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8. LANGUAGE IN MATHEMATICS EDUCATION RESEARCH

INTRODUCTION

In the past few decades, language has become an active focus of investigation in educational research, including research in mathematics education. Such a focus is a symptom of a relatively recent paradigmatic shift whose chief characteristics are a new understanding of the student and an increasing awareness of the complexities of learning contexts, such as, notably, the complexities arising from cultural and linguistic diversity. This paradigmatic shift appeared as the field attempted to move away from the two main models that emerged and evolved during the educational progressive reform of the early 20th century (see also Lerman, 2006).

The first of the two main models was the "transmissive model." With its intellectual origins in behaviourism, this model was promoted by bureaucratic pedagogues who focused on implementing mass education to efficiently address the demands of industrial and business production (Tyack, 1974). Two of the contemporary heritages of this model are a methodical and detailed curriculum and the obsession with systematic "objective" assessments. The second main model was the "child-centered" educational model. Intellectually rooted in a romantic pedagogy, this model focused on the child's interests and intellectual potential. "Progressivism," as this model came to be known, promoted the idea that "knowledge is … [a] personal acquisition, obtained by learning from experience" (Darling & Nordenbo, 2002, p. 298) and meant "promoting discovery and self-directed learning by the student through active engagement" (Labaree, 2005, p. 277).

Although language is not absent from these models, it does not appear as a central research problem. And when it does appear, it is generally related to problems surrounding the investigation of students' conceptualizations. Language is considered as a kind of *window* to see indirectly what is happening in the student's mind as, for instance, in Piaget's conservation tasks. When students' conceptualizations are perceived to be incorrect, language is often then seen as an obstacle or barrier to the effective communication of the desired knowledge or structures. Language, however, is clearly more than a window or an obstacle; language, talk, text and the production and interpretation of symbols are integral to the creation of learning, teaching and assessment, particularly in mathematics. In Piaget's conservation tasks, for example, language is not simply a neutral conduit for conveying thoughts between experimenter and subject; the tasks are constituted through linguistic processes. For language to move to the forefront as an educational research problem on its own, it was necessary to move beyond the conception of humans as Cartesian problem solvers promoted by progressive models. This move, from which emerges the idea of *homo communicans* and that opens up new spaces in which to conceive of the student in new terms, was not merely accidental. It responded to fundamental changes of a social, cultural, historical, and economic nature. As Paul Kelly puts it in his book *Multiculturalism Reconsidered*,

> With the retreat of European empires […] and, much more significantly, with the collapse of the old European empires following the Second World War, there has been a transformation of that earlier colonialist legacy […]. European states—especially the old colonial powers such as Britain, France, Holland, Belgium and, to a lesser extent, Spain and Portugal—became multicultural states as a consequence of colonial retreat […] In the British case, the retreat from empire began a process by which immigration from former colonies transformed the country into a multiethnic and multiracial society. (Kelly, 2002, p. 2)

The result is that today "All modern states face the *problems* of multiculturalism even if they are far from endorsing multiculturalism as a policy agenda or official ideology" (Kelly, 2002, p. 1; emphasis in the original). Although multiculturalism was a predominant feature of life in Ancient Greece and Rome, contemporary multiculturalism with its central interest in language is truly new. As Gress (1999) points out, "The [ancient] Greeks never learned foreign languages" (p. 565). He goes on to say that

> For the Greeks of the archaic and classical eras—from Homer to Alexander—encounters with the other were encounters with the marvelous or the dangerous. They took place in the framework of an evolving anthropology of curiosity and difference, accompanied and complemented from Herodotus on by an overarching dichotomy of Greek versus barbarian. (Gress, pp. 562–563)

To understand contemporary multiculturalism's interest in language we should add that the paradigmatic shift alluded to above has also been entangled with changes in new forms of production and colonization brought forward by global capitalism where "money, technology, people and goods […] move with increasing ease across national boundaries" (Hardt & Negri, 2000, p. xii). These new global forms of production have been accompanied by a variety of unprecedented kinds of virtual interaction and communication. Within the new global context of production and modes of human interaction, individuals from other cultural formations have ended up acquiring a central place—an ontological one, in fact—in the manner in which individuals have come to understand themselves. However, "the appearance of the *Other*," as we may term it, has not been a neutral experience. It has brought with it new questions about identity, power, ethnicity, multiculturalism, multilingualism, etc.

Perhaps the contemporary global context of production in which we live is leading us to experience a somewhat similar historical phenomenon as the one the 16th century Spaniards experienced when they confronted the multitude of communities of what is now called the American continent. That is, when they discovered a substantially different *other* and, along with it, they also discovered that gods, customs, morals, language, and worldviews may have a different order than the one they grew up with and knew. In his book *The Conquest of America*, the Bakhtinian specialist Todorov (1984) points out the strong need that Christopher Columbus felt to rename all things. For Columbus, language was an instrument through which things were possessed and individuals subjugated. Talking about the first island he found in his travels, Columbus said, "I gave [to the first island] the name of *San Salvador*, in homage to His Heavenly Majesty who has wondrously given us all this. The Indians call this island Guanahani" (Todorov, 1984, p. 27). And he went on to tell the King the names he had given to the other islands. Todorov comments:

> Hence Columbus knows perfectly well that these islands already have names, natural ones in a sense (but in another acceptation of the term); others' words interest him very little, however, and he seeks to rename places in terms of the rank they occupy in his discovery, to give them the *right* names. (Todorov, 1984, p. 27; emphasis in the original)

Naming things—which Columbus did through notarial acts written ceremoniously in front of the perplexed natives—provided him with a means to possess things and people. The difference between us and Columbus and the conquerors is that we are asking questions about power and culture within an array of new sensibilities. How, in our contemporary multicultural settings—in a culturally diverse classroom, for instance—could language not be an instrument of subjugation and possession? We will come back to this question later. For the time being, let us summarize the previous comments by noting that the invention of *homo communicans*—that is, the constitutive insight that what humans are is deeply entangled with, and rooted in, the individuals' historical and cultural communicative relationships with others—has not been embedded in epistemic questions only (e.g., how we name things, how we know things) but also in questions of alterity, power, identity, culture, and politics.

In this chapter, we review PME research on language from the past 10 years and offer a critical appraisal of this work. To begin, in the next section we set out an overview of the relevant research, looking at both the major themes that have appeared, as well as the different theoretical approaches to language that have been deployed. In the remaining sections, we discuss three themes in more depth: the role of language in mathematics conceptualization; cultural dimensions, such as the role of language in mediating between the individual and society, and, in particular, questions of power and authority in mathematics education; and language diversity in learning and teaching mathematics.

In the first PME Handbook, there was no chapter explicitly devoted to language as a focus of research. Questions of language are most salient in Lerman's (2006) chapter on socio-cultural research and in Gates's (2006) chapter on equity and social justice. Lerman's (2006) categories of socio-cultural research, for example, include:

- Cultural psychology, including work based on Vygotsky, activity theory, situated cognition, communities of practice, social interactions, semiotic mediation.
- Ethnomathematics.
- Sociology, sociology of education, poststructuralism, hermeneutics, critical theory.
- Discourse, to include psychoanalytic perspectives, social linguistics, semiotics. (p. 351)

It is apparent even from these brief characterizations that language is pretty central both explicitly (e.g., social interactions, discourse, semiotics) and more implicitly (e.g., as a key aspect of both Vygotskian and poststructuralist theory). Meanwhile, Gates (2006) includes a brief discussion of "Language, discourse and critical consciousness" as part of a section on the third decade of PME. In this section, he highlights contributions on language, the politics of discourse, and critical studies, with most emphasis on the issue of multilingual classrooms.

It seems, then, that in the first 30 years of PME, questions of language can best be described as an emerging theme: both Lerman and Gates highlight their absence in the early days of PME and their increasing presence in the third decade. In the past 10 years of PME, however, there are more than 150 research reports, contributions to research forums and plenary lectures devoted to language-related topics. Given the linguistic turn described in our introduction, it is perhaps no surprise that PME research has attended to the kinds of questions we have mentioned.

For this chapter, then, we have compiled a corpus of contributions to PME conferences from 2005 to 2014, consisting of research reports, plenary presentations and research forums. Research reports are, of course, the primary form of contribution to PME and most reflect the work of the members. Plenary presentations represent substantial contributions that discuss specific topics in more depth. Research forums offer multiple perspectives on a given topic and, although individual contributions can be somewhat brief, the overall contribution of a research forum can be substantial. We will refer to all three forms of contribution as 'papers'. We did not include short oral presentations, posters, discussion groups or working groups, since these activities are only represented by brief, single-page reports that lack important detail.

The corpus consists of papers that explicitly address language issues, or for which language is a relevant feature. Papers that explicitly address language, for example, include contributions on multilingual mathematics classrooms, the role of mathematics classroom interaction in learning or teaching mathematics, or the nature of mathematical discourse. In some papers, some aspect of language is identified as a factor within a broader research focus, such as the role of classroom discussion within a paper focused on teaching for equity in mathematics outcomes. In total, the corpus consists of 153 papers, for which we conducted two classificatory analyses. The first analysis looked at the substantive focus of each paper. The second analysis looked at the theoretical framework used in each paper. In the rest of this section, we summarize the outcomes of these analyses, in order to situate the thematic discussions which make up the rest of the chapter.

