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Discussion and Conclusions

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Abstract After an introductory section that addresses the nature of semiotics, the editors discuss themes that highlight issues that have arisen from and that illustrate what has been accomplished in the varied chapters of this monograph. The final section provides some suggestions, based on these issues, for further research on the various threads that pertain to the potential significance of semiotics in mathematics education. The editors believe that there is room for both theoretical development and further empirical studies designed in resonance with these theories, in order to address the full potential of semiotics in areas of research that have not yet received widespread attention.

19.1 The Nature of Semiotics Revisited

Dictionaries—because they provide definitions according to common usage—do not always address the latest theoretical developments in thinking about terms, such as those relevant to semiotics, that are the concern of researchers in fields that address human knowledge and its implications for education. This is so particularly in mathematics education, which is the field of this monograph. Nevertheless, some aspects of dictionary definitions address the core of certain words. For example, here are some definitions that are pertinent:

sign (noun): any indicative or significant object or event; a symbol; token (Funk and Wagnall).
represent (transitive verb): to bring before the mind (Webster); to serve as the symbol, expression, or designation of; to set forth a likeness or image of (Funk and Wagnall).
representation (noun): the act of representing, or the state of being represented; the stage or process of mental conservation that consists in the presenting to itself by the mind of objects previously known (Funk and Wagnall).
mediate (transitive verb): to be the means of (Webster); to serve as the medium for effecting (a result) or conveying (an object, information, etc.); to occur or be in an intermediate relation or position (Funk and Wagnall).

Semiotics is the science of signs; it is clear that the nature of a *sign* is that it *stands for* some entity, and that this representation is the essence of the term.

Now, based on the writings of Peirce (1931–1958), the sign frequently is described as the triadic relationship among sign vehicle, object, and interpretant (e.g., Presmeg 2002; Sáenz-Ludlow and Kadunz 2016a; Sáenz-Ludlow and Presmeg 2006). However, there are also powerful voices that express different opinions. Some philosophers suggest that Peirce never thought about the sign in the triangle that Ogden and Richards later used, charging it with introducing a dualism of mind and world (Dewey and Bentley 1949/1999). Some semioticians, too, consider the triangular diagram over-simplified, though giving it some credit when theorizing the production of signs (Eco 1976). The triad could imply an interpreted *iconic* likeness of sign vehicle to its object, in which case the relationship could involve an image, a diagram, or a metaphor (Kinach, Chap. 13). But the triad could also imply something of the order of indexicality. As Menz and Sinclair point out (Chap. 17), during *creative* mathematizing with diagrams the signifying relationship appears to be indexical, something of a pointing or indicative nature. Although philosophers hoped that objective expressions—all those built on "the principles, theorems, proofs, and theories of the 'abstract sciences'" (Husserl 1913b, p. 81)—and subjective–indexical expressions could be distinguished, some social scientists later noted the indexical properties of *all* forms of signs (Garfinkel 1967). Pragmatic philosophers thus encourage us to investigate signs (language) in actual use rather than trying to figure out some unchanging structural relation, such as that between sign (signifier) and meaning (Dewey and Bentley 1999/1949; Wittgenstein 1997/1953).

Along this pragmatic line of thought, it is worth recalling that Peirce (1992, 1998) was one of the first semioticians to insist on the advantage of thinking of semiotics as a *process*. Although his work starts with the triadic definition of a sign, the force of his approach is the continuous relationship between a sign and a new sign, a process that goes from signs to signs. What this view means is that a sign is not a static entity; it is rather an entity that is part of a chain. If the defining feature of a *sign* is that it *stands for* some entity, and *representation* is the essence of the term, we need to understand this representation not in terms of a mapping but in a fluid and dynamic manner. The focus shifts from object-representation to representing, something that using a more modern terminology we may call *signifying*. In this view signs are dynamic entities arising according to specific forms of signifying.

As some of the dictionary definitions hint, the something or entity indicated by a sign vehicle may have a variety of characteristics, including those of a purely ideal nature. The nature of the object to which a sign refers often changes from one semiotic approach to another. For example, in semiology, the signifier–signified relationship such as between the word "tree" and the image of a tree is psychological (de Saussure 1967). In other approaches, this entity, which is not *necessarily* a physical object, is assumed to have ontological presence irrespective of human observation. Other semioticians and philosopher-anthropologists have defined the sign as a relation between two parts of the *material* continuum, each being shaped in its own way (Eco 1984; Latour 1993). All in all, the important aspect for understanding human behavior through semiotic lenses is the manner in which the different parts come to be made to correspond and thus how such things as diverse as a mathematical equation or function come to be related to the functioning of a winch, pulley, or table of paired data. And as we have seen, the question of how signs signify as well as the question of the nature of the object the sign stands for are two questions that receive different answers in the varied range of semiotic approaches.

