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Introduction

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Abstract This introductory chapter presents an overview of the role of semiotic issues in the teaching and learning of mathematics, as these issues are characterized and elaborated in the chapters of this monograph. Several threads are represented in the four sections of this book: the evolving sociocultural perspective is addressed in Sects. 1 and 4; Sect. 2 addresses linguistic and textual aspects of signification, and Sect. 3 represents Peircean perspectives that were recognized as important in our field more than two decades ago, which continue to have relevance.

1.1 Introduction

All the chapters in this monograph grew out of presentations by the authors in Topic Study Group 54 (TSG 54), *Semiotics in Mathematics Education*, of the Thirteenth International Congress on Mathematics Education (ICME-13), held in

Hamburg, Germany, 24–31 July 2016. The four regular sessions of TSG 54, and the associated Oral Communications in the Congress timetable, were well attended by scholars who had been working in the field of semiotics in mathematics education for decades, as well as interested newcomers to the field. This variety of experience in the topic is reflected in the chapters in this book, which is thus intended both as an introduction to the field, and a reasoned appraisal of research on semiotics in mathematics education, its significance, old and new theoretical developments, where it has been and where it might be going.

1.2 What Is Semiotics and Why Is It Significant for Mathematics Education?

Semiotics is related to *semantics*, according to one dictionary (Funk and Wagnall 2003), and may be defined as "The relation between signs or symbols and what they signify or denote." The same dictionary defines the verb *signify* as follows: "1. To make known by signs or words; express; communicate; announce; declare. 2. Hence, to betoken in any way; mean; import. 3. To amount to; mean." The adjective *significant* has this definition: "1. Having or expressing a meaning; bearing or embodying a meaning. 2. Betokening or standing as a sign for something; having some covert meaning; significative. 3. Important, as pointing out something weighty; momentous."

These definitions point to a number of elements intrinsic to the nature of the activity with which semiotics is concerned, *semiosis*: it involves *signs*; these signs have *meanings*, which suggests that there are interpretations, and that consciousness is involved in some way—and also communication. Semiosis is "a term originally used by Charles S. Peirce to designate any sign action or sign process: in general, the activity of a sign" (Colapietro 1993, p. 178). A sign is "something that stands for something else" (p. 179); it is one segmentation of the material continuum in relation to another segmentation (Eco 1986). *Semiotics*, then, is "the study or doctrine of signs" (Colapietro 1993, p. 179). Sometimes designated "semeiotic" (e.g., by Peirce), semiotics is a general theory of signs or, as Eco (1988) suggests, a theory of how signs signify, that is, a theory of signification (see Presmeg et al. 2016, p. 1).

The significance of semiosis for mathematics education lies in the use of signs; this use is ubiquitous in every branch of mathematics. It could not be otherwise: the objects of mathematics are ideal, general in nature, and to represent them—to others and to oneself—and to work with them, it is necessary to employ sign vehicles,[1]

[1]A note on terminology: The term "sign vehicle" is used here to designate the signifier, when the object is the signified. Peirce sometimes used the word "sign" to designate his whole triad, object [signified]-representamen [signifier]-interpretant; but sometimes Peirce used the word "sign" in designating the representamen only. To avoid confusion, "sign vehicle" is used for the representamen/signifier.

which are not the mathematical objects themselves but stand for them in some way. An elementary example is a drawing of a triangle—which is always a particular case—but which may be used to stand for triangles in general (Radford 2006).

Semiotics has long been a topic of relevance in connection with language (e.g., de Saussure 1959; Vygotsky 1997). However, it is in the last few decades that its potential has been realized for mathematics education research. In the early 1990s, David Kirshner may be credited with the introduction of semiotics, in the form of Saussurean semiology, Peircean semiotics, and semiotic chaining, to many researchers in mathematics education in the USA. He organized the Annual Meeting of the North American chapter of the International Group for the Psychology of Mathematics Education in Baton Rouge, Louisiana, and Whitson (1994, 1997) delivered the keynote address, in which he also provided an entry to the semiotic activity of Walkerdine (1988). Semiotics gained the attention of many researchers interested in furthering the understanding of processes involved in the learning and teaching of mathematics (e.g., Anderson et al. 2003; Presmeg 1997, 1998, 2003, 2006a, 2006b; Radford 2013; Radford et al. 2008, 2011; Sáenz-Ludlow and Kadunz 2016; Sáenz-Ludlow and Presmeg 2006a). In the Vygotskian tradition, *semiotic mediation* was used as a powerful research lens (Mariotti and Bartolini Bussi 1998).