For the first analysis, a general emergent classification, conducted with the help of NVivo software, examined research topics within the corpus. This analysis highlighted four main conceptual categories . Of course, conceptual categories may overlap. Table 1 provides the main conceptual categories along with their corresponding common core:

Table 1. Main categories and their core

Conceptual category	*Core*
Cultural dimensions	Focus on the relationship between individual and society; language, mathematics, and culture; cultural discursive routines; and multilingualism.
Language and conceptualization	focus on language and conceptualization; language in collective participation and in embodiment; representations and symbol use, and Vygotskian semiotics.
Mathematics as discourse and mathematics discourse	Focus on mathematics discourse or mathematics as discourse; the investigation of students' discourse and teachers' discourse.
Theoretical approaches to language	Focus on theoretical approaches to language; problems of hermeneutics, the theoretical relationship between language and thinking, and the role of language in the construction of knowledge.

Figure 1 shows the distribution of papers according to these categories. In the NVivo software terminology, a "source" corresponds to a document made up of excerpts coming from papers of a same PME conference. An excerpt (or "unit of sense," comprising usually one or more paragraphs from a paper conveying the general focus and meaning(s) of the paper) is called a "reference." The ongoing analysis of references gives rise to conceptual categories (called "nodes" and "sub-nodes") that NVivo displays in the form of a "tree" (see Figure 1). Thus, within the general category (or "node") "Language and conceptualization" is a sub-node called

"Representation and symbol use." From Figure 1 we see that, from the pool of the 153 surveyed papers, 13 references fall under "representation and symbol use" and that the 13 references come from 7 PME proceedings ("sources"). The topic that has the biggest number of references is "ideology, power, agency, and gender." It contains 28 references coming from the 10 PME surveyed proceedings. The NVivo distribution of nodes provides us with a possible view of the research landscape on language in mathematics education research.

Name	Sources	References
Cultural Dimensions	0	0
Cultural discursive routines	3	4
Ideology, Power, Agency, Gende...	10	28
Individual -Society Relationship	4	5
Language, Math and Culture	5	12
Multilingualism	1	1
Effect on T&L, Supporting T&...	7	10
Language-Math proficency	5	7
Task Design	1	2
Values and expectations	1	1
Language and conceptualization	6	20
Collective participation & Intera...	9	13
Embodiment	8	13
Language as a thinking tool	1	1
Math and everyday languages	8	20
Representations and symbol use	7	13
Vygotskian Semiotics	5	9
Math as discourse and math disco...	5	16
Students' discourse	2	2
teachers' discourse (effect, prof...	7	11
Theoretical	1	1
Critical Discourse	1	1
Hermeneutics	1	1
Language and Constr Knwlge	1	1
Thinking and Language	2	2

Figure 1. NVivo tree showing main nodes and sub-nodes, as well as sources and references in nodes and sub-nodes

Although this categorisation has guided our work in this chapter, we do not discuss every category or subcategory, preferring to restrict ourselves to areas in which the field has developed the most.

For our second analysis, we attempted to identify the principal theoretical orientation for each paper. This process was not always straightforward; some papers had a rather general theoretical basis involving references to a variety of ideas and authors, while a few papers had no identifiable theoretical framework at all. Nonetheless, the majority of papers referred to one or two key sets of ideas as the basis for the research they reported, in some cases fairly briefly as part of a literature review, in other cases more elaborately. Some papers, of course, were entirely devoted to theoretical considerations. We further grouped the theories into higher-order categories, although distinctions between the different groupings are not necessarily especially clear. Any approach with fewer than five instances was recorded as 'other'. The results are summarised in Table 2.

Table 2. Theoretical orientations in PME papers on language topics from 2005–2014

Theoretical orientation	
Sociocultural	48
Discourse analysis	22
Sociopolitical	11
Informal/everyday language	9
Teachers' practice	9
Constructivism	7
Embodied cognition	5
Other	23
Total	134

The most striking observation arising from this fairly crude analysis is the prevalence of sociocultural theory as the basis for much PME research on language in mathematics education. This finding is particularly striking given Lerman's (2006) charting of the then recent rise of sociocultural perspectives across all PME research reports, not just those focusing on language. This work falls largely within the Vygotskian tradition, in which language is understood as a tool, and as mediating between subject and object in the production of mathematical meaning (e.g., Berger, 2005).

In more recent years, Sfard's development of Vygotskian theory in particular has formed the basis for numerous PME papers. Sfard (e.g., 2008) argues that mathematical thinking is an individual form of mathematical communication, reflecting Vygotsky's claim that development occurs first intermentally and then intramentally. Sfard's approach develops this idea in terms of participation in mathematical discourse as forming the basis for individual mathematical cognition, with learning conceptualised as change in discourse. Sfard has subsequently proposed a categorisation of mathematical discourse into four aspects: endorsed narratives, routines, word use and visual mediators. This work has informed almost 20 research contributions at recent PME meetings, including work on dynamic geometry environments (Sinclair & Kaur, 2011; Berger, 2011; Ng, 2014), fractions learning (Wille, 2011), the concept of limit (Güçler, 2011), and the concept of square root (Shinno, 2013). Sfard's work on identity in mathematics has also informed a number of contributions (e.g., Nachlieli, Heyd-Metzuyanim, & Tabach, 2013).

Several other interpretations of sociocultural theory have been proposed and used in the past 10 years. Radford has an approach that draws on semiotics, embodied cognition and dialectical materialism but which is fundamentally rooted in Vygotskian theory (Radford et al., 2005; Radford, Miranda, & Guzmán, 2008; Radford, 2011, 2014). Others have drawn on Gee's (discursive) theory of cultural models (Setati, 2006; Kleanthous & Williams, 2010); activity theory (e.g., Ohtani, 2007); and communities of practice (e.g., Hunter, 2008). Finally, some papers draw on Bakhtinian concepts, often in combination with Vygotskian theory (Mesa & Chang, 2008; Radford, Miranda, & Guzmán, 2008; Williams & Ryan, 2014) although not in all cases (e.g., Barwell, 2013).

The second most frequent theoretical orientation groups together various forms of discourse analysis. This category includes: papers drawing on positioning theory, such as Herbel-Eisenmann and Wagner's (2005) analysis of textbooks, Sakonidis and Klothou's (2007) analysis of students' written work, or Skog and Andersson's (2013) investigation of pre-service teachers' discourse; papers drawing on discursive psychology, such as Barwell's (2007, 2008) analyses of how mathematical thinking is constructed in the discourse of mathematicians and of mathematics education researchers; papers drawing on Halliday's systemic functional linguistics and his notion of mathematical register, including Leung and Or's (2007) study of students' explanations, Herbel-Eisenmann, Wagner and Cortes's (2008) analysis of lexical bundles, and Gol Tabaghi and Sinclair's (2011) study of pre-service teachers' diagramming practices.

The socio-political orientation covers contributions that mainly draw on sociological theories, including Fairclough's critical discourse analysis (e.g., Thornton & Reynolds, 2006; Le Roux & Adler, 2012; Le Roux, 2014), Goffman's participation frameworks (Hegedus et al., 2006), and Bernstein's theory of framing and pedagogical practice (Knipping & Reid, 2013). The total shown for socio-political papers is likely to be somewhat understated, since several other papers, particularly listed under discourse analysis or socio-cultural theory suggest at least socio-political leanings, even if the theoretical framework is not explicitly socio-political in nature (we comment more on this issue later in the chapter). This kind of orientation is relatively recent, following the changes to the PME constitution around ten years ago, which allowed research to address topics in addition to psychology for the first time.

The remaining categories are less represented and sometimes less well theorised. Several papers were based on a general theoretical distinction between everyday or informal language and mathematical language (e.g., Amit & Jan, 2006; García-Alonso & García-Cruz, 2007; Bardelle, 2010). Another group of papers focused on teachers' practices (Chen & Chang, 2012), or teachers' knowledge or understanding in relation to their teaching (Adler & Ronda, 2014), or look at the orchestration or conceptualization of mathematics classroom discussion (e.g., Kahn et al., 2008; Morera & Fortuny, 2012; Wang, Hsieh, & Schmidt, 2012). A handful of papers were based on the theoretical notion of embodied cognition (e.g., Bjuland, Cestari, & Borgersen, 2008; Edwards, 2010; Warren, Miller, & Cooper, 2011).

Finally, 'other' incorporates a wide variety of theoretical orientations to language that occurred relatively rarely. Some notable examples include Lunney Borden's (2009) use of decolonizing methodologies; Heinze et al.'s (2009) use of Cummins' theories of bilingual education to investigate the performance of language minority students in Germany; and Shinno's (2013) analysis of semiotic chaining.

The common thread that runs through the majority of theoretical frameworks adopted in PME research on language in the past 10 years is the idea that language is central to the processes of mathematical thinking, learning and teaching and, as such, is the link between the individual and the social. In this work, language is neither the means of *transmission* of mathematical knowledge, nor the learner's means of expression of their individually constructed schemas. Rather, it is through language that both learners and teachers are historically and culturally constituted *as* learners and teachers of mathematics. As we shall discuss in the remaining sections of the chapter, the predominant theoretical orientations necessitate, often implicitly, or at least, often without being fully developed, a central place for otherness, often termed alterity. In the next sections, we look in more depth at three main thematic foci for PME research on language in the past 10 years: ways of conceptualizing language and mathematics; cultural dimensions of language and mathematics; and language diversity in mathematics education.