Several of the chapters in this book grapple with the ontological question of how and what signs signify. In line with other authors, Morgan (Chap. 8) and Otte (Chap. 9) eschew the Platonic approach to mathematical objects. In his chapter, Otte provides useful instances of the *complementarity* of sense and reference, and of structure and function. He points out that man is a symbolic being, and that every thought is a sign. Similarly, Kadunz (Chap. 7) points out that the translational approach to semiotics that he espouses does not need the assumption of "an objectively existing mathematical instance." In Chap. 12, Salazar reminds the reader that semiosis—an action that Peirce understands as the irreducible relation of three subjects, such as sign vehicle, object, and interpretant (Eco 1976)—and noesis—intentionally oriented thinking (Husserl 1913a)—are inseparable. The representational correlate of semiosis is *production*, the purpose of noesis (e.g. judging) is the conceptual *apprehension* of an object and its correlate (outcome) is the noema (what is judged).

It is a welcome development that non-representational theories that take into account the sociocultural nature of human signification (see especially the chapters in Parts I and IV of the book), or the richness of linguistics in this regard (Part II), are being promulgated in many of the chapters. They shift the attention from a static

view of representation to a dynamic view of sign production and meaning-making. Based on work in mathematics education and the social studies of sciences, some educational scholars have suggested conceptualizing what tend to be called mathematical representations in terms of the concept of *inscriptions*, which orients investigators to the use of diagrams, graphs, equations, and other forms of mathematical expression (Roth and McGinn 1998; Presmeg 2006a). The shift goes hence from the diagram per se and what it may signify to the *use* of the diagram and the broad spectrum of signifying possibilities that arise in using it in activity.

A contentious point is the role in semiotic processes of artifacts and cultural devices, which traditionally have been conceptualized as *mediators* of teachers' and students' activity (see, e.g., Bartolini-Bussi and Mariotti 2008). Roth and Jornet (2017) contend that there is no necessity to invoke the concept of mediation, particularly in cultural historical approaches to cognition and learning. Resorting to a materialist viewpoint, Radford (2015) has suggested that signs and artifacts be understood as *part* of the teachers' and students' activity and as *the material fabric of thinking*. In this view, signs and artifacts do not mediate activity: they are key components of it in that they contribute to configure, organize and re-organize classroom activity and teachers' and students' thinking.

The various chapters in this book suggest that, in the complexity of human teaching and learning, there are a variety of questions that arise, and thus it is appropriate to endorse theories that are appropriate to the questions and issues: no single theory can take into account the need for different grain sizes in data analysis (see Chap. 16, in which Vygotskian and Peircean theories are applied in analysis of the same data set), or the affordances offered by different conceptual frameworks that are suitable for the multitude of empirical research questions that concern mathematics education (e.g., Chap. 10, Semiotic Conceptual Analysis; or Chap. 12, Dynamic Figural Register).

In the following section, we select some of the specific threads that arise in the chapters of this monograph, and discuss these with a view to highlighting what has been accomplished to date. The final section takes the discussion further, pointing out possible future directions for theory and empirical research, and identifying lacunae in current research foci that might profitably be addressed using semiotics as a lens.

19.2 What Has Been Accomplished?

The following subsections address specific threads from the chapters.

19.2.1 Theoretical Developments

In recent years, semiotics has undergone a series of developments that resonate with developments in other areas of inquiry in the social sciences. Such developments

have led to revisiting the concept of knowledge and knowing and the role that signs play therein. It is in this context that there has been a shift of paradigms. In the first paradigm individuals were conceived of as producing ideas in their minds as they acted in the world. Signs in this paradigm were the conduits in which thought was cast. Actions, words and semiotic activity in general were explained in terms of ideas that justified the actions, the words, and the symbols to which individuals resorted in their endeavors.

In the second paradigm, ideas are conceived of as not necessarily prior to action and semiotic activity. Actions and semiotic activity are fused in the processes out of which individuals achieve deeper forms of reflective awareness, and their ideas become clearer through more and more sophisticated expression. In this paradigm, signs, tools, and thought are entangled. As a result, the semiotic lenses to understand and provide accounts of teaching and learning include signs and tools. The tool is not independent of cognition and learning, just as cognition and learning are not independent of the tool. Both are integral and irreducible parts of the same category. Any behavior we may analyze, therefore, has aspects of both. As the research on aptitude has shown, understanding the role of personal characteristics in the behavior requires knowing environmental characteristics, and understanding the role of the environmental characteristics in performance requires knowing the relevant personal characteristics (Snow 1992). In this paradigm, in considering the role of technology, we may be wise not to give special status to computing or other more recent forms (e.g. virtual reality) to be theorized differently from other things that are part of, determining and being determined by, current activity. Just as GeoGebra (Salazar, Chap. 12) or Virtual Math Teams (Swidan and Prusak, Chap. 18) do, pencils, diagrams (Menz and Sinclair, Chap. 17), rulers, slide rules, Dienes blocks, and natural or formal languages (e.g. C++) may (but do not have to) have a function in activity and change the task from activities in which they are not present. In this context, human behavior appears as the result of the irreducible characteristics of the person–environment unity/identity. These remarks are not intended to diminish the investigation of tools and technology. On the contrary, we believe that it is useful to investigate cognitive and semiotic systems in which different types of technology are an integral part. For one thing, this investigation provides our community with materials for theorizing these systems. But also it can help us understand the deep relationships between tools, signs, and thought.

