most contemporary educational theories. In the TO, the individual is a cultural-historical entity that goes beyond the skin; it is *relational* through and through. It is profoundly linked to an ensemble of material and immaterial relationships with other parts of nature—including social relationships—and is based on culturally and historically constituted conditions of life.



Joint labour as the chief ontological and epistemological category of the theory of objectification leads us to consider classroom activity as the unit of analysis. Yet, the role of language, signs, artefacts, and the body is not dismissed in the processes of knowing and becoming. In the TO, language, signs, artefacts, and the sentient body are understood not as mediators but as *part* of the individuals' activity. And since thinking and activity are considered here as intertwined (Vygotsky in Zavershneva, 2010), language, signs, artefacts, and the body are also considered as part of thinking. They are part of the *material texture* of the individuals' thinking.<sup>3</sup>

At a practical level, the concept of joint labour allows one to conceive of classroom teaching and learning not as two separate activities, one carried out by the teacher (the teacher's activity) and another one carried out by the student (the student's activity), but as a single and same activity: the same teachers-and-students' joint labour. The teacher does not appear as a possessor of knowledge who is delivering or transmitting knowledge to the students or as someone scaffolding strategies to the students. Nor do the students appear as passive subjects receiving knowledge. In the classroom activities that we seek to promote in our work with teachers and students, the teacher and the students *labour together* towards the production of a *common work* — e.g., the sensuous appearance in the classroom of a co-variational algebraic way of thinking about sequences. It is in the production of this *common work* that the students are conceived of as encountering, and becoming gradually aware of, culturally and historically constituted forms of mathematics thinking.

### **An example of teaching-learning algebra**

I would like to refer here to an example that comes from a Grade 4 class (9-10-year-old students) where the students were dealing with a sequence generalization problem. The problem was based on the following story:

For his birthday, Marc receives a piggy bank with one dollar. He saves two dollars each week. At the end of the first week he has three dollars, at the end of the second week he has five dollars, and so on.

The teacher provided the students with bingo chips of two colours (blue and red) and numbered plastic goblets intended to represent Week 1, Week 2, etc., and invited the students to model the saving process until Week 5. Then, drawing on the model, the teacher invited the students to find the amount of money saved at the end of Weeks 10, 15, and 25.

With some help, the students produced the model shown in Figure 1. After some discussion, the students came up with a “doubling strategy”: they found the number of bingo chips in Week 5, doubled this amount and removed 1 bingo chip:

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<sup>3</sup> This theoretical position leads to a particular conception of embodiment (see “sensuous cognition” in Radford, 2013)

1. Krysta: So, we should do . . . That (see Pic 1 in Fig. 1) times two. So 11 . . .
2. Albert: 11 plus 11 . . . 22.
3. Krysta: 22 . . .
4. Albert: Well, wait . . . No. It would be 11 plus 10 because . . . (Pointing to the blue bingo chip) We always start with the . . . [blue chip] (see Pic 2 in Fig. 1).

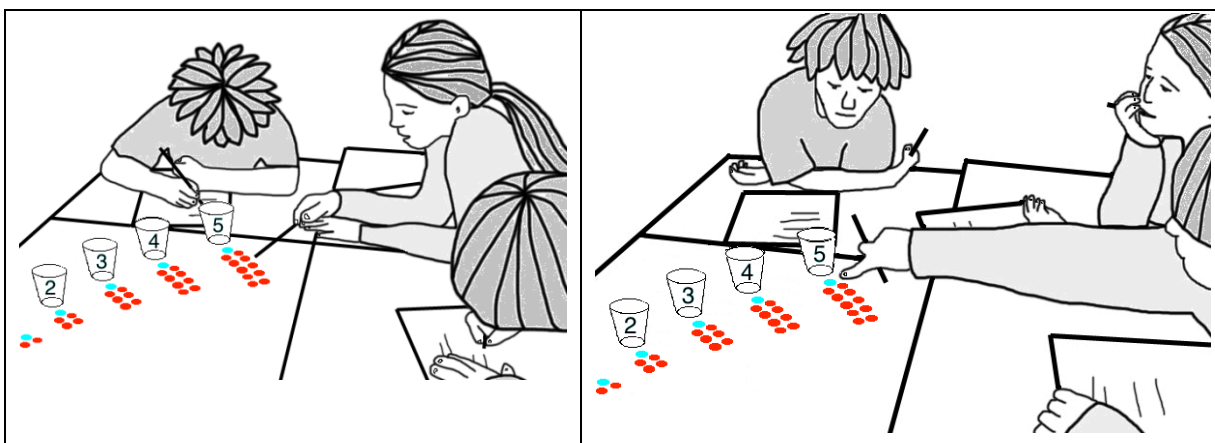


Figure 1. Left, Krysta pointing to the bingo chips in front of Week 5. Right, Albert pointing to the blue bingo chip.