WAYS OF CONCEPTUALIZING LANGUAGE AND MATHEMATICS

In this section, we discuss PME research that examines language in collective participation and in embodiment, representations and symbol use, and Vygotskian semiotics. We focus, in particular, on the role that is ascribed to language in the students' and teachers' mathematics conceptualization. Although there seems to be an agreement that "Language is an important tool in the construction of mathematical knowledge" (García-Alonso & García-Cruz, 2007, p. 258), we still need to understand how mathematics education researchers conceive of the relationship between language and conceptualization.

Natural Language and Mathematical Language

Several papers in our corpus deal with the problem of the relationship between natural language and mathematical language. Various terms of have been used for natural language, including 'informal language' and 'colloquial language'. Some of these papers stress the *influence* of natural language on the students' understanding of mathematical concepts. For instance, Fernández Plaza, Ruiz Hidalgo, and Rico Romero (2012) show that the students' mathematical concept of limit of a function at a point is influenced by colloquial uses of terms such as "to approach," "to tend toward," "to reach," and "to exceed" (2012, p. 235).

In a study dealing with the concept of monotonicity, Bardelle (2010) refers to the students' frequent "misuse of mathematical language" (p. 183) and the students' lack of awareness that mathematical terms have a specific scientific meaning:

> [The] interviews show that Matteo and Filippo understand the concept of monotonicity of a function but they cannot answer correctly because they do not realize that the term 'increasing' is a scientific one and hence it has just one well determined meaning. Matteo and Filippo give their own interpretation of the term. (Bardelle, 2010, p. 181)

In another investigation, Bardelle (2013) shows also the influence of natural language on the mathematical understanding of universal statements (e.g., "Not all A is B"): "the interpretation of verbal statements in a mathematical setting may happen to be based on everyday context and not on a mathematical one" (p. 71).

Expanding on Bardelle's work, Ye and Czarnocha (2012) carried out an investigation that "confirms, in a spectacular fashion, the impact of natural language on the mathematical understanding of negation by identifying, during the student interview, a source of misconception initiated from incorrect French/English translation" (Ye & Czarnocha, 2012, p. 235).

It is, therefore, clear that there is an influence of natural language on students' mathematical conceptualizations and that one of the problems is that students do not seem to be aware of the fact that the meanings of natural language do not necessarily coincide with those of mathematical language. Drawing on the work of Shuard and Rothery (1984), García-Alonso and García-Cruz (2007) suggest a distinction between "(1) those terms which have the same meaning in both [everyday and mathematical] contexts; (2) those terms whose meaning changes from one context [to] the other; and (3) those terms which are only seen in a mathematical context" (p. 258). Bearing this typology in mind, they carried out an investigation of four popular textbooks among high school teachers, and analyzed the meaning of 27 terms pertaining to statistical inference in everyday use as well as in the mathematical context (e.g., "population," "sample," and "confidence level"). They concluded that, often, "definitions that appear in the textbooks do not correspond to their mathematical meaning but instead to the one in their everyday use" (p. 263). The problem is thus not only the students' but also the textbook authors'.

The co-occurrence of mathematics and everyday language in the classroom, not only in its oral dimension but also in its written one, has led some researchers to investigate the impact of natural language on the understanding and performance of students (see, for example, Ilany & Margolin, 2008). Bergqvist (2009, p. 146) noted that "In order to read texts in mathematics it is necessary to be able to recognise which category words belong to in order to be able to interpret them correctly." Bergqvist endeavoured "to identify PISA mathematics items for which student performance is influenced by reading ability" (Bergqvist, 2009, p. 145).

Let us try to pose the problem in a more general manner. To do so, let λ_1 and λ_2 be two semiotic systems (a contemporary natural language and a contemporary language of "mathematics," respectively). To a semiotic system λ we can associate the "concepts" or "ideas"' i that individuals express, convey, and manifest with and through λ. Thus, i_1 is the system of ideas associated to with λ_1 and i_2 is the system of ideas associated with λ_2. With a few notable exceptions (e.g., Baber & Dahl, 2005; Lunney Borden, 2009; Edmonds-Wathen, 2014), PME language researchers seem to be, to an important extent, asking questions not about the relationship between λ and i, but about the *influence* of λ_1 in i_2 (Bergqvist, 2009; Fernández Plaza, Ruiz Hidalgo, & Rico Romero, 2012) or the *interference* of λ_1 in λ_2 and i_2 (Bardelle, 2010, 2013; Ye & Czarnocha, 2012).

For Makar and Canada (2005), the problem revolves around the pedagogical use teachers can make of the students' use of λ_1 and i_1 in moving towards λ_2 and i_2 Their research is about the concept of variation with prospective teachers. In a task from a post-interview, the prospective teachers were showed "weights for 35 different muffins bought from the same bakery, and asked what subjects thought their own (36th) muffin might weigh. The set of data for the 35 muffin weights were shown in both a boxplot and a histogram" (Makar & Canada, 2005, p. 276). Makar and Canada note that the subjects resort to terms of natural language to convey ideas of distribution (e.g., "bulk of this data," "concentration of data," data "really clustered," or, as in other interviews, "scattered" or "bunched" data, when the interviewed subjects referred to data presented in dot plots). They conclude by saying that the:

> informal use of language needs to be given a greater emphasis in research on statistical reasoning […] There are several reasons for this. For one, teachers need to learn to recognize and value informal language about concepts of variation and spread to better attend to the ways in which their students use this same language. Secondly, although the teachers in this study are using informal language, the concepts they are discussing are far from simplistic and need to be acknowledged and valued as statistical concepts. Thirdly, the scaffolding afforded by using more informal terms, ones that have meaning for the students may then help to redirect students away from a procedural understanding of statistics and towards a stronger conceptual understanding of variation and distribution. (2005, pp. 279–280)

At the practical level, PME mathematics education researchers seem to recognize that natural language may be both a source of interference and a support in the development of mathematical language and ideas. Even "vague language" may prove to be important: “vague language fosters construction of new mathematical ideas” (Dooley, 2011, p. 287; see also Tatsis & Rowland, 2006).

Previous studies have focused on the identification of linguistic functions to which students resort to express mathematical ideas *in natural language*. For instance, in research about pattern generalization of figural sequences, Radford (2000) identified two such functions, termed *deictic and generative action functions of language.* Radford focused on students’ sentences like “OK. Alright, look. You . . . one has to add (*pointing to a figure on the paper*) . . . you *always* add 1 to the *bottom*, right?” He argued that the deictic function and the generative action function of language were at the root of the students’ mathematical generalization. Through terms like “top” and “bottom,” the deictic function of language provides students with the possibility to *notice* and *refer* to key parts of a *perceptual* term in order to imagine *non-perceptual objects* and their mathematical properties. The argument is that perception is somehow oriented by the meaning of deictic linguistic terms, suggesting thereby potential manners by which to look at, and attend to, objects in our environment. The “generative action function” refers to

> the linguistic mechanisms expressing an action whose particularity is that of being repeatedly undertaken in thought. In this case, the adverb ‘always’ provides the generative action function with its repetitive character, supplying it with the conceptual dimension required in the generalizing task. The relevance of generative action functions can be acknowledged by noticing that, in our example, generality is objectified as the *potential action* that can be reiteratively accomplished. (Radford, 2000, p. 248)

In other words, in λ, the adverb “always” plays a similar role as the universal quantifier $\forall$ plays in λ_2.

Consogno, Gazzolo, and Boero (2006) identified an additional linguistic function, which they termed the Semantic-Transformational Function (STF) of natural language. It refers to

> the construct that accounts for some advances of [the students’] conjecturing and proving process. The student produces a written text with an intention he/she is aware of; then he/she reads what he/she has produced. His/her interpretation (suggested by *key expressions* of the written text) can result in a *linguistic expansion* and in a *transformation of the content* of the text that allow advances in the conjecturing and proving process. (pp. 353–354; emphasis in the original)

The Relationship between λ_1 and λ_2

Naturally, the fact that students can start thinking mathematically within λ_1 (the semiotic system of a natural language) does not mean that λ_2 (the semiotic system of a contemporary language of “mathematics”) can be dismissed. And reciprocally: it would be a mistake to think that a mathematical activity within λ_2 is independent of λ_1: there is a limit to what can be mathematically expressible within λ_1. Natural languages have not been created to calculate and to carry out relatively complex computations. Nor have they been created to investigate theoretical properties of Banach spaces or abstract topologies, for instance. The standard contemporary mathematical language to which students are exposed in school mathematics has acquired, since the Renaissance, an operational dimension it never had before. There was a rupture indeed in the conception of language in the Renaissance that led to the development of two different paths. On the one hand, there was a humanist trend that sought to remove from language the barbaric dimensions of scholastic Latin and other previous linguistic formations. The humanistic trend ended up in a research program whose goal was a simplification and purification of language, the identification of the various parts of discourse, a systematic approach to grammar, and a general theory of the structures of thought (Cassirer, 1963). Grammar, "was taken to provide access to the bases of thought itself" (Reiss, 1997, p. 23). On the other hand, the Renaissance witnessed the emergence of a new scientific language epitomized in the works of Galileo and the abacist mathematicians. The chief characteristic of this language was to reason in an operational manner.

Although both conceptions of language in the Renaissance take different directions, they each rest on a formidable cultural abstraction. On the one hand, there is a progressive development of the idea of a general grammar that in its reasonability, that is, in its appeal to a supposedly general and universal reason, applies to any particular language. On the other hand, there is a search for an efficient language where unknowns, variables, and parameters, and their operations can be carried out regardless of the reference—a minimalist language in which the subject vanishes.