In the second paradigm, the question of the social is reformulated in new terms. In the first paradigm—a paradigm that was informed to a great extent by idealist and subjectivist understandings of knowledge and knowing—the social is not foregrounded; a discrete set of individuals coordinate actions in order to accomplish something. In the second paradigm, an individual's action is *transindividual* as it occurs with others (present or virtually present through language or social context). Knowing appears here as something that is social in nature through and through. In this line of thought, cultural-historical perspectives have emphasized the relation between teacher and student as something deeply social and as an extremely important root of cognition. For cultural-historical theorists, the relation between the teacher and the student is cognitively important because it *is* the higher

psychological function that can later be found in the behavior of the student (Vygotsky 1989; Vygotskij 2001). The relation is important here because, as Vygotsky describes, what has become one in the behavior of the teacher now is unfolded across the relation and the two persons involved. It is the "experimental unfolding of a higher process (voluntary attention) into a *small drama*" (p. 58). Thus, when Stott (Chap. 5) writes about space of joint attention, then this "space" includes the relation between student and teacher. This relation is not something abstract. Indeed, the word or any other form of sign involved in the teacher-student interaction *is* an essential part of the relation. The same happens in the mathematical game that kindergarten students play (Radford, Chap. 2). The space of joint action is a space where an understanding of the mathematical rule is disclosed through verbal and embodied forms of social relations.

The social relations that are encompassed in classroom interaction have been conceptualized in the theory of objectification as one of the two main lines or axes around which classroom activity is organized. Social relations are defined as part of the forms of *human cooperation* that underlie all activity. In the aforementioned first paradigm (the idealist-subjectivist one), forms of human cooperation often appear through utilitarian lenses: individuals exchange ideas in accordance with a logic of self-interest. "Classroom communication is good if the student learns more than he or she would if working alone. This is still the 'me-perspective': I am willing to transact with you if, at the end, my wealth (here knowledge) increases" (Radford 2012, p. 109). The second paradigm makes room to envision social relations and, more generally, classroom forms of human cooperation in a different way. The theory of objectification makes here a contribution in terms of delineating a communitarian ethic that, through pedagogical action, emphasizes a sense of the social as embodied in solidarity, answerability, and responsibility, moving away from individualist forms of human cooperation fostered by neoliberal conceptions of the school (Radford 2014). The question, however, is not merely to come and be able to participate in certain forms of non-alienating activity. The question is that the activity that individuals produce is the same one that produces them. The result of this dialectic movement, which appears in particular in Chap. 2, is a function of the activity's forms of human cooperation to which the pedagogical project resorts (either implicitly or explicitly).

The other line or axis around which classroom activity is organized in the theory of objectification is the one of the *modes of knowledge production*. They include the technological or material dimension (tools, signs, diagrams, etc.) and the various *savoir-faire* through which individuals produce their means of subsistence and fulfill their needs. Strictly speaking, the forms of human cooperation and the modes of knowledge production do not emerge in situ, that is in the classroom, as something out of the blue. They have been produced and refined in the course of historical and cultural developments. It is these historical and cultural dimensions that the concept of *semiotic systems of cultural signification* try to capture. Human activity unfolds, indeed, (often implicitly) organized by a fluid and ever-changing super-symbolic structure from which individuals draw ideas of right and wrong both in their dealings with what is to be known and in their dealings with others.

Since both dealings involve language, symbols, and material culture, the result is that signs (and semiotics) are not the repository and conveyers of ideas (as idealism suggested), nor are they the mere channels out of which ideas and knowledge are formed. Inescapably, signs and semiotics appear as imbued in an ethical and political view. There is always an ethical and a political stance behind knowing and becoming. In this line of thought, the question is not about representing knowledge, but how we come to signify through signs. That is, how we signify things in the world and how we come to signify ourselves and are signified by others through the activities in which we engage.

19.2.2 Methodological Concerns

The question of *grain size* needed in different research designs is one that is salient in some of the chapters but latently present in all. Krause and Salle (Chap. 16), in their investigation of gestures and their role in the formation of *Grundvorstellungen* (GV, "mental models that carry the meaning of mathematical concepts or procedures"), use lenses of both Vygotskian and Peircean semiotics in their analysis. Vygotskian theory proved to be more suitable for the grain size of their research data: in this theory, gestures bridged between *primary* and *secondary* GVs, whereas in Peircean theory it appeared that signs led to the shaping of the immediate mathematical task, but the given data were insufficient to answer the question of *how* gestures led to the solving of the problem. According to the authors, a broader view encompassing scenes from an earlier point in the research might have been helpful in the Peircean approach.

The issue of grain size appears also in Morgan's use of systemic functional linguistics (SFL, Chap. 8) to explore social semiotic issues. However, the theoretical stance of SFL and associated methodology, adapted to different grain sizes, proved to be suitable both for the institutional assumptions involved in the fine-grained analysis of how an examination question is written, and in Morgan's second, society-level, focus involving a thematic analysis of official documents.

Another issue addressed in some chapters is methodology used in research that aims at the *constitution of typologies*. In the chapters of both Salazar (Chap. 12, constitution of the dynamic figural register, using Duval's theory) and Mathews et al. (Chap. 14, signification pathways, based on Peirce's semiotics), a fine-grained analysis of theoretical positions (Chap. 12) or qualitative empirical data in a naturalistic study (Chap. 14) resulted in typologies that move the field forward. However, in the case of the study by Mathews et al., the semiotic chaining framework that was a starting point for the methodology required modification and greater specification before it became useful for the research purpose. In the study by Kinach (Chap. 13) involving primary school teacher preparation, the semiotic chaining model broke down, in the sense that it could be applied only to the first three of five phases in an interactive computer game developed for the research purpose.

