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The epistemic, the cognitive, the human: a commentary on the mathematical working space approach

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Abstract This article is a commentary on the mathemati- cal working space (MWS) approach and draws on the arti- cles contained in this ZDM issue. The article is divided into three parts. In the first part I discuss the place of the MWS approach among the French theories of didactique des mathématiques. In the second part I outline what I think are the central ideas of the MWS approach. I conclude the arti- cle with a sketch of what seems to me to be its accomplish- ments and challenges, focusing mainly on the epistemolog- ical and cognitive stances that the MWS approach conveys in order to elucidate the manner in which this approach theoretically assumes that things are known and learned.

Keywords Cognition · Epistemology · Activity Theory · Conceptions of the student · Collective versus individual learning

1 Introduction

One of the most distinctive traits of the French *didactique des mathématiques* is its focus on mathematical content and its suitable organization in teaching and learning. It is not surprising that concepts such as “praxeologies” and “fundamental situations” are at the core of two of the most influential French theories—the Théorie Anthropologique du Didactique (Chevallard, 1985, 2006) and the Théo- rie des Situations Didactiques (Brousseau, 1997). It is not surprising either that, in this context, epistemology comes to play a central role (Artigue, 1990; Brousseau, 1989;

Chevallard, 2006; Glaeser, 1999): an epistemological per- spective and the ensuing epistemological analyses are con- sidered to contribute to shedding some light on questions of the genetic structure of knowledge and knowing.

Yet, a focus on mathematical content, its epistemol- ogy and didactic organization is far from doing justice to the general panorama of questions tackled within the vari- ous approaches that constitute the *didactique des mathé- matiques*. Vergnaud’s (1985, 1990) and Duval’s (1995, 1998, 2000) work have both a definite *psychological* and *semiotic* orientation. Artigue (2013a, 2013b) and Trouche (Guin & Trouche, 2004; Trouche, 2003), to mention two, have called attention to the crucial role of diverse kinds of artefacts in teaching and learning leading to what can be termed an *instrumental* approach.

The mathematical working space approach (MWS) draws on the French tradition of *didactique des mathéma- tiques*. But where does it stand vis-à-vis the diverse general trends of the French tradition? The