The teacher came to see the students' work and engaged in the conversation:

5. Teacher: (*Trying to make noticeable to the students the co-variational structure*) What do you remark about Week 5 (*She shows the glass corresponding to Week 5*) and (*Pointing to the red bingo chips says*) the number of bingo chips? (*Making the same actions*) The fourth week and the number of bingo chips?
6. Albert: (*Hesitantly and at the same time interested says*) It's always twice . . .
7. Mrs. Giroux: (*Repeating*) It's always twice.

The mathematical variables started being noticed. They started becoming objects of consciousness. However, their co-variational algebraic nature remained unnoticed. Joint labour reaches here a tension that derives from the contradictory ways in which the terms of the sequence have been so far perceived (an arithmetical one, based on doubling, and an algebraic one, based on a co-variational approach to the problem). This contradiction is not a flaw of a didactical design: it is the very motor that keeps the activity unfolding. To encounter algebraic thinking as featured in the teacher's didactical project, the teacher and the students have to keep working together and to try to make the algebraic approach to appear in the classroom and to become an object of the students' consciousness. Its appearance is a bit like the appearance of Beethoven's 7<sup>th</sup> symphony: for it to become an object of consciousness it has to aurally appear in a hall concert. Since mathematics is simultaneously visual, tactile, aural, material, artefactual,

gestural, and kinesthetic, it can only come into life here through the sensuous and artefactual joint labour of the teachers and the students.

After some discussion and failed attempts at making noticeable the algebraic structure of the bingo chips' visual arrangement, the teacher came back to an analysis of Week 5:

8. Teacher: (*Taking with her hand again the goblet of Week 5*) What did you do here?
9. Albert: (*Takes a long breath and hits the desk with the pen, while the teacher holds the goblet of Week 5; see Pic 1*) OK.
10. Teacher: (*Still holding the goblet, speaks softly*) 5 . . .
11. Albert (*In sync with the teacher's gesture that points next to the red chips, see Pic 2*) Times 2 . . .
12. Krysta: (*Who has followed the discussion*) Times 2 equal . . .
13. Teacher: (*Pointing at the blue bingo chip, see Pic 3*) Plus 1.
14. Albert: (*Almost at the same time*) Plus 1.
15. Teacher: (*Now pointing to an empty space where Week 10 would be, see Pic 4*) 10?
16. Albert (*The teacher points silently at the place where the red bingo chips should be, Pic 5*) Times 2.
17. Krysta: (*At the same time*) Times 2.
18. Teacher: (*Points silently at the place where the blue bingo chip should be, see Pic 6*)
19. Krysta: Plus 1.
20. Albert: (*Looking at the teacher*) Minus 1?, times 2, minus 1?, plus 1?

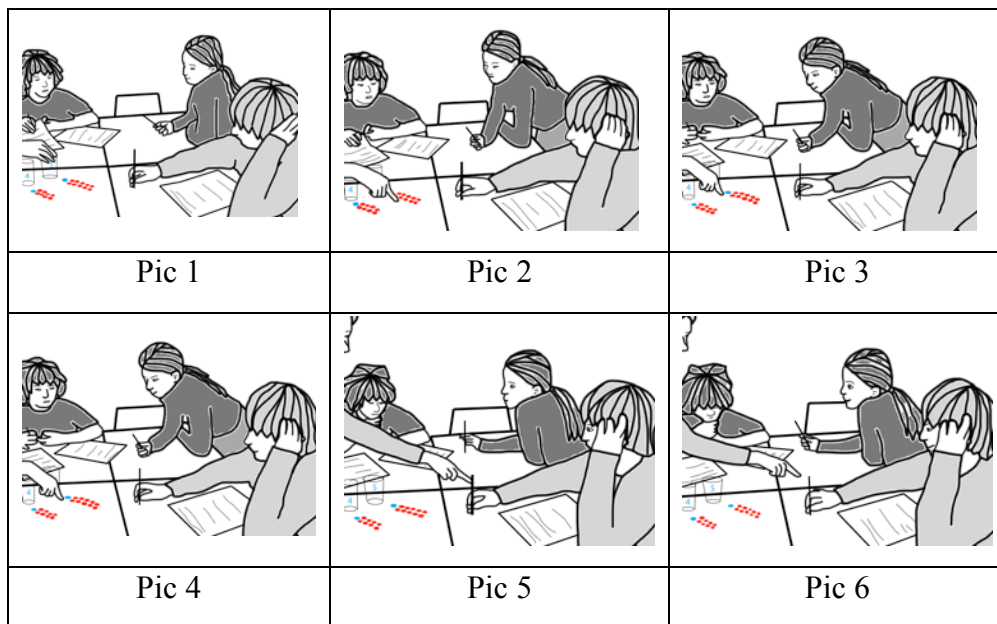


Figure 2. Joint labour and the appearance of an algebraic approach to solve the problem