It is clear in all the chapters of this monograph that research purposes determine the nature of the methodologies used, based on appropriate theoretical considerations. However, it is noteworthy that with the exception of Chap. 17 (Menz and Sinclair), none of the chapters use quantitative analysis of data sets, or psychometric designs that require the use of statistics. It remains an open question whether research designs that require such methods are useful in research on semiotics in mathematics education. It is possible that qualitative research designs are more suited to the issues pertinent to semiosis in mathematics education, in which *paradigm cases* (of individuals or groups) are likely to shed light on questions of teaching and learning, even when sociocultural lenses are applied. Even Menz and Sinclair's chapter is based largely on the paradigm case of three research mathematicians. It is clear that issues of "joint attention" (Shvarts, Chap. 5), "togethering" and "moments of poēsis" (Stott, Chap. 5) and many of the issues investigated and reported in other chapters, are more amenable to qualitative rather than quantitative research methodologies, notwithstanding the current trend for mixed methods to be used in mathematics education research (Bikner-Ahsbahs et al. 2015).

The traditional question of grain size obtains—or should obtain—a new sense within cultural-historical theories (e.g., Chaps. 2–6). This is so because the minimum unit of analysis, as made explicit in cultural-historical activity theory, is *societal* activity. This unit is based on the recognition that what characterizes humans as distinct from other animals are the specific relations that constitute society. Cultural-historical psychologists thus consider personality to be the ensemble of societal relations (Leont'ev 1978; Vygotsky 1989). As a result, every analysis of a relation between two people also makes for an investigation of society; every analysis of societal relations also constitutes an investigation in available psychological functions and personality traits (Roth 2012a). The issue is particularly important in the case of the birth of signs (Chap. 3, Roth). Whereas signs may be created by individuals, they *inherently* have *social* character. But they may not be universal but be characteristic only of a group; but universality is a suprasensible (ideal) potential of anything material exchanged (Marx and Engels 1962, 1978), a point taken up in Chap. 3 (Roth). Thus, if investigators adhere to the cultural-historical approach, they (have to) accept the inherently social, thus shared and societal nature of the investigated phenomena—though, admittedly, investi- gators may fail to point out or exhibit this feature. For example, Salinas-Hernández and Miranda (Chap. 4) conduct their investigation in a 12th-grade physics course. But what the students generally say and do, or any particular student specifically, is not an individual construction. This is so because their talk is designed *for* others, using a language that they have learned *from* others, and which, in their talk, returns *to* the other. Whatever students say, thus, is a reflection of the culture, a mirror of society, a microcosm thereof (e.g., Zeyer and Roth 2009).

A very different approach to method arises when research focuses on the ways in which phenomena come about—i.e. the work in which the birth of a sign is founded (see Roth, Chap. 3)—rather than the outcomes of this work. The distinction maps on to that made above between noesis, an event, and noema, its outcome. Chapter 3

exhibits the work children do to assist each other in making some aspect visible (perceivable), whether this is an aspect of the hidden object or the different shapes they make from plasticine. It has been pointed out that research using formal methods (such as psychometric analysis)—specified in the methods sections of research reports—focuses on the identification of noema, social or material things, whereas ethnomethodology focuses on the work that real people do to constitute and make visible the situationally relevant social and material things (Garfinkel 1996). Formal methodologies include quantitative (mathematical, experimental) studies as well as some qualitative (interpretive) studies (Bikner-Ahsbahs et al. 2015), and these differ incommensurably from ethnomethodology and other naturalistic methodologies that investigate the methods the people themselves employ to make and make visible social and material *things* (i.e. the noema). In this light, naturalistic studies appear to be particularly suited to empirical research on semiotic issues—and indeed, many of the empirical studies described in chapters in this book fall into this genre.

19.2.3 The Role of the Teacher

The role of the teacher is one of the elements that has been recently under intense discussion in mathematics education research. There seems to be a need to reconceptualize this role in teaching and learning. In the chapters included in this book, the role of the teacher takes various forms.

In Chap. 2 (Radford), it is the sensitive intervention of the teacher that breaks the impasse between Carl and Jack in the dice game and allows them to move forward. Rather than facilitator, the teacher actively participates in the creation of a social space where the rule of the game regains focus. Carl's and Jack's interaction becomes again *reorganized* by the rule. This reorganization makes room for the unfolding of a cultural form of intersubjectivity in which individuals take positions and answer, in a responsible manner, to each other. The teacher plays a preponderant role here in order to make sure that the students' answers are coherent with the operating semiotic systems of cultural signification (Radford 2008; see also Chap. 2) that convey ideas (mostly implicitly) about how individuals should behave. And as we saw, the joint creation of the social space by the teacher and the students, where the rule can be again attended to, occurs in difficult moments that are full of tensions and emotions—tensions and emotions of which the teacher is, like the students, a part. In Chap. 3 (Roth), the teacher points out to Melissa, Jane, and Sylvia that there is *one* object in the shoebox, and therefore the group should decide on one plasticine model for this object: Melissa's cube model *dies* in favor of the flatter rectangular model of Jane and Sylvia, in an example of the gap-crossing needed to reconcile two structured portions of the continuum.