As characterized by Presmeg et al. (2016), the study of signs has a long and rich history. However, as a self-conscious and distinct branch of inquiry, semiotics is a contemporary field originally flowing from two independent research traditions: those of Peirce (1931–1958), the American philosopher who originated pragmatism, and de Saussure (1959), a Swiss linguist generally recognized as the founder of contemporary linguistics and the major inspiration for structuralism. In addition to these two research traditions, several others implicate semiotics either directly or implicitly: these include *semiotic mediation* (the "early" Vygotsky 1978), *social semiotics* (Halliday 1978), various theories of representation (Goldin and Janvier 1998; Vergnaud 1985; Font et al. 2013), relationships amongst sign systems (Duval 1995), and more recently, theories of embodiment that include gestures and the body as a mode of signification (Bautista and Roth 2012; de Freitas and Sinclair 2013; Radford 2009, 2014; Radford et al. in press; Roth 2010). Components of some of these theories are elaborated in this book.

As a text on the origin of (Euclidean) geometry suggests, the mathematical concepts are the result of the continuing refinement of physical objects that Greek craftsmen were able to produce (Husserl 1939).[2] For example, craftsmen were producing rolling things called in Greek *kylindros* (roller), which led to the mathematical notion of the cylinder, a limit object that does not bear any of the imperfections that a material object will have. Children's real problems are in

[2]Husserl distinguished two aspects of signs, namely *expression* and *indication* (Husserl 1970; Zagorianakos 2017). It is beyond the scope of this monograph to explore the implications of Husserl's phenomenological distinction here; however, both of these aspects of signs are highly relevant in the issues addressed in this book. Expression relates to intention and the grounding of ideation, whereas indication relates to communication and is the essence of semiotics.

moving from the material things they use in their mathematics classes to the mathematical things (Roth 2011). This principle of "seeing an A as a B" (Otte 2006; Wartofsky 1968) is by no means straightforward and directly affects the learning processes of mathematics at all levels (Presmeg 1992, 2006a; Radford 2002). Thus semiotics, in several traditional frameworks, has the potential to serve as a powerful theoretical lens in investigating diverse topics in mathematics education research.

1.3 Sociocultural Perspectives on Semiosis

Sociocultural perspectives on semiosis emphasize the social, cultural, and historical dimension of signs. In these perspectives signs are understood not as artifacts to which an individual resorts to represent or present knowledge, but as artifacts of communication and signification. Signs are not considered as mere expressions of the individual's thought; they appear rather as entities through which the individual orients her actions and reflections and shapes her experience of the world.

The origin of this non-representational view of signs goes back to the early Vygotsky, who considered the sign as a sort of psychological instrument deeply related to the way we conduct ourselves in society. The essence of sign use, Vygotsky argued, "consists in man's [sic] affecting behavior through signs" (1978, p. 54). Vygotsky was particularly interested in the role of language. In a notepad dated 1926 he defines language as follows:

> Language is not the relation between a sound and the denoted thing. It is the relation between the speaker and the listener, the relation between people directed toward an object, it is an interpsychic reaction that establishes the unity of two organisms in the same orientation toward an object. (Vygotsky in Zavershneva 2010, p. 25)

More than a representation device, language is a *nexus* between individuals.

At the end of his life, Vygotsky was moving away from the instrumentalist view of signs to a view where meaning and signification acquired a more prominent role and where consciousness was understood in semiotic terms. Vygotsky wrote: "Consciousness as a whole has a semantic structure" (1997, p. 137). By this, Vygotsky meant that consciousness is not something metaphysical but our actual link to the world. He continues: "*We judge consciousness by its semantic structure, for sense, the structure of consciousness, is the relation to the external world*" and concludes that "*Speech produces changes in consciousness.* Speech is a correlate of consciousness, not of thinking" (p. 137; emphasis in the original).