The extinction of the subject is one of the most impressive accomplishments of the contemporary mathematical semiotic system. Such a semiotic system, that endlessly keeps scaring students and sometimes teachers as well is voiceless. Yet it cannot work alone. As Vergnaud notes, "No diagram, no non-linguistic symbolism, no algebra can fulfill its function without a linguistic accompaniment, even if it remains internal or inner only" (2001, p. 14). In short, even in its most developed form, λ_2 depends on λ_1: "Natural language is a metalanguage of all symbolisms" (p. 14). Natural language and the language of mathematics play different roles. With their own specificities, each one of them provides individuals with access to different layers of mathematical consciousness. They provide individuals with different forms of expressiveness and aesthetic experience.

How has this relationship been understood by PME researchers interested in language issues? As we have already noted, the predominant theoretical perspective used in PME language research draws on sociocultural theory, and for the most part the relationship between colloquial and mathematical discourse is framed by ideas from this theoretical tradition.

For Sfard (2010), the route to the development of mathematical language is through changes to colloquial language:

> If mathematics is a discourse, then learning mathematics means changing forms of communication. The change may occur in any of the characteristics with the help of which one can tell one discourse from another: words and their use, visual mediators and the ways they are operated upon, routine ways of doing things, and the narratives that are being constructed and labelled as "true" or "correct". Since uses of words and mediators create a tightly knit web of connections, we should probably consider this system in its entirety, even when interested in only some of its elements. In research on learning any mathematical concept, therefore, nothing less than the whole discourse of which the given concept is a part would suffice as a unit of analysis. (p. 218)

Sfard's approach construes individual learning in terms of change in individual communication, including thinking, which she considers to be communication with oneself. Her approach has been adopted and developed by many contributors to PME over the past 10 years.

For example, Sánchez and García (2011) examined the think-aloud responses of 14 pre-service primary school teachers to a set of nine questions about the properties and definitions of regular quadrilaterals. Sánchez and García analyzed the students' responses by looking for moments of 'commognitive conflict' (using a portmanteau word coined by Sfard to underline the fusion of communication and cognition in her theory). According to the theory, moments of commognitive conflict will arise due to the differential use of language in colloquial and mathematical discourse. Sánchez and García were able to show that such moments did arise for the participants in their study, and related them to the 'confrontation' of mathematical and socio-mathematical norms. For example, one such confrontation was:

> between the [Mathematical Norm] related with defining expressed in the criterion of minimality and the [Socio-Mathematical Norm] 'everything you see in a figure that goes with the presentation of a task has to necessarily indicate something'. It leads students to incorporate descriptive features/ aspects, coming from the task presentation, in some of their responses that are neither necessary nor relevant (for example, length of the side). (p. 110)

This position appears to be based on a couple of important assumptions: first, that there is a clear separation or dichotomy between colloquial and mathematical language; and second, that teachers can make use of students' colloquial language to bridge to mathematical language and meaning.

Barwell (2013), however, argued that the relationship between colloquial and mathematical discourse (for which he used the terms 'informal' and 'formal' language) must be seen as dialogic. In particular, he argued that the implicitly linear sense of development from informal to formal mathematical language is problematic. Referring to data from a class of 10–12-year-olds, he concludes:

> A dialogic perspective on formal and informal language in mathematics classrooms highlights a relationship between formal and informal that is not uni-directional. Rather than steady progress from informal to formal, these students work at both. The teacher, too, must make skilful use of varying degrees of formality. Of course, students need to learn formal mathematical language as part of learning mathematics, but this does not mean that informal language disappears; nor is it simply a scaffold to reach more formal language. Both are necessary; they will always be in tension. (p. 79)

Embodiment

In truth, the situation is more complex than insinuated above. As research on embodiment suggests, in the classroom processes of conceptualization, students and teachers resort to more than colloquial and mathematical languages. They resort to gestures, body posture, kinaesthetic actions, artefacts, and signs in general. Instead of being epiphenomenally surplus to teaching and learning, these embodied and material resources are an important part of classroom activity. As Warren, Miller, and Cooper (2011) report, "the use of gestures (both by students and interviewers), self-talk (by students), and concrete acting out, assisted students to reach generalisations and to begin to express these generalisations in everyday language" (p. 329).

The proper cognitive and epistemological understanding of embodiment and material culture has been the object of an active line of research in PME. At the theoretical level, Edwards, Rasmussen, Robutti, and Frant (2005) led a working session in PME 29 to discuss the role of conceptual metaphor and conceptual blends, and language and gestures in the construction of mathematical ideas and in teaching, learning, and thinking. In the same PME conference, Arzarello and Edwards (2005) organized a Research Forum on "Gesture and the Construction of Mathematical Meaning." The Research Forum led to a Special Issue in Educational Studies in Mathematics (Edwards, Radford, & Arzarello, 2009) where the need of a "multimodal approach" is argued:

> Crucial to the production of knowledge is the individual's experience in the act of knowing and the fact that this experience is mediated by one's own body. However, this return of the body to epistemology and cognition does not amount to a disguised form of empiricism. Conceptual ideas are not merely the impression that material things make on us, as Hume (1991) and other 18th century empiricists once claimed. The return of the body is rather the awareness that, in our acts of knowing, different sensorial modalities—tactile, perceptual, kinesthetic, etc.—become integral parts of our cognitive processes. This is what is termed here the multimodal nature of cognition. (Radford, Edwards, & Arzarello, 2009, p. 92)

A great deal of research on multimodality has revolved around the understanding of the relationship between gestures and language in the students' conceptualizations (e.g., Askew, Abdulhamid, & Mathews, 2014; Edwards, 2010, 2011; Edwards, Bolite Frant, & Radford, 2010; Edwards, Bolite Frant, Robutti, & Radford, 2009; Hegedus, Dalton, Cambridge, & Davis, 2006; Ng, 2014; Radford, 2011; Radford, Bardini, Sabena, Diallo, & Simbagoye, 2005; Robutti, Edwards, & Ferrara, 2012). Arzarello and his collaborators have investigated the role of gestures in the evolution of students' mathematical signs. Thus, in Arzarello, Bazzini, Ferrara, Robutti, Sabena and Villa (2006), the authors investigate "the genesis of written signs starting from specific gestures, progressively shared within the group." They suggest that gestures have various functions: "understanding the situation, looking for patterns or rules, anticipating and accompanying productions of written representations, drawings and symbols necessary to solve the problem" (p. 73).

There has also been an interest in understanding the role of the teacher's gestures on the students' gestures and conceptualization. For instance, in their PME 32 paper, Bjuland, Cestari, and Borgersen (2008) asked the following research question: "What kinds of communicative strategies does an experienced teacher use in her dialogues with pupils, introducing a task that involves moving between different semiotic representations?" (p. 185) They found that: "The [teacher's] gestures make the connection between the semiotic representations, figure and diagram" (p. 185). In the same PME conference, Radford, Miranda, and Guzmán (2008) dealt with a similar problem, cast in terms of the role of multimodality in the classroom evolution of meanings. Following the idea of conceiving of gestures as signs that constitute a genuine semiotic system on its own (Radford, 2002), Radford, Demers, Guzmán, and Cerulli (2003) suggest seeing gestures as embodying different views, voices, and meanings, much like words in natural language. Their analysis shows how, in a very subtle way, the students' gestures come to echo, with their own *intonation*, the teacher's gestures. The echoing of the teacher's gestures and the personal intonation that students bring forward opens up possibilities to *generalize* previous gestures. Within this context, gestures in particular, and multimodality in general, are conceived of as *polyphonic*, and the joint teacher-students classroom transformation of meanings appears as *heteroglossic*:

> Borrowing a term from M. M. Bakhtin, we want to call the transformative process undergone by the students' meanings as *heteroglossic*, in that heteroglossia, as we intend the term here, refers to a locus where differing views and forces first collide, but under the auspices of one or more voices (the teacher's or those of other students'), they momentarily become resolved at a new cultural-conceptual level, awaiting nonetheless new forms of divergence and resistance. (Radford, Miranda, & Guzmán, pp. 167–168)

In general terms, we can reformulate the question of languageandconceptualization as follows. Instead of a relationship between two semiotic systems (natural and mathematical languages) and their corresponding (interrelated) conceptualizations alluded to in the previous section, conceptualization emerges in activities underpinned by a range of perceptual, tactile, kinesthetic, and other sensorial multimodal channels in dialectical interaction with semiotic systems (natural languages, mathematical languages, gestures, diagrams, etc.).

The systemic understanding of such interaction and the political forces that underpin the evolving relationships require more research that may complete the substantial number of PME investigations dealing with representations and symbol use (e.g., Misailidou, 2007; Verhoef & Broekman, 2005; Walter & Johnson, 2007), language and conceptualization (Armstrong, 2014; Baber & Dahl, 2005; Mellone, Verschaffel, & Van Dooren, 2014; Mesa & Chang, 2008; Meyer, 2014; Planas & Civil, 2010; Ruwisch & Neumann, 2014; Viirman, 2011) or classroom discourse (e.g., Asnis, 2013; Berger, 2005; Gholamazad, 2007; Le Roux, 2014; Sfard, 2010). Such a systematic understanding could also benefit from the interesting question of the role of society and culture in conceptualizations in natural and mathematical languages (e.g., Lunney Borden, 2009; Clarke & Mesiti, 2010; Clarke, Xu, & Wan, 2010; Edmonds-Wathen, 2010; Morgan & Tang, 2012).