This excerpt is an example of the teacher and students' joint labour —a spatial-temporal dynamic system that is created by the students and the teacher as they engage in the pursuit of

the object of their joint labour (encountering an algebraic system of thinking). It features a social process that is at the same time a process of objectification (as the students are encountering something culturally relevant) and subjectification (as the students are coming into presence and positioning and being positioned in a mathematical practice).

Of course, the students are not familiar with it and as a result cannot recognize it yet. To encounter it, and to be able to recognize it, is the object of their joint labour. But how can one pursue something we don't know? Learning is the disclosure of that which so far was *desire*.

Joint labour is made up of the energy that the teacher and the students discharge, energy that is sensible and sensual, material and ideational, discursive and gestural, and that is a fluid carrier of half-confessed and half-understood desires, intentions and motives. Wrapped in this energy, the teacher moves her hand silently to indicate with an indexical gesture the imaginary position of the blue chip (see Pic 6) and Albert says "Minus 1? Times 2 Minus 1? More 1?" attesting to the fact that the co-variational algebraic manner by which to see the variables is becoming intelligible to his consciousness. At this point of the activity/energy/joint work, Albert's encounter with key aspects of algebraic knowledge is taking place. Albert still has to perceive better the nuances of the algebraic variables and how they relate to each other. It did not take long. During the general discussion, which started right after the end of the previous excerpt, the teacher invited Albert to explain how to find out the number of bingo chips in Week 4. He said: "4 times 2 . . . plus 1, 4 times 2 plus 1 equals . . . 9."

### **The ethical dimension**

I argued above that the classroom activity of the child-centred and transmissive pedagogies end up in alienation. The idea of classroom activity as joint labour provides us with an alternative to envision new cultural-historical non-alienating forms of teaching and learning. To go further further, we need to redefine the two axes that organize all classroom activity: the axe of the forms of classroom knowledge production, and the axe of the forms of human collaboration.

In the TO, the forms of classroom knowledge production are driven by collective endeavours informed by history and culture where the teacher and the students work together to reach deep levels of mathematical conceptualization. Knowledge is not constructed or transmitted but *encountered* through collective sensuous processes of objectification. The forms of human collaboration are informed by a *communitarian ethic* that is coherent with the educational project that subsumes the theory of objectification mentioned at the beginning of this paper, namely the dialectical creation of reflexive and ethical subjects who critically position themselves in historically and culturally constituted mathematical practices, and who ponder new possibilities of action and thinking.

The ethical forms of human collaboration are motivated by a general attitude toward the world and serve to configure the teachers' and students' joint labour in the classroom. These critical and communitarian ethical forms of relating to others blur the borders that separate the

teachers from the students. Teachers and students labour in concert as one. The classroom appears as a public space of debates in which the students are encouraged to show openness toward others, responsibility, solidarity, care, and critical awareness. The classroom indeed appears as a space where teachers and students come into presence. That is to say, the classroom appears as a space of encounters, dissidence, and subversion, where teachers and students become individuals who are more than in the world—they are individuals with a vested interest in one another and in their joint enterprise; individuals who intervene, transform, dream, apprehend, suffer, and hope *together*.

### **Relations and connections with the theories of Anna Sfard and Michèle Artigue**

I find plenty of possible relations and connections between the TO, Commognition, the ATD, and the Joint Action Theory in Didactics (JATD).

Certainly, the theoretical orientations and the philosophical traditions that inform those theories are not necessarily the same; their research questions may be motivated by different problems. Mathematics education theories, I believe, bear the imprint of the intellectual traditions from where they emerge and are profoundly shaped by the educational and societal problems to which they attempt to respond. But I do not think that mathematics education theories should have the same intellectual traditions or face the same problems in order for us to find affinities between them and to engage in a fruitful dialogue. There is much to know about each other that is important to get to better know oneself. Bakhtin once remarked that “I am conscious of myself and become myself only while revealing myself for another, through another, and with the help of another. The most important acts constituting self-consciousness are determined by a relationship toward another consciousness (toward a *thou*)” (Bakhtin, 1984, p. 287).