Vygotsky's non-representational view of signs leads to a conception of semiotics that opens an interesting path in which to investigate the problems underlying education in general and mathematics education in particular. Consciousness and thinking are not merely the production of the individual. Consciousness and thinking come into life against the backdrop of their sociocultural context. But this context is not a mere facilitator of consciousness and thinking. Consciousness and thinking do not merely *adapt* to the context, they are modified by it and, in a dialectic movement, they come to modify the context from which they emanate. In the dialect materialist framework in which Vygotsky sets the problem of consciousness and thinking, both the individual and culture are coterminous entities in perpetual flux, one continuously becoming the other and the other the one.

This is so because signs and semiotic systems more generally are bearers of a worldview that includes mathematical, scientific, aesthetic, legal, and ethic components through which individuals organize their world (Radford 2008). As a result, the apparently transparent and neutral manner in which students encounter mathematics and other disciplines in the school has an unavoidable ideological valence. For instance, the Cartesian graph, which is featured in several chapters in this monograph, conveys a conceptual view according to which things in the world can be related and referred to the same point (the Cartesian origin). It stresses a relational view of phenomena attended to in terms of variables and their relationship. In opposition to other kind of graphs, such as maps, what a Cartesian graph depicts is not the elements of the considered phenomena but specific mathematical relationships between them—their covariation. Behind a Cartesian graph lies thus a general view of the world, where things are thought of in certain culturally and historically constituted ways. Implicitly, they organize and orient the kind of experience that students and teachers make of the world, creating thereby sociocultural conditions for the emergence of specific forms of mathematical thinking and learning. The same can be said of other semiotic systems too (e.g., the alphanumeric symbolism of algebra). Through them, our view of the world becomes *naturalized*. The world appears in specific ways. This is what the ideological valence of signs means.

Of course, the ideological valence of signs expressed in the worldview that the signs unavoidably carry and the concepts to which they refer cannot be revealed to the students spontaneously, that is, in an immediated or unmediated manner. A student can spend hours looking at a Cartesian graph without necessarily understanding what this complex mathematical sign means and is meant for. A sign *as such* is no more than that: a sign. To signify, to reveal its conceptual power, a sign has to become part of an *activity*. It is not hence through signs as such that students make the experience of mathematics their own, and that they encounter the culturally and historically constituted forms of mathematical thinking in the school or the university. Mathematics can only be disclosed to the students through sign-based activity. It is through material and concrete sign-based activity that students learn mathematics and that teachers teach it.

One of the differences between sociocultural perspectives on semiotis resides in how they conceptualize the learning activity and the role they ascribe to signs. Some perspectives emphasize the *discursive* dimension of activity, while others emphasize its *intersubjective* and *ethical* relational dimension and the evolving object/motive of the activity. Some perspectives consider signs as *mediators* of activity (Bartolini Bussi and Mariotti 2008), others consider signs as *part* of activity and as part of the material texture of thinking (Radford 2016a, b).

1.4 Language and Text Orientations

The importance of language for the learning of mathematics is a widely studied subject (Hoffman 2005; Sáenz-Ludlow and Presmeg 2006b; Schreiber 2013). A search query on the subject “language” in the journal *Educational Studies in Mathematics* resulted in more than a thousand responses (June 2017). As an example, reference should be made here to two anthologies published in recent years or forthcoming (Moschkovich 2010; Barwell 2017) and to a recent review of language in mathematics education published during the past 10 years in the Proceedings of PME (Radford and Barwell 2016). A much smaller result followed the request for “semiotics and language,” to which the contributions in the second section of this volume are devoted. What is the relationship between text and language, the written and the spoken (Radford 2002; Kadunz 2016)?

In the brevity of this introduction, let us concentrate on two of the most important authors. On the one hand, we focus here on Peirce (see also the next section of this introduction), whose semiotics can be seen as paradigmatic for the importance of the written when doing mathematics. On the other hand, in contrast let us consider the work of the philosopher Ludwig Wittgenstein, who, with his theory of language play, has made a significant contribution to the philosophy of language, as used also in didactics of mathematics (Vilela 2010; Knijnik 2012).

A semiotic view of the learning of mathematics—if one chooses a Peircean orientation—is mainly determined by interpreting the use of visible signs. What approaches and questions open up when a linguistic approach is added to this theory of signs? Which parallels can be found between the formulations of Peirce and Wittgenstein? A simultaneous use of both approaches for questions involving the didactics of mathematics took place a few years ago (Dörfler 2004, 2016).