CULTURAL DIMENSIONS OF LANGUAGE AND MATHEMATICS: AUTHORITY, POWER, AND COLLECTIVE DISCOURSE

In this section, we bring together the question of language as it appears in discussions where the focus is on ideology, power, agency, and gender, including the relationship between the individual and society; the question of language, mathematics, and culture; and cultural discursive routines. The topic of language diversity is addressed in the next section.

There is a growing sensitivity in PME research about the manner in which language embeds, conveys, perpetuates, and shapes ideological stances and social relations, like power. There is also a growing sensitivity in understanding the often subtle mechanisms through which language affords or constrains agency, and structures views about gender. Although the questions about ideology, power, agency, and gender are not necessarily related to multilingualism, it is in multilingual contexts that they often become more salient.

As mentioned previously, in our count, discussions about ideology, power, agency, and gender appear centrally in 28 papers. One of the main concerns is the manner in which students position themselves and also how they come to be positioned by current classroom practices, discourses, and texts (Herbel-Eisenmann & Wagner, 2005; Esmonde, Wagner, & Moschkovich, 2009; Moschkovich, Gerofsky, & Esmonde, 2010; Skog & Andersson, 2013). Another important concern is to describe and understand inclusive discursive practices and practices that exclude or marginalize students (e.g., Hunter, 2013; Hunter, Civi1, Herbel-Eisenmann, & Wagner 2014; Moschkovich, Gerofsky, & Esmonde, 2010). This “political/ideological” line of inquiry rests on a broad conceptualization of language that goes beyond the investigation of the relationship between language and the development of mathematical understanding to focus on “how language in the mathematics classroom illustrates power relationships” (Thornton & Reynolds, 2006, p. 273).

Power relations can appear in the manner in which communication happens in the classroom (e.g., Adler, 2012; Brown, 2011, Civil, 2012; Chapman, 2009; Hussain, Threlfall, & Monaghan, 2011; Radford, 2014; Wagner, 2014), but also in more subtle ways, as for instance in how teachers assess their students' achievements (Sakonidis & Klothou, 2007), how authority is asserted through lexicological choices (Herbel-Eisenmann, Wagner, & Cortes, 2008), or in how students' activity is constrained by recourse to the passive voice and nominalisations (Morgan & Tang, 2012). Behind the "political/ideological" line of inquiry is, of course, a conception of teachers and students that—at the most general level—rests on beliefs about the relationship between the individual and society, and about the nature of power and authority. As two theoreticians of power in classrooms noted a few years ago, "different understandings and practices of authority have been shaped for over a century by conflicting ideological belief systems" (Pace & Hemmings, 2007, p. 10). How, then, do language-minded mathematics education researchers publishing in PME proceedings tackle the question of power and authority? The answer is both difficult and easy.

The answer is difficult in the sense that in the PME language papers dealing with power there is rarely any specific theorization of the meaning of power and authority. A relatively elaborated instance appears in Herbel-Eisenmann, Wagner, and Cortes's (2008) paper, where the authors refer to Pace and Hemmings (2007), who define authority as "a social relationship in which some people are granted the legitimacy to lead and others agree to follow" (p. 6; emphasis in the original). Pace and Hemmings's definition—inspired by Max Weber's work and more precisely by Mary Haywood Metz (1978)—highlights an asymmetrical relation between the manner in which individuals act towards each other, and the social distinction between those who are granted legitimacy to lead and those who are expected to follow. The definition, however, is too abstract. Authority is eradicated from its context. Furthermore, the only explanation that is given for the existence and practice of authority is that authority serves to maintain a "*moral order*" (2007, p. 6; emphasis in the original) which, to make things worse, is equated with "shared purposes, values, and norms intended to hold individuals together and guide the proper way to realize institutional goals" (2007, p. 6). This definition of authority turns out to be very rationalist, simplifies the idea of moral order as something transparent and politically neutral, and portrays individuals as merely consenting and negotiating agents.

At the same time, the question about how language-minded mathematics education researchers publishing in PME proceedings tackle the question of power and authority has a relatively easy answer. It is easy in the sense that through the papers we see that power and authority are thematized along the lines of a *reaction* to transmissive teaching. Let us explain.

In transmissive teaching, the teacher appears as the holder of authority and the students as those who follow the authority of the teacher. The implicit conception of authority and power of transmissive teaching takes as its starting point the idea that the cultural mission of the teacher is to ensure that knowledge, values, and norms are properly passed on to the students. Likewise, the cultural mission of the student is to receive or appropriate this knowledge, values and norms. “In this view,” Henry Giroux notes, “authority is frequently associated with unprincipled authoritarianism” (Giroux, 1986, p. 25).

The remedy against the affliction of authority is usually found in the students’ freedom and autonomy. Freedom and autonomy—the two chief Western categories that have defined the idea of the human subject since the emergence of manufacturing capitalism in the 16th century (Beaud, 2004; Kaufmann, 2004; Radford, 2012)—are considered to provide the basis for students’ escape from authority, and the central condition for students’ emancipation and authentic learning.

This story is not new—and this is something on which we would like to insist, as it is only by understanding the educational story behind authority and its antithetical position, i.e., freedom and autonomy, that we believe we may be able to go beyond the predicaments in which the political/ideological research on language seems to be immersed today. Authority on the one hand, and freedom and autonomy on the other, were the axes around which the proponents of the two main models of the 20th century pedagogical reform mentioned in the introduction envisioned and organized their corresponding pedagogical programs. In the case of the transmissive model, authority provided the hierarchical relationship between teachers and students that was required to put in motion a specific form of knowledge production and reception. In the case of the progressive model, authority appeared as something to be overcome through the nurturing of the student’s freedom and autonomy (see, e.g., Neill, 1992). In searching to promote the student’s freedom and autonomy, progressive educators built their pedagogy through a dichotomy between teachers and students. This dichotomy offered the conceptual and methodological basis for their pedagogical action.

We should not jump to the conclusion that this is past history. The two main pedagogical programs of early 20th century educational reform have not disappeared. On the contrary: both have evolved under the influence of new societal and historical demands. The progressive model has moved from a discourse entrenched in the student to a discourse about students. However, the move from the singular to the plural, that is, the move from a child-centered pedagogy to a children-centered one, where collective discourses are emphasised, does not amount to a change of view of the learner. The move, as we shall see, is cosmetic, not ontological. More profound changes are noticeable in the transmissive program. In its search for efficiency and alignment with neo-liberal global capitalism’s forms of material production, the transmissive program has undergone a profound refinement. It has developed sophisticated technologies of control to monitor students’ achievement (e.g., through regional, national, and international tests) and the teachers’ implementation of a technical, prescriptive curriculum. Ironically, the curriculum of the transmissive model is not shy about advocating for students’ engagement in their learning. One of the best examples is the Ontario mathematics curriculum. Yet, in practice, students’ engagement remains more often than not a purely rhetorical move. We do not need to go far to find other examples. Referring to the American educational context, the historian of education, David Labaree, argues that today “It is hard to find anyone in an American education school who does not talk the talk and espouse the principles of the progressive creed” (Labaree, 2005, p. 277). However, as Labaree notes, “We talk progressive but we rarely teach that way. In short, traditional methods of teaching and learning are in control of American education” (Labaree, 2005, p. 278). And referring to the endless war between progressives and bureaucratic, efficientist, transmissive pedagogues, he concludes that “The pedagogical progressives lost” (Labaree, 2005, p. 278; see also Kantor, 2001).

The lost war of the progressive model is a recurrent theme in many PME papers, even if the theme is not formulated explicitly in this way. Brown (2011), for instance, having in mind not only the UK context in which he works, but also the contemporary educational context at large, complains that teachers find themselves working under governmental demands that seek to promote prescriptive curricula that favour some social groups. “Specifically,” Brown (2011) notes, teachers “work to curriculums that mark out the field of mathematics in particular ways that favour certain priorities or groups of people” (p. 190), confining students and teachers to the sphere of cultural reproductive agents. Wagner (2014) makes a similar point: “I consider it unfortunate that mathematics classroom practices tend toward closed dialogue in which children are not invited to see the possibility of multiple approaches and possibilities” (p. 63). And he did not miss the opportunity to complain about the lack of autonomy with which students are left in traditional transmissive classrooms: “Teachers too frequently fail to raise the possibility of students’ autonomy” (p. 63).

It is, however, in empirical papers that the reaction to the traditional transmissive model is most salient. It is there that the question of students’ participation (or the lack thereof) comes to the fore (e.g., Høines & Lode, 2006; Hunter, 2007; Hodge, Zhao, Visnovska, & Cobb, 2007).

These empirical papers also show a great concern for understanding the role that teachers may play in promoting students’ dialogical participation in collective discussions (e.g., Hunter, 2008; Mesa & Chang, 2008; Chapman, 2009; Gilbert & Gilbert, 2011; Sánchez & García, 2011; Morera & Fortuny, 2012; Toscano, Sánchez, & García, 2013; Adler & Ronda, 2014; Cavanna, 2014; Hung & Leung, 2012; Thornton & Reynolds, 2006). For instance, Thornton and Reynolds (2006) investigate the extent to which Grade 8 Australian students have opportunities to express themselves and submit ideas to the classroom. A closer look at the analysis shows that the students’ opportunities for participation are still carried out against the background of the teacher-students dichotomy championed by the progressive reformers. Thornton and Reynolds (2006) contrast Noemi’s classroom—that is, the classroom they investigated—to many of the TIMSS 1999 video classrooms, which “featured reproductive discourse, with the apparent goal of students being to guess what was in the teacher’s mind” (p. 275). They remark: “In Noemi’s classroom students see themselves as active participants in learning, who have power over both the mathematics and the discursive practices of the classroom" (p. 277). They go on to say: "Power is located with students" (p. 277). With power on the side of the students, the teacher's authority has finally vanished.