Anna Sfard and Michèle Artigue have been two among the most important scholars in our field who have made a crusade of the need to overcome separatism in intellectual traditions. I have learned a great deal from their laudable and exemplary attitude. In the introduction to a special issue on mathematical discourse, Anna pleads for concrete actions “to break out from the grip of the separatist tradition, the tradition of using different, often unbridgeable discourses for dealing with different aspects of learning” (2012, p. 1). Michèle has played a prominent role in the birth and development of what came to be known as *Networking Theories*, that is, in the creation of a space where different research traditions and orientations could meet, talk to each other, and work together. It is no small honour for me to be here today with them at this Symposium.

There are several relations that, in my view, can be found among the aforementioned theories. I mention here two.

### **Language and discourse**

First, the theories discussed at this Symposium have a common interest in language and discourse. Language and discourse appear clearly in two of the chief components of the TAD: as “technological discourse describing, explaining and justifying [a] technique . . . and [as] theory conceived as a more or less explicit and organized discourse backing up the technology” (Artigue, 2018). And, of course, language and discourse are at the heart of Commognition where mathematics is considered to be a discourse (Sfard, 2012). In the TO, language plays a fundamental role in the investigation of processes of objectification and subjectification, in the students’ attainment of higher levels of consciousness and forms of alterity.

## History

The ATD, Commognition, and the TO entertain a special relationship to history. Not only the didactic processes we are interested in investigating unfold in time (as in the ontogenetic discursive developments described in the work of Anna or in the constitution of a theory that stems from a task described in the work of Michèle), but they are also informed by history.

Then, there are also more specific points of convergence between a theory and another. The levels of hierarchy of elementary algebraic discourses seem to play a somehow similar role in Commognition and the TO. In the latter, an effort has been made to distinguish types of generalization in mathematical thinking. The idea is that levels of conceptualization correspond to levels of consciousness, and levels of consciousness can be distinguished in terms of the semiotic means that the students and the teachers use in processes of objectification. You cannot achieve the same level of consciousness about a mathematical object, its components, and the relationship between these components, through gestures alone, as if you were to use a more complex unit of gestures-language-mathematical symbols or as if you were to use pure alphanumeric symbolism. There is hence an intriguing connection here between these two theories that I think is worth investigating. And, of course, there is also the question of vocabulary; that is, the theoretical terms we use in our theories to refer to some theoretical content. Anna has been using the term objectification for many years. I do not think that this term refers to the same thing in the TO. It is worth mentioning, indeed, two related but different German terms that Hegel uses in his philosophy and that have much to do with the concept of objectification that I am proposing. The terms are *Gegenstand* and *Objekt*. Usually, both terms are translated as *Object*, although their meaning is not the same.

*Objekt* refers to an object in general, independent of the individual. On the contrary, *Gegenstand* means an object of consciousness, mediated by consciousness and continuously changing with respect to it. The difference is, then, that *Gegenstand* refers to an intentional object—an object of consciousness or thought—while *Objekt* refers to the object in its logical, non-subjective sense—an object (not necessarily material) in the world, an historical-cultural object (Inwood, 1992, p. 204).

There are, then, two clearly different senses of objectification. Objectification in the sense of *Objektivierung*—which is inscribed in the idea of making an object, as in doing its reification, its *thing*-ification, and the idea of objectification that am I proposing here. These two senses are at the same time different from a third sense of objectification that appears in certain ontological approaches, in which the term refers to something objective independent of the human. The idea of objectification that I propose is rather a phenomenological idea that is situated in the *Vergegenständlichung* sense: a process that includes the work or effort in apprehending an object (Objekt) already there (temporary, changing object, always relative to culture); a process in which the subject expresses herself through precisely the work or effort she makes to apprehend the object (Objekt). Hegelian scholars who try to preserve the difference use in their translations objectification (as reification) and ob-jectification in the *Vergegenständlichung* sense. I tried to use ob-jectification (and even ob-jectification) for a while, but some colleagues found it curious and amusing, so I gave up.

A last point that I find very interesting in the Joint Action Theory in Didactics is the concept of “joint.” I see the efforts to keep this concept compatible with the concepts of the Theory of Didactic Situations. Reading the paper that Michèle shared with us was not only an enjoyable enterprise but one that originated a series of questions. Are the two concepts equal, similar, or different? And if they are not equal, to what extent are they similar?

As I mentioned before when I made reference to Bakhtin, this contrast, as well as all the contrasts that can emerge from this dialogue between theories, is important in order to better grasp our own concepts, and better understand ourselves.

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