Documented parallelism between the central concepts of Peirce and Wittgenstein can be found mainly outside mathematics didactics. In this respect, starting several years ago, Gorlée (1994, 2012) presented a series of publications dealing with Peirce’s semiotics as a tool for the analysis of translation questions. In “Semiotics and the problem of translation” (1994), she dedicated a chapter to the relationship between notions of Peirce and Wittgenstein. What are the similarities between Peirce’s semiotics and Wittgenstein’s philosophy of language? Wittgenstein presented one of his most far-reaching tools, namely, the *language game*, in his philosophical investigations (1953–1968). In this formulation, among others, he included the following:

> Giving orders, and obeying them;
>
> Describing the appearance of an object, or giving its measurement;
>
> …
>
> Forming and testing a hypothesis;
>
> Presenting the results of an experiment in tables and diagrams;
>
> …

> Solving a problem in practical arithmetic;
>
> Translating from one language into another.
>
> … (Wittgenstein 1953–1968 Part 1 paragraph 23; Gorlée 1994, p. 97)

In particular, he counted mathematical activities among these games. On the one hand the participation in a language game is characterized by the obeying of rules. On the other hand, the language game is embedded in a form of life: "Though primarily language-based, language-games do not function in a social vacuum, but are inscribed in so-called 'forms of life'." A form of life is "… a pattern of meaningful behavior in so far as this is constituted by a group" (Finch 1977, p. 91; Gorlée 1994, p. 99).

Practicing a language game is practicing an activity within a form of life which, according to Wittgenstein, combines language and reality. If we follow Umberto Eco (1979), the concept of the form of life provides a first bridge between Wittgenstein and Peirce. "Eco identifies the cultural system as a whole with the dynamic process of semiosis, and therefore, cultural units with Peircean interpretants." (Gorlée 1994, p. 100). In Peirce's terminology, the meaning of a sign is another sign which leads to a never ending process of interpretation embedded in our sociocultural life, which can be seen as a certain kind of practice.

Another similarity between the theories of Peirce and Wittgenstein can be found when we look at Peirce's concept of "ground." For Peirce, ground seems to be a kind of context that determines how a character represents a designated object. As Gorlée portrays it, then, "this ground is an abstract but knowable idea serving as justification for the mode of being manifested by the sign" (1994, p. 101). This embedding in a context corresponds in some respects to Wittgenstein's concept of "inner motivation". This motivation is the "ground," in which a language game has to be played within the framework of the corresponding rules. Similar comparisons, which can be only hinted at here, concern the Peircean concepts of firstness, secondness and thirdness, contrasted with concepts from Wittgenstein`s language games (Gorlée 1994). What is common to these opposites is the fact that for Peirce and for Wittgenstein, the interpretation of signs (semiosis) as well as the activities within the context of a language game are more focused on the process than on the result. After this excursion into a certain philosophy of language let us return to pure Peircean semiotics, in the next section.

1.5 Peircean Semiotics, Including Semiotic Chaining and Representations

Throughout the 1990s and in the early 2000s, the issue of how representations of various kinds played a role in mathematics education was a significant focus for researchers. There were Working Groups on this topic in the meetings of the International Group for the Psychology of Mathematics Education (PME) and its North American affiliate (PME-NA), resulting in an edited volume of papers from these conferences (Hitt 2002). Several authors in the current monograph were represented in this volume (Otte, Presmeg, Radford, Sáenz-Ludlow). In this early work, researchers were groping for theoretical formulations that went beyond a dualistic view of mathematical representation and captured the complexity involved in learning mathematics using its signs. A Peircean semiotic perspective provided one such conceptual lens.

Although a representational perspective on semiotics has largely given way to evolving sociocultural views (see Sects. 1 and 4 of this monograph), Peircean semiotics still has a foundational role to play in semiotics defined as “the relation between signs and symbols and what they denote” (Funk and Wagnall 2003). Further, semiotic chaining based on Peirce’s semiotics has historical significance in the field of mathematics education on account of its contribution to research in this field since the early 1990s (Whitson 1994, 1997; Presmeg 1997, 1998, 2003, 2006a, b; Sáenz-Ludlow and Presmeg 2006a), and it still continues to provide a viable research lens for the teaching and learning of mathematics (Sect. 3 of this monograph).