In Chap. 4 (Salinas and Miranda), there is a hint from the researcher, but it is the interactive technology that stands in lieu of the teacher in the students' processes of objectification dealing with the meanings of Cartesian graphs. The interaction with

the technology enables the students to check their predictions using the software. The students' description of the motion of a tennis ball down an inclined plane was *preceded* by their understanding of the mathematical relations of physical variables of space and time in the graphs; this understanding evolved as the students formulated new hypotheses and interpreted the software outcomes. The teacher appears here as a kind of virtual entity whose contours change as the students ask questions and the software provides answers.

In the dual eye-tracking synchronized research of Shvarts (Chap. 5), the role of the parent is crucial in establishing the joint attention with the child that enables the child to apprehend the *cultural way* of using coordinates on a Cartesian plane. Indeed, phylogenetically speaking, the higher psychological function of attention is the result of a lengthy culturally and historically evolved way of attending things, which is materialized here through the concrete relations with other people, generally a parent. Ontogenetically speaking, the birth of the capacity to orient to the same objects and to signal whether joint attention has been achieved lies early in the life of a child (Vygotsky 1997), exemplified in a recent analysis of reading multimodal texts involving a mother and her one-year-old boy (Roth 2016). Shvarts gives examples both of the ambiguity of visual and gestural presentations (and verbal terms) that cause misunderstanding, and in a nice contrast, the seamless joint attention that results from the adult's sensitivity to the child's anticipations, as the parent *followed* the child's attention, in an outcome that was satisfying to both of them. Finally, resonating with Shvarts's study, in Chap. 5 Stott gives examples in an after-school program, of the importance of the role of the facilitator in establishing the *space of joint action*, and *togethering*, that are necessary for cultural meanings. In Stott's first example, this space of joint action with Anathi and Thembela leads to the cultural meanings that are the goal of the activity; in the second example, with Akhona and Kuhle, the facilitator does not succeed in intervening in *attention catching*, and the absence of togethering prevents the moment of poēsis that is apparent in the first example.

All five of these chapters highlight the indispensable role of the teacher or surrogate teacher and the various forms this role takes, depending on the activity, the object of the activity and the tools and artifacts involved in the classroom activity.

19.2.4 The Affordances of Interactive Technology

In several of the chapters, interactive computer software plays a pivotal role in accomplishing the goals of the research. For instance, GeoGebra is an essential tool in both Chap. 12 (Salazar) and Chap. 18 (Swidan and Prusak). It has been shown empirically for some time now that the affordances of such software (compared with paper-and-pencil methods) change the processes of learning mathematics quite radically (e.g., Yu 2004, using a conceptual framework based on semiotics).

In the case of Salazar's research, which employs the theoretical formulations of Duval concerning formation, treatment and conversion of semiotic registers, the use of a *dynamic representation environment* such as that provided by GeoGebra is essential. Duval did not take into account the differences that the use of computer software introduced in examining formation, treatment, and conversion. Thus Salazar's examination of the *dynamic figural register* in such an environment, highlighting the reconfiguration of quadrilaterals to study their area, moves this field forward both theoretically and empirically. Indeed, from a cultural-historical perspective, Salazar and Duval are investigating different phenomena, different forms of activity, for as soon as a tool is introduced into a person–environment unit, then cognition, consciousness, sense, and the related affects change. As research on the relation between cognition, technology, and work shows, even a minor variation in a tool—such as a 5-centimeter displacement of a dial in an airline cockpit—changes the cognitive system as a whole (e.g., Soo et al. 2016).

Swidan and Prusak's research (Chap. 18) is also concerned with the study of quadrilaterals, but from the viewpoint of the inherent inclusion relationships in a hierarchy of quadrilaterals. The new field of Computer-Supported Collaborative Learning (CSCL) is central in their research methods, because the aim of their study was to investigate the role of CSCL in the processes of students becoming *fully* aware (as the final stage of a triad of processes, starting with unawareness, followed by latent awareness) of selected inclusion relationships of quadrilaterals. They concluded that the research design (partially) supported the objectification of these relationships. The authors note that the computer environment was limiting in that it did not provide access to certain kinds of data: the students could not gesture or communicate verbally (or at least the researchers did not have access to these data sources) in this environment, in which communication was in a chat room with written explanations.

In the case of Kinach's research (Chap. 13), an "area game" designed for prospective elementary school teachers also invoked the affordances of technology: the design was intended to provide the prospective teachers with an introduction to semiotic chaining in the Peircean framework, as a basis for their pedagogy, as they worked through five levels of the children's game. In practice, the chaining broke down after level three; but nevertheless the research provided access to Peirce's three principles of diagrammatic reasoning, and highlighted some of Krutetskii's (1976) problem types.

Technology was also an essential component of Shvarts's (Chap. 5) methodology. In this case, two synchronized eye-trackers and an external video camera enabled the researcher to capture the gazes, gestures (with a thin stick pointed at the computer screen), and verbal interactions of mother and child as both looked at the same computer screen and engaged with tasks involving Cartesian coordinates of points. The power of this methodology was illustrated in the micro-mechanisms that were identified in the processes of joint attention, and objectification of the canonical way of working with such coordinates.