Chapman (2009) offers us a similar view. As in the case of Thornton and Reynolds (2006), she poses the problem against the backdrop of the war between traditional mathematics classrooms and reformed classrooms. In a clear and succinct way, she summarizes how discourse is conceptualized in current reform mathematics education perspectives: "Discourse, as promoted in current reform perspectives of mathematics education, is not about classroom talk intended to convey exact meaning from teacher to student; instead, it is about communication that actively engages students" (Chapman, 2009, p. 297). Of course, there is nothing wrong with this. As Giroux notes, "student experience is the stuff of culture, agency, and self-production and must play a definitive role in any emancipatory curriculum" (1986, p. 36). To see the teacher-students dichotomy appear we have to consider the following part of the citation that we highlight in italics: "…instead, it is about communication that actively engages students *in a way that allows them to construct new meanings and understandings of mathematics for themselves*" (Chapman, 2009, p. 297; our emphasis). The second part of the citation tells us who is in control of the means of classroom knowledge production. It reveals that the conception of classroom discourse is still based on the teacher-students dichotomy. It is the students who, through their engagement in classroom communication, have to understand mathematics *for* themselves. This is what empowerment seems to be about.

Lee (2006) also stresses the need for students to take control of the means of classroom knowledge production. She pleads for an approach that engages students in classroom discourse and that is oriented towards helping them express and explain their ideas, so that "They take *ownership* of *their* ideas and become able to *control* and use them" (Lee, 2006, pp. 7–8; our emphasis).

In sum, contemporary progressive (or reform) views of mathematics classroom interaction revolve around the old progressive idea of students' participation. Although this is certainly a commendable idea, we see that students' participation is understood against the backdrop of a dichotomy between teachers and students. This dichotomy, the progressive pedagogues feel, is required in order to guarantee the overcoming of the teacher's authority. At the epistemological level, the dichotomy serves to define a specific form of knowledge production in the mathematics classroom, which is based on the idea that students have to gain control over, and ownership of, knowledge and its mechanisms of production.

How does the teacher understand the operating dichotomy that promises to set the students free from authority? Noemi, the teacher in Thornton and Reynolds's (2006) investigation, says:

> My aim in my Mathematics classroom is for students to regard Mathematics as an art which belongs to them, a means of regarding and interpreting the world,

> a tool for manipulating their understandings, and a language with which they can share their understandings. My students' aim is to have fun and to feel in control. My role is primarily that of observer, recorder, instigator of activities, occasional prompter and resource for students to access. Most importantly, I provide the stimulus for learning what students need, while most of the direct teaching is done by the students themselves, generally through open discussion. (Thornton & Reynolds, 2006, p. 278)

As we can see, the teacher conceptualizes herself as a *resource*, providing the students with occasional stimuli. In other PME papers, the teacher appears as a "facilitator" (Chapman, 2009, p. 298) or "guide" (Hodge, Zhao, Visnovska, & Cobb, 2007, p. 42) of the subjective expression of the students. There is a generalized patriarchal view of the teacher, who is reduced to playing a shepherding role—teachers appear as scaffolders, observers, and room-makers-for-students-to-think-and-act. They are there to promote student achievement and established forms of academic success. But the progressive model does more than that: most importantly, it provides teachers with technologies of subjectification to conceive of themselves as shepherds and facilitators.

We can try to go further and ask the question about how the teacher conceptualizes the students. The previous cited passage provides us with some interesting elements with which to answer the question. Understanding knowledge—mathematics, in this case—as something that can be *possessed*, the teacher conceives of the students as potential possessors. The teacher wants the students to regard mathematics as something that "belongs to them" (Thornton & Reynolds, 2006, p. 278).

Let us notice that this stance is not typical of teachers like Noemi. As we have seen, researchers also expect the students to understand mathematics *for* themselves; they are expected to take *ownership* of *their* ideas. The same goes for the Theory of Didactical situations, where teachers are advised not to show the students the answer. As Brousseau notes, if the teacher shows the student how to solve the problem, the student "does not make it her own" (Brousseau, 1997, p. 42). Since how to solve the problem is not "her own," in this line of thinking the student cannot be said to have achieved a genuine mathematical understanding.

In brief, the progressive (reform) model and the theories and pedagogies it has inspired tend to look at the students through the lenses of the students-as-private-owners paradigm. That is, the students are conceived of as subjects of a specific form of "knowledge production that equates doing and belonging: what belongs to the students is what they do by themselves. What they do not do by themselves does not belong to them" (Radford, 2014, p. 5; rephrased). Within this context, understanding is featured as the epistemic equivalent of belonging: Understanding is the product of the students' own cogitations and deeds. The students' understanding is the product of their own labor—not the teachers'. How indeed—the question runs—could students understand something that they did not themselves produce?

In the same way as we labor in society to acquire and possess things, students labor in the classroom to possess/understand knowledge. Hussain, Threlfall and Monaghan (2011) attempt to introduce a new approach to mathematics teaching and learning: “This paper introduces an approach to mathematics teaching and learning which we feel transcends the usual teacher-centered versus student-centered dichotomy by integrating two kinds of mathematics classroom discourse, the authoritative and the dialogic” (2011, p. 1). The solution that they envision is based on a partition of authority—sometimes authority rests with the students, sometimes it rests with the teacher. They continue:

> It is proposed that mathematics teaching and learning should engage students in dialogic communicative approaches to empower them to articulate their ideas and to take more responsibility, but that in order to enable students to build mathematics competences effectively it is also proposed that the teacher should at times involve periods of authoritative discourse on topics prompted by the dialogic discourse. (Hussain et al., 2011, p. 1)

The question of authority is again posed against the background of the opposition of teacher and students. The solution exists in the alternation of authority, a compromise between the two camps at war—the transmissive (traditional) and the progressive (reform) camps.

In his PME 38 plenary talk, Radford (2014) suggested a dialectical approach that puts at the center the idea of teaching and learning as a single process in which teachers and students work together—an idea captured in the term joint labour:

In joint labour teaching and learning are fused into a single process: the process of teaching-learning—one for which Vygotsky used the Russian word obuchenie. In this sense, teachers and students “are simultaneously teachers and students” (Freire, 2005, p. 76). They are simultaneously teachers and students, but not because both are learning (Roth & Radford, 2011). They are, of course. However, the real reason is because teachers and students are labouring together to produce knowledge. (pp. 10–11)

Here knowledge is neither something that teachers possess and pass on to the students (the transmissive model) nor something that students acquire through their own personal deeds (the progressive model). Knowledge is not something to possess; like music, it is a kind of evolving space to attend (“fréquenter” as Guillemette, 2015, p. 76 says), visit, and enjoy. More precisely, knowledge is a diverse cultural-historical set of potentialities that, through the teacher-students’ joint labour, enables actions, imaginations, interpretations and new understandings.

This perspective moves away from the conception of the teacher as a shepherd discussed previously:

> regardless of how much the teacher knows about [mathematics], she cannot set [mathematical] knowledge in motion by herself. She needs the students—very

> much like the conductor of an orchestra, who may know Shostakovich's 10th Symphony from the first note to the last, needs the orchestra: it is only out of joint labour that Shostakovich's 10th can be produced or brought forward and made an object of consciousness and aesthetic experience. (Radford, 2014, p. 11)

Although teachers and students do not play the same role, they work together. They need each other. "Teachers and students are in the same boat, producing knowledge and learning together. In their joint labour, they sweat, suffer, and find gratification and fulfillment with each other" (Radford, 2014, p. 19).

LANGUAGE DIVERSITY IN MATHEMATICS EDUCATION

The perceived increase in language diversity in contemporary classrooms must, for education, be one of the most salient legacies of colonialism and globalization. There are two aspects to this legacy. First, the increasing movements of people around the world, initially as a result of colonial policies, more recently as a result of globalization, mean that classrooms now rarely fit the presumed ideal in which all students speak one andthe same language. In developed countries, thesecircumstances have often come as something of a shock, leading to concepts like 'superdiversity' (Vertovec, 2007; see Barwell, 2016, for a more extended discussion in the context of mathematics education) as societies and, in particular, education systems struggle to come to terms with the presence of multiple languages and cultural backgrounds. The second aspect of the legacy of colonialism and globalization, however, is that the Eurocentric view of 'normal' societies as unilingual, with one language unifying one nation, is finally itself being overturned. A plurilingual view of society is no surprise to the 'rest' of the world, where living with multiple languages is the norm. Much as the peoples of the Americas must have been surprised to learn that they were 'Indians', so the 'discovery' of language diversity implies a complex and problematic relationship with otherness.

A focus on language diversity, including topics such as mathematics learning in multilingual classrooms, in bilingual education programs or of immigrant second language learners have featured at PME for some time. Indeed, in his paper at PME29, the first year of our current survey, Barwell (2005) reviewed research reports with a focus on language diversity from the previous 10 years. He identified 13 research reports in that period, indicating a good level of interest in topic of growing prominence. In our current survey, we have identified 21 papers addressing this topic, suggesting a degree of growth in work in this area. These papers cover a range of national contexts (Australia, Canada, Catalonia-Spain, Germany, Malaysia, New Zealand, Philippines, South Africa, Tonga, USA) and sociolinguistic settings, including bilingual classrooms, indigenous learners, immigrant learners, and multilingual societies. This work addresses several interrelated topics.