The essence of Peirce’s semiotics is his use of triads (see Chap. 11 by Sáenz-Ludlow for a fuller treatment of this topic). “But it will be asked, why stop at three?” Peirce asked, and he replied that unlike a triad, which adds something to a pair, “four, five, and every higher number can be formed by mere compilations of threes” (Peirce 1992, p. 251). His triad of signs as composed of *object* (signified), *representamen* (signifier) and the essential component of *interpretant* made possible the chaining of signs, since each sign as a whole is subject to further representation and interpretation, in a never-ending process of potential signification. Presmeg (1998, 2002, 2006b) used the metaphor of Russian nested dolls to describe this process, which was useful in linking home cultural practices of students with the mathematics that they learned in school (see also Presmeg 2007). Sáenz-Ludlow and colleagues used elaborated versions of Peirce’s triads and the chaining of signs in fine-grained analyses of the processes involved in teaching and learning geometry (Sáenz-Ludlow and Kadunz 2016, and see Chap. 11 of this monograph). Sign vehicles characterized by Peirce’s triad of *icon*, *index*, or *symbol* were the basis for an analysis of connections among early processes in the teaching and learning of trigonometry at high school level, obviating the compartmentalization that is often a hindrance in such learning (Presmeg 2006a).

Semiotic resources including gesturing and tools; developments in theoretical frameworks that involve these aspects, are discussed in the next section.

1.6 Semiotic Resources Including Gestures and Tools

In most general terms, the sign has been defined as a relation between one portion of the material continuum, which serves as sign vehicle for a relationship with other portions of the continuum (Eco 1986). Thus, any material thing—scribbles with pens, characters printed by a machine, sounds coming from a mouth, or tools used for doing things—constitutes a segmentation of matter that may be part of a relation between things. Such relations among material things indeed reflect relations between persons; in turn, human relations are reflected in the relation between things (Marx and Engels 1978). The relation between material things comes to be attributed to one of these as a suprasensible (ideal) characteristic, namely, its value in economic (Marx and Engels 1962) or verbal exchange (Rossi-Landi 1983; Roth 2006); human relations, once represented in the individual, have become higher psychological functions and personality (Vygotsky 1989). Indeed, the human body, being material, may be part of a material configuration and thus serve as the sign vehicle for other things. It is therefore not without surprise to read that "as subjects, we are what the shape of the world produced by signs makes us become" (Eco 1986, p. 45).

Any human action may become significant and, thereby, become part of a signifying relation. Thus, for example, when asked to describe and explain an experiment they have done, physics students invited the teacher, "Look!," and then redid the experiment (Roth and Lawless 2002). When students in a mathematics class have their heads down, writing in a notebook, this bodily configuration and writing itself may be treated as the sign of their engagement with the task. Work-related body movements that are taken to stand for something are denoted by the term ergotic gestures [gestes ergotiques] (Cadoz 1994; Roth 2003). Interestingly, in speaking, more is happening than the production of sound words that are somehow referring to or invoking something else. The very act of speaking may be significant as an act generally or as a speech act specifically (Schütz 1932). "Did you say something?" is a query to find the significance in the former case, whereas "What did you say?" is a query to find out the significance in the latter case.

In individual development, there actually is a movement from ergotic gestures to symbolic gestures [gestes symboliques], which, in humans, may morph into or be replaced by productions such as sound-words or hand/arm movements that take their place (e.g., the stinky finger, a square formed by an appropriate configuration of thumbs and index fingers of two hands). This movement first was described in the case of children learning to gesture: an infant may be seen by the mother as reaching for an object; she takes the object and puts it into the hand of the infant; and finally, the infant moves hand and arm intentionally to point at objects (Vygotsky 1989). The same transformation also was reported among the bonobo chimpanzees, where part of the infant's movement involved in the mother's picking up the infant later, deployed in a frozen form, is treated by the mother as a sign that the infant wants to be picked up (Hutchins and Johnson 2009). In school science, the same trajectory has been described beginning with the initial ergotic gestures, which then turned into symbolic gestures using part of the equipment or substituted artifacts (tools), all of which eventually were replaced in verbal descriptions and diagrams (Roth and Lawless 2002). Materials, artifacts and tools have communicative and thus both social and psychological function. In the context of using graphs as part of lectures, hand/arm movements initially appeared to carve out the space, exploring possible placements of curves, before some of these movements actually produced the graph expressing a mathematical function (Roth 2012). That is, gestures may indeed pave the way for linguistic and conceptual development (Iverson and Goldin-Meadow 2005). It also has been shown that the relation between symbolic gestures and what they stand for may change in the course of time, thereby changing (developing) signifier–signified relations.