19.2.5 Research on Diagramming

Although diagrams enter into the mathematical components of data sources of several chapters in this monograph, there are two chapters (Chap. 11, Sáenz-Ludlow; and Chap. 17, Menz and Sinclair) that engage with processes of diagramming specifically, using different theoretical lenses and investigating different issues. Sáenz-Ludlow's chapter, on *Iconicity and diagrammatic reasoning in meaning making*, is largely theoretical in nature, based on Peirce's (1931–1958) semiotics, and provides two convincing examples of the power of diagrammatic reasoning. Sáenz-Ludlow highlights Peirce's definition of a diagram as an "icon of possible relationships." For Peirce, *all* knowledge is the product of the self-corrective activity of the human mind, and diagrammatic reasoning is an inferential process that addresses the relationships among sign vehicles, their interpretants, and the interpreted objects in question. Sáenz-Ludlow explores the three triads at the root of the ten-fold classification of sign vehicles and their relationships according to Peirce. Mathematical diagrams are considered to be epistemological tools for inferential thinking: dynamic interpretants give way to logical interpretants, which in turn lead to chains of inferences. Cultural-historical scholars point out that the diagrams are not merely tools in but constitutive of thinking in the same way as words and thinking are integral parts of the same communicative activity, thinking and speaking being two dynamic processes that are flexibly related (Vygotsky 1987). Thought completes itself in the word, which, as part of the material reality, determines further thinking. The same is the case for all forms of signs, including the relationship between diagrams and reasoning: reasoning *becomes* what it is in the diagram, and diagrams determine the situationally appropriate reasoning. This general relationship between signs and reasoning, paired with Sáenz-Ludlow's concrete analyses, help us better understand how initial *transformand* diagrams give way to more meaningful and useful *transformate* diagrams in the processes of diagrammatic reasoning. Sáenz-Ludlow's analyses also show specific ways in which visualization is indispensable in the process. In this complex theoretical formulation, diagrams have the potential to have both iconic and indexical properties (and symbolic, metaphoric and metonymic ones too, Presmeg 1992).

In a different theoretical formulation, in their chapter on *Diagramming and gesturing during mathematizing*, Menz and Sinclair challenge the traditional formulation of diagrams as representations of mathematical objects and relations. Peirce (1992, 1998) was one of the first semioticians to conceive of semiotics in a dynamic way. He called attention to the role of diagrams in *processes of inference and abduction*. For Peirce a diagram is made up of signs (indexical and others) that serve to emphasize relations that come to mind in the investigation of a problem. For Peirce a diagram is first of all a kind of iconic sign, not in virtue of a resemblance with something but in virtue of the relationships that it brings to the fore—

e.g., he argued that algebraic formulas are icons. Focusing on the context of mathematical invention, Menz and Sinclair emphasize the indexical nature of diagrams. They base their arguments on the writings of Châtelet, who, as a result of his deep investigation of the thinking of great mathematicians according to the records that they left, described diagramming as a *material* practice of mathematical invention. Metz and Sinclair illustrate this theoretical formulation convincingly in the case of three research mathematicians, Fred, Colin, and Victor, working collaboratively on problems of Topological Graph Theory. About 200 diagrams were identified during nine meetings of the mathematicians. The life-cycles of the diagrams were analyzed according to three phases, namely, *manufacturing*, *communicating*, and a final phase of *dénouement*. From their data, the authors identified five elements in the life-cycles of the mathematicians' diagrams: firstly, the diagram is emerging ("is born," as Roth, Chap. 3, might say); then a mathematician adds to or erases from an existing diagram; next, the diagram is refigured by tracing edges and vertices; the fourth element is a mathematician drawing a known diagram by retrieving it from memory; finally, a mathematician might be directed to draw a diagram by his colleague.[1] Although the indexical nature of these diagrams is clear from this data set, the diagrams might also be considered to have iconic properties. For instance, in the case of the diagram drawn from memory, memory must not be thought of in terms of representations, but in terms of the movements or actions themselves that have become habitual. The movements producing mathematical diagrams and those gesturing over and about the diagrams produced before or as a result are the same—as shown in a semiotic study of lecturing with graphs in university physics courses (Roth 2012b). Iconicity appears here not as a property of the sign itself but rather as a property of its production: "as the projection of an earlier experience onto a new one" (Radford 2008, p. 94)—something that makes this experience at the same time old and new, "similar and different" (p. 94), as in the various hand movements and the ensuing figures inscribed on the blackboard.

More generally, because the emergent and evolving meaning of signs changes continuously as semiotic activity unfolds, the nature of signs moves constantly around from being indexes, to being icons, and to being symbols; in fact, signs can be more than indexes, or icons, or symbols only; signs can be one, two or three things all at the same time. However, it is worth remembering that there are many roles of diagrams in the teaching and learning of mathematics. One open question is the influence of *prototypical images* on students' and mathematicians' thinking in all areas of mathematics (e.g., prototypical images of octahedron and torus in Menz and Sinclair's study, or triangles and pentagons in Sáenz-Ludlow's), both as affordances and as constraints (Presmeg 1992, 2006a).