Several contributions examine aspects of students' mathematics learning in the context of language diversity, looking at how their mathematical understanding is linked to practices like code-switching (e.g., Manu, 2005; Planas, Iranzo, & Setati, 2009; Planas & Civil, 2010) and the challenges of word problems given in an 'imported' language (Verzosa & Mulligan, 2012). There has also been work seeking to understand students' perspectives on learning mathematics in a language other than their home language (Setati, 2006), and the perspectives of 'local' students on practices designed to support immigrant learners in their mathematics classes (Planas & Civil, 2008).

Another strand of research continues the search for a link between language proficiency and mathematics achievement. Some of the early work on this topic was reported in PME in earlier decades (e.g. Clarkson, 1996; Clarkson & Dawe, 1997). Recent papers include two quantitative studies conducted in Germany (Heinze et al., 2009; Prediger et al., 2013), as well as Essien and Setati's (2007) investigation of the effects on mathematics scores of an intervention designed to improve a group of South African students' proficiency in English.

Several researchers have reported their work with teachers to develop more effective tasks or teaching methods (Poirier, 2006; Nkambule, Setati, & Duma, 2010; Hunter, 2013) and Civil (2008) has also reported on similar work with parents. Lim and Ellerton (2009) reported teachers' views as part of their examination of changes to language policies in Malaysia.

Finally, three papers have examined the relationship between grammatical structures of indigenous languages and the related affordances for mathematical thinking and learning (Lunney Borden, 2009; Edmonds-Wathen, 2010, 2014).

This work reflects the kinds of tensions arising in mathematics classrooms in contexts of language diversity discussed by Barwell (2012a, 2012b, 2014), including tensions between home and school languages, between formal and informal mathematical language, and between language for learning and language for getting on in the world. Barwell draws on Bakhtin (1981) to theorize these tensions as reflecting an inherent tension in language. Bakhtin uses the metaphor of centripetal and centrifugal forces to conceptualize the nature of language both as diverse and constantly new and different (called *heteroglossia*), and as striving to reflect an ideal of purity and perfection (known as *unitary language*). Hence, most of the papers mentioned above subscribe to an idea of mathematical language as a stable, unified register or discourse, when it can instead be seen as multiple, diverse and unstable.

Bakhtin's understanding of language is based on relationality and, in particular, dialogue. Thus heteroglossia is not simply the presence of difference, but rather the relations and interactions between these differences. For Bakhtin, these interactions are dialogic in nature, meaning that they involve more than one perspective at once. Dialogue arises between languages, discourses, utterances or voices and is what make meaning possible. Fundamental to this view of language is the role of alterity. Difference requires otherness but, as we have seen, difference is also the source of an unavoidable tension within language. Whenever students must learn mathematics in a second language, or a language they do not use at home, they are learning mathematics with an Other's language (Barwell, 2013). And language, in Bakhtin's theory, is not just language, it is ideology—a worldview. Thus, learning mathematics in another language, or in multiple languages, is not just a question of getting through the language to the mathematics that lies beneath; rather, each language, or a particular variety of language or languages, offers a different mathematics (Edmonds-Wathen, 2014). A key question for the work reviewed in this section, then, is: How do PME researchers interested in language diversity deal with the fundamental issue of otherness in their research?

There are, inevitably, a variety of responses to this question apparent in the different papers. In some cases, the learner is the Other. For example, in Heinze et al.'s (2009) carefully designed quantitative study conducted in Germany, the goal was to understand the relationship between the language proficiency of immigrant students and performance in mathematics, such as in a high-stakes mathematics test. The assumption is (reflecting, we presume, the national policy context in Germany) that many immigrant students do not speak good German and should learn to do so in order to succeed in mathematics. Heinze et al. found some links between proficiency in German and mathematical performance. The students' proficiency in their home language, was not evaluated, however, despite much research showing that home language proficiency can also be an important factor in school success (e.g. Cummins, 2000). Immigrant students are characterized in terms of 'foreignness'—they are either migrants, or their parents are migrants, or they speak a foreign language at home (Heinze et al., 2009). (The study also found no difference between migrants and non-migrants on basic arithmetic performance.)

The othering of immigrants is also apparent in Planas and Civil's (2008) paper. They worked with a secondary school mathematics teacher who was implementing 'reform' teaching practices, which included problem-solving and collaborative group work. Planas and Civil report on interviews with some of the 'local' students, which reveal how they see immigrant students as language learners rather than mathematics learners:

> Helena [high achiever]: They put us in small groups and they say that this way we will learn more mathematics, but the real reason is that they do it so that those from outside get a chance to practice our language. I don't think this is right because I think that these decisions should be based on the mathematics. (Planas & Civil, 2008, p. 125)

Moreover, while there was interest from local students in the alternative mathematical methods displayed by the immigrant students, the prevailing view was that the immigrant students should learn 'our' methods.

It seems that the key basis for the construction of immigrant students as other is the perception that they do not speak the classroom language 'correctly' or are not proficient or simply speak differently. Khisty (2006) discusses this issue in some depth, in the context of Spanish/English bilingual students in the USA (not necessarily immigrants). She proposes a sociocultural view of learning in which learning mathematics is understood as socialization into the language of the mathematics community. She uses this perspective to look for explanations for underachievement:

> Academic discourse competence in this broader sense is acquired through active participation in the community that uses that discourse, and through interactions with a more capable other (Vygotsky, 1986). The lack of discourse competence suggests academic failings. […] Without the academic discourse or language, students are systematically excluded or marginalized from classroom curricula and activities. (Khisty, 2006, p. 436)

She also argues that the "denigration" (p. 437) of students' home language amounts to an additional form of alienation from school and from mathematics and "silences students' voice" (p. 437). Khisty's argument is one of the more carefully developed positions apparent in the papers in this section. Nevertheless, it is not without some underlying tensions, at least when viewed from the perspective of Bakhtin's theory. In particular, it is based on a view of mathematics and mathematical discourse as something students should learn. The nature of mathematical discourse is not itself questioned; students should learn it and will benefit from it. It appears that the students simply need to learn mathematical discourse, and hence the educational problem is to create suitable conditions (reflecting the progressive view of education). In fact, learning mathematical language also means learning a particular worldview; it means becoming a particular kind of person and could thus be seen as a kind of colonization of the mind. This tension is an example of the problem of moving beyond both transmissive and progressive approaches to teaching and learning mathematics.

An alternative approach to alterity is to assume from the start that language has a political dimension. Setati (2006), for example, assumes that "The political nature of language is not only evident at the macro-level of structures but also at the micro-level of classroom interactions. Language can be used to exclude or include people in conversations and decision-making processes" (p. 98). In her interviews with five South African students about the language they preferred to use to learn mathematics, three preferred English and two did not express a preference. The students all spoke four or five different languages. For Setati, a preference for English can be related to the political role of English; the students saw English as an international language and therefore as a "route to success" (p. 99) and in some cases preferred it even when they acknowledged that they would understand mathematics better if it were taught in one of their home languages.

Civil (2008) in her work with Mexican-American parents also sees language as political. Her study reveals how the language policies in South-West USA which enforce a strong preference for English in schooling serve to marginalize the parents in her study. They are less able to attend class when their children are young, or to support their children in mathematics. They also noticed that their children were often grouped with other learners of English, so reducing their chance to interact with English-speakers, and they often studied mathematics they had previously learned in Mexico.

In both these studies, then, it is language itself that is seen as the Other. In Setati's study, English, although widely used in education in South Africa, is seen as the language of 'international' and of 'social goods.' In order to succeed, the students felt that they must learn this language, even to the detriment of their understanding of mathematics. In Civil's paper, the parents report how the use of English (in a particular way), positions them as different, and so as less capable. In both studies, English is colonizing students of mathematics and, as a result, may marginalize and alienate them. Indeed, in the case of the students in South Africa, they may be alienated from the very languages they speak at home. Again, then, there is a tension, between the many ways students have of talking about mathematics, including the different languages they may know (*mathematical heteroglossia*), and the educational ideal of a single language of instruction for mathematics.

A third approach to alterity is to attempt to understand the Other better. Three papers reported studies focused on analyzing the linguistic structure of other languages, particularly indigenous languages in Canada and Australia. Edmonds-Wathen's work (2010, 2014) draws on the concept of linguistic relativity, which assumes that the structures of language influence ways of thinking. For Edmonds-Wathen, this principle applies to mathematics. In the first of her papers, she reports on her work in a remote community in the Northern Territory, Australia, in which mathematics is taught in an indigenous language called Iwaidja. She sets out how spatial language in English is structured very differently from in Iwaidja (Edmonds-Wathen, 2010). In the second paper, she looks at the structures relating to number in various languages around the world to show how presumed universal features of mathematics are actually culturally and linguistically specific.