In the mathematics education literature, we can find the term "body language" (e.g. Evans et al. 2006). But this notion, though "common in everyday language, is not a useful concept here because body movements and positions are neither structured nor used like language" (Roth 2001, p. 368). Hand/arm movements may be located somewhere along a continuum ranging from idiosyncratic movements that accompany speech (gesticulation) to highly structured sign language, with language-like gestures, pantomime, and emblems lying between the two extremes. Sign language consists of hand/arm movements that indeed have a relatively fixed syntax and semantics (lexicon): it is language in a strong sense. Emblems take specific places in and in lieu of linguistic expressions. Gesticulations accompany speech but are not subject to syntax and semantics, so that the same movements may appear in many different contexts contributing in very different ways to communication. Finally there are body movements that are completely incidental: grooming gestures and body positions and configurations. None of these forms deserve to be classified as language in the linguistic sense.

It is useful to distinguish different functions of hand/arm and other body movements. Movements may have deictic (pointing) function, stand in an iconic relation with something else, or constitute a rhythmic feature denoted as beat gesture (McNeill 2005). Although a pointing gesture does not have mathematical content, it may nevertheless have an important function in making manifest a gestalt in the environment that is part of the sense-making process (e.g. Radford 2009). Iconic gestures may be sign vehicles for the relation with other concrete portions of the continuum, such as when a lecturer moves a hand in a straight line while talking about a linear function drawn on a chalkboard (Núñez 2009). Or they may stand for an idea, such as that of a mathematical limit (itself modeled on material limits), when the speaker holds one hand still while approaching it with the other (McNeill 1992). Although not immediately apparent, beat gestures, too, may have important functions in mathematical teaching|learning events, for example, supporting grouping and counting (Roth 2011).

In the past, mathematical knowing was considered in terms of mental constructions and conceptual frameworks. More recently, it was recognized that body movements generally and hand/arm-produced gesticulations more specifically manifest knowing. Some research is based on the conviction that there are two underlying cognitive systems, whereas others consider there to be one cognitive system that manifests itself in two different, sometimes contradictory ways (see the review by Roth 2001). Even more recently, embodiment and enactivist perspectives have attempted to emphasize the role of the body in human communication and knowing (e.g. Núñez 2009; Proulx 2013). Both approaches, however, have been subject to critique because of the underlying Cartesianism (Sheets-Johnstone 2009). This Cartesianism is overcome by a Marxian-Spinozist approach in which body and mind (thought) are two manifestations of the same underlying substance (Roth 2017). Consider the example of a circle. By inscribing a circle with a pen on a piece of paper, the body is in a state identical with the circle outside of the body (Spinoza 2002). Indeed, the body comes "into a state of real action in the form of a circle" (Il'enkov 1977, p. 69), and the associated awareness (consciousness) is the idea of and fully adequate to the circle. Even seeing a circle as such is fully adequate, because the eyes have to move along (but saccading to and away from) the circular line to produce the visual experience of a circle (Yarbus 1967). Drawing a circle and knowing a circle have become indistinguishable.

1.7 Conclusion

This introductory chapter gives the reader a preliminary overview of the diversity of theoretical formulations of semiotics as a field of scholarship, and of the power of semiotics as a research lens in investigating the complexities of learning and teaching mathematics. At present, semiotic theories have proved valuable mainly in fine-grained qualitative research studies involving students' learning of mathematics, the relationships involved in activities towards this end, and the role of teachers and teaching in this regard, at various levels and in diverse social contexts. The chapters in this book exemplify the efficacy of semiotic theories as lenses in such research. However, as attested by Morgan's chapter, there is also the potential for semiotics research in the wider fields of institutional contexts and policy research. Because of its relevance in the human endeavor of creating and learning mathematics, and all that entails, semiotics will continue to have significance in mathematics education and its research.

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