[1]See also Roth (2015) where the continued changes in thinking and writing of mathematical graphs and equations, associated with continued erasure and rewriting, are theorized in the field as the birth of understanding arising from the excess of graphical movements.

19.2.6 Types of Signification

In view of the rich variety of theories of signification offered in the chapters in this monograph, in this section we honor the view that there are different types of ways that signs may signify. Just as mental imagery may signify in different ways in mathematical activity (Presmeg 1997, 1998, 2006b), the signs identified in the chapters of this book have different contexts, purposes, and roles in mathematics education. As in Shakespeare's words, "A tale told by an idiot, Full of sound and fury, Signifying nothing" (Macbeth, Act 5, scene 5), for *some* students mathematics may have no personal signification, despite its sound and fury. Signification implies *meaning*.

The meaning of activities for the learning of mathematics is a relevant topic in mathematics education at least since Jean Piaget (e.g., Piaget and Inhelder 2013/1958): numerous researchers have presented texts on research questions in this area (Boaler 2002; Wittmann 1995). Semiotics offers several other views on mathematical activities. In the Peircean tradition, besides diagrammatic reasoning (see Sáenz-Ludlow, Chap. 11), which focuses on Peirce's use of signs, the pragmatic maxim introduced by Peirce in (1878) and slightly changed by William James (1907) is a relevant theoretical principle to explain the meaning of concepts in common usage and in mathematics in particular (Sáenz-Ludlow and Kadunz 2016b). Concentrating on the pragmatic maxim also builds a bridge to the ideas of Ludwig Wittgenstein (see Kadunz, Chap. 7), another philosopher widely known but rarely used in mathematics education (Dörfler 2016). On the one hand Wittgenstein cannot be seen as a semiotician. On the other hand we can find numerous hints in his posthumously published book *On Certainty* (1975/1969) very similar to the pragmatic maxim. Boncompagni (2016) stated the pragmatic maxim as follows:

> Roughly, it is the connection between the meaning of a conception and its practical effects, thanks to which we can claim to know something completely only insofar as we know its effects in factual and/or behavioural terms. (p. 140)

Peirce sees *action* not as a concept to be studied in detail, but as a tool to interpret various different situations. For him, action is a part of a life practice, a behavior (habit), which is controlled by action and shows itself at the same time. What is the meaning of this maxim for learning mathematics? If we follow Peirce, it is a tool to achieve clarity about ideas and concepts, including those that determine mathematics. It is remarkable that, in Peirce's view, "One of the goals is to illustrate that certain hypotheses or concepts do not bear any cognitive content at all" (Hookway 2012, in Boncampagni 2016, p. 43). The pragmatic view focuses mainly on the *outcome* of activities. Peirce's definition of meaning is "translation of one sign into another system of signs" (CP4.127). This definition resonates with Eco's (1984) characterization of semiosis as one segmentation of the material continuum in relation to another segmentation.

Although they espouse different traditions, Kadunz (Chap. 7) and Roth (Chap. 3) both recognize the principle of a field transverse to the flow of ideas, as in the

metaphor of a river. Roth, in the tradition of the later Vygotsky, calls it a communicative field. Kadunz points out that the German word for translation, *Übersetzung*, implies a leaping across: a good translator "leaps to the other river bank" (Chap. 7). As in the pragmatic maxim, *use* is another focus in the worldview of Kadunz that has resonance in Roth's chapter, in which sign vehicles are adumbrated in terms of the buying and selling of commodities: the buyer is interested in the *use-value*; the seller in the *exchange value*. A human relation is involved. The transaction is future-oriented (as in the pragmatic maxim): *use* matters. Roth writes of "lines of becoming" (Chap. 3). Two questions are involved in the foregoing, as it relates to mathematics education: (a) What is signified? (b) How is it signified?

Some indication of an answer to these questions is given by Krause and Salle (Chap. 16). In their use of a framework of *Grundvorstellungen*, "mental models that carry the meaning of mathematical concepts and procedures," they invoke theories of both Vygotsky and Peirce. *What* is signified by Victor's coordination of gestures, speech, and inscriptions is indicated in his mental models that proceed from an initial "growth point" to a "catchment" (a confluence of growth points) that leads him to the idea of *function as object*, in the context of linear equations and mobile telephone contracts. Thus function as object is what is signified by this catchment. *How* it is signified, using a Peircean frame, was not able to be identified with the dataset that they presented. It would require data from an earlier point in time.

Thus there are many aspects that concern signification in the teaching and learning of mathematics. The use-value may be identified in mental models. But in the learning processes involved, "individuals come to position themselves in differential, polyglossic, and ideological ways" (Radford, Chap. 2). Otte, too, writes that "man is a symbolic being" (Chap. 9), characterizing semiosis as meaningfulness, not objective reference. What is signified, according to Radford, is uniqueness, being in flux, shaped by a common cultural ground of ideology, leading to a student as a "continuous, moving sign in the making" (Chap. 2). Then the student *is* the sign. Radford's (2008) theory of objectification, involving *semiotic systems of cultural signification* (SSCS), is used in the empirical research reported in several of the chapters of this monograph (Salinas-Hernandez and Miranda, Chap. 4; Shvarts, Chap. 5; and others more implicitly). SSCS and the identity of a student in flux are not incompatible with Sáenz-Ludlow's Peircean analysis (Chap. 11) of dynamic, continuous sign-building in diagrammatic reasoning. The focus is on Peircean chains that "stretch out in both directions" (Otte, Chap. 9) without limit. The Peircean analysis may be better suited to hinting at the details of *how* the signification proceeds, but the SSCS view is better able to capture the mathematical *identity in flux*, of a student learning in a cultural context, with all the emotions and affect that belong to this process. It is not only a question of the grain of the research; Shvarts's study, for example, is extremely fine-grained—and provides a detailed account of *how* a case of objectification of a cultural view of Cartesian coordinates takes place through joint attention (see also Stott, Chap. 6). It is also a question of the *focus*, in view of the conceptual framework of the research, which enables and constrains the research questions. Perhaps Otte's (Chap. 9) principle of