In her paper, Lunney Borden (2009) describes some of her work with Mi'kmaw schools in Nova Scotia, Canada. Her experiences illustrate the alienating effects of an English-language perspective on mathematics. For example, she describes how the English concept of 'middle' is not easily translatable into Mi'kmaw, so that a student asked in English to show the middle of something may appear not to understand the mathematical notion, when in fact it is language that is most relevant. Edmonds-Wathen characterizes well the deeper issue at stake in all three papers:

> It is difficult to avoid a deficit perspective in a discussion of people not using numbers because Western culture and mathematics education values quantification so highly. Nevertheless, it also does learners a disservice if their prior learning and conceptual development is not taken into account by mathematics educators. This is particularly relevant for remote Indigenous

> Australian children who enter a compulsory school system that is largely designed and taught by English-speaking non-Indigenous people who learnt their own number words from their parents within their own cultural milieu. (Edmonds-Wathen, 2014, p. 437)

Much as Columbus named the new world in his own image, as part of the process of conquest and appropriation, so mathematics has also been named by Eurocentric thinkers. Recognizing that this naming can itself be a form of colonization, however, makes it possible to consider alternative positions. In Edmonds-Wathen's and Lunney Borden's work, the Other is relative; the 'English-speaking non-Indigenous people' are others to Iwaidja people or Mi'kmaw people and vice versa. The Other is no longer singular, identified with the oppressed, the marginalized, the alienated; there is a relation—Bakhtin's ideas would suggest a dialogic relation—through which each constructs the other, although of course this relation is not necessarily equal (see the section on cultural dimensions).

Given the complex relationship between language, mathematics teaching and learning, and alterity, what (again) can teachers do? And how, for that matter, can researchers conduct their research in a way that does not marginalize and alienate (if this is even possible)?

For some, the answer to both these questions can be found in the concepts of voice and dialogue. Khisty (2006) explicitly draws attention to the role of student voice in supporting mathematics learning and concludes by raising questions about how teachers position themselves in relation to students' home languages:

> Do teachers and others understand and appropriately consider the political implications of which language is used and how? Do they view it as a learning resource or as something that does not have a place in mathematics classrooms, that should be ignored? Do they genuinely value the home language, do they recognize that differential status among students, including language status, is detrimental to students' learning, and do they seek ways to equalize language status? Do they seek ways to validate what students' have to say even when they do not speak the dominant language of instruction? (p. 438)

Khisty's questions point towards approaches to teaching that involve dialogue between languages, as well as between the voices of students, the teacher and mathematics. Nkambule, Setati and Duma (2010), for example, working in a South African classroom of 46 Grade 11 students analyzed what happened when the teacher used dual language versions of mathematics problems. All of the students and the teacher spoke multiple languages and were grouped according to the main language they used at home. The mathematics problems were presented in English and one of isiZulu, isiXhosa, Sepedi or Sesotho. Nkambule et al.'s analysis shows how the use of multiple languages supported the students to invoke 'horizontal mathematization'; that is, to make links between the mathematics in problems and their own experiences of similar situations. The study sets out a teaching strategy that values students' home languages, as well as their interpretations of the mathematics problems.

Poirier (2006), in a contribution to a research forum, describes her contribution to mathematics curriculum development with a school board in Nunavut, the Canadian province with a majority Inuit population. She recognizes the dangers of the situation:

> If we want to re-examine the Inuit curriculum and develop learning activities adapted to the Inuit culture, the researcher who is not a member of that community can not do that alone. The risk of developing activities that will not be suitable, or well-adapted, is too great. (p. 110)

She describes how, to mitigate these risks, she worked collaboratively with a team of four Inuit teachers and three Inuit teacher trainers. Her approach is highly dialogic, with the team exploring Eurocentric and Inuit mathematical concepts and ways of thinking, each in relation to the other. She reports the comments of a member of the school board:

> This research proposal is also a unique project in the history of KSB research specifically addressing curriculum questions in a minority, bicultural, and bilingual situation. As described in your paper, the dual phenomena with two cultures in contact in a learning environment, and in a school setting using the subject of math, is like an unexplored expedition to a foreign area of the universe of learning. (Betsy Annahatak, Curriculum development department, Kativik School Board, September, 2002). (Poirier, 2006, p. 112)

These remarks suggest that a degree of dialogue was established, although there remains an underlying sense of tension arising from the dominant nature of Eurocentric mathematics and European languages.

Lunney Borden (2009) has perhaps gone furthest towards a fully dialogic approach. Having taught for many years in Mi'kmaw schools, she drew on decolonizing methodologies, engaging in discussions with Mi'kmaw elders to develop an acceptable approach to her research. An important aspect of decolonizing methodologies is questioning the way research itself—frequently a colonizing activity—is conducted. The outcomes of her research, then, not only challenge Eurocentric notions of mathematics, but challenge Eurocentric approaches to research.

Mathematics education is still mostly conceived of in terms of unquestioned forms of alterity. What is transmissive education, if it not a form of colonization of the mind? Perhaps less obviously, progressive education can be seen in the same light: the imposition of a particular view of students, teachers and mathematics. The starting point for the development of a more dialogic approach is the awareness of the value of the Other, and an acceptance of heteroglossia as a normal state of affairs. This position suggests the need not just for a more effective approach to teaching mathematics in the context of language diversity, but also the need for a more ethical approach.

CONCLUSION

In this chapter, we have surveyed research on language published in PME conference proceedings from 2005 to 2014. We have discussed some of the main trends, such as language and conceptualization, questions surrounding authority and power, and language diversity. In this conclusion we ask the question: What is missing in current research on language?

What is missing, we think, is the constitution of a language of critique that may help us move from the two models of the early 20th century educational reform that continue to inform educational practice today. We have lived for more than a century pulled by a transmissive conception of education and a children-centered notion of education that, in the end, has been engulfed by schooling tailored to respond to the needs of contemporary capitalist forms of production. It is against the backdrop of the century-long struggle of these two models of educational reform that an important line of research on language has been moving for some time towards questions of power, authority, student participation, and equity. These questions have often been dealt with along the lines of a neo-liberal "redistributive" pedagogy. That is, a pedagogy that seeks to re-order the structures of knowledge and power in order to ensure "equal opportunities for all to learn through accessing both the mathematics curriculum and qualified teachers" or "equality of mathematical achievement outcomes across student groups" (Hunter, 2013, p. 97).

Although commendable on several counts, this pedagogy falls short of questioning the societal forces that produce inequalities and oppression. It fails to question, for instance, the mathematics curriculum, its political and economical orientation, and the kind of subjectivities it favors. While this critique has been made by Walkerdine (1988) and Giroux (1989) some 30 years ago (and developed in more recent work by, for example, Appelbaum, 2012; Valero, 2007; Walshaw, 2014), it is not well developed in PME research on language (or in PME research in general). Yet, it is within a redistributive pedagogy that questions of power or language diversity are often formulated in the PME proceedings: they are often formulated as the search for pedagogical actions that capitalize on minority group languages to lead the members of these groups to dominant mathematics. Language diversity becomes a tool to attain, maintain, and affirm Western mathematics. What is missing here, we suggest, is a critical language that could help us understand that the tensions between languages, or between forms of language, are not simply the source of pedagogical or ontological challenges: they are political, through and through. Such a critical language should help us transcend the shortcomings of redistributive pedagogy and to go beyond the conception of knowledge as something politically neutral to be possessed, the conception of students as private owners and teachers as technical implementers of a prescribed curriculum (shepherds, scaffolders, observers, instigators, helpers, etc.). As one of the reviewers put it, “so-called reform classrooms risk to privilege privileged students again.” Instead of conceiving of teachers as curriculum technologists whose role is to promote conventional forms of academic success, we argue for a conception of teachers as intellectual practitioners who critically problematize the knowledge and values that they and the students bring to, and co-produce in, the classroom. We argue for a conception of teachers as critical agents who acknowledge the fact that classrooms are first of all places of conflict and resistance and that it is out of conflict and resistance that subjectivities are formed and transformed, the teachers' included. Such an approach would connect the research in our first theme (on language and mathematical conceptualization) with research in our second and third themes (on language, power, authority and language diversity).

What remains to be done to address the challenges we have highlighted in PME research on language in mathematics education, we think, is the elaboration of a new emancipatory conception of knowledge, authority and power. To do so, we need to start working from a non-substantialist perspective. That is, we need to think of knowledge, authority, and power not as “things” that people have or lack. We might be better off thinking of authority and power as rather a set of fluid and always moving relations that are enacted as individuals engage in human life. Authority and power are at the heart of the social practices of the division of labor and the tensions that result from the manner in which persons, groups of persons, and communities envision, define, and pursue their individual-societal purposes and truths. It is through human practices that authority and power are produced (not in situ, but historically). In turn, authority and power come to shape, embrace, and orient these practices, thereby making it possible that “certain forms of subjectivity, certain object domains, certain types of knowledge come into being” (Foucault, 2000, p. 4). What is also missing in PME research on language and discourse, then, is a vision of teachers and students where authority is not an authoritarian relationship but rather a communal social and cultural construction “that expresses a democratic conception of collective life, one that is embodied in an ethic of solidarity, social transformation, and an imaginative vision of citizenship” (Giroux, 1986, pp. 22–23). Power and authority should rather serve as methodological lenses to critically reflect on the school values that we promote, nurture, and convey, as well as the kinds of rationalities and ways of knowing that we privilege. By looking at power and authority in this way, we may become reflectively able to notice those that we exclude, allowing us to envision more encompassing inclusive and just courses of action. Such a conception of authority and power may also allow us to rethink the positions, stances, and ideologies we come to embrace and promote in the school and beyond. We need to rethink the forms of classroom knowledge production and the forms of human collaboration that could be consonant with an emancipatory critical pedagogical agenda.

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