complementarity is a useful one to adopt in the context of theories that inform a particular research question. Salazar (Chap. 12) successfully integrates Peirce's qualisign, sinsign, and logisign with Duval's registers of formation, treatment, and conversion respectively, in her configuration of the *dynamic figural register* in a GeoGebra environment. European researchers have been leaders in the "networking" of theories that together can provide a broader picture of a phenomenon (Bikner-Ahsbahs et al. 2015).

19.3 Suggestions for Future Research on Topics Concerning Semiotics in Mathematics Education

Whether or not a principle of complementarity of theoretical positions is embraced, the foregoing issues, and the rich variety of theoretical positions on semiosis adopted in the chapters of this monograph, suggest further research in a broad array of different areas.

Firstly, is there scope for more *system-wide* research involving semiotics in mathematics education? Morgan's research (Chap. 8) indicates that indeed that is the case. Using social semiotics and systemic functional linguistics, she addresses three questions: (a) What is mathematics? (b) Who or what does mathematics? and (c) What do teachers and students do? Both her examples are fine-grained in their analysis, but they manifest the versatility of ways that system-wide issues may be addresses through these lenses. Examination of the wording of past national mathematics examination questions highlights the subtlety of messages signified, in characterizing what mathematics is and who should do it (the first two research questions). In the second example (question 3), official government documents are analyzed to decode what is considered "good practice" in teaching mathematics—agency, performance, support, etc. are all key words that signify a view of teaching that is rewarded by the system. This kind of study addresses both kinds of research, namely, *descriptive-analytic* (what *is*), and *normative* (what could be different).

With a different focus, there is much scope for further investigation of Radford's question (Chap. 2), What is the nature of the human activity that, at school, produces teachers and students? His research, based on the classroom constitution of mathematical subjectivities, suggests that *what* is learned, and *how* it is learned, are the "threads out of which subjectivities are made," involving *togethering* (Radford and Roth 2011). Specifically, Roth (Chap. 3) indicates three possible areas of investigations of signification in this regard: (a) creation of signs; (b) transparent reading of signs; and (c) interpretation of signs. Roth has addressed all three of these foci in his writings, but Chap. 3 is devoted to the first, the birth of signs. Chapters 4, 5, and 6 are all involved in examining the micro-mechanisms that address the mathematical subjectivities of students in processes of learning, and the field is ripe for more research in this area. There is also scope for further research on attention and awareness in progressive objectification as characterized by Swidan

and Prusak (Chap. 18), who introduced in their research Roth's (2015) three-fold distinction of being unaware, being latently aware, and culminating in full awareness.

There are several theoretical formulations in the chapters of this monograph that could fruitfully be put to the test as lenses in contexts other than those described. One such formulation is that of Menz and Sinclair (Chap. 17), using Châtelet's theory in research on diagrammatic reasoning in the creation of new mathematics by three research mathematicians. Since Châtelet's theory was founded on the written work of famous research mathematicians, it would be interesting to investigate the scope of his theory in the classroom teaching and learning of mathematics at all levels. Are the five steps in the life-cycle of diagrams, which Menz and Sinclair identified, replicable in other contexts? Sáenz-Ludlow's research (Chap. 11) also concerns diagrammatic reasoning, but within a Peircean framework: there is much scope for the further investigation of the usefulness of *transformate* diagrams in teaching and learning geometry. Further theoretical formulations that might be tested as lenses in different contexts are as follows: Peirce's three-step plan for chaining diagrams (Kinach, Chap. 13), the Dynamic Figural Register (Salazar, Chap. 12), the typology of Signification Pathways, co-produced by teachers and learners (Mathews, Venkat, and Askew, Chap. 14), and Vygotsky's view of gradual perceptual change in the Object-Meaning Ratio (Abtahi, Chap. 15).

Finally, because linguistics addresses a large area of semiotic expression, there is need for further research that involves elements of translation in mathematics education (Kadunz, Chap. 7). Kadunz asks what conditions must be met in order that key terms can be translated? This is still a focal question that might be addressed using Semiotic-Conceptual Analysis (Priss, Chap. 10), the framework that Priss used in investigating why formal language is difficult for students to learn. The conceptual lens of SCA introduces the fine grain of four structural aspects in questions of language: incompleteness, polysemy, synonymy, and iconicity.

It is clear from the rich diversity of theoretical formulations, and the numerous issues still pertinent to semiosis in mathematics education, that signification will remain an important aspect of research in this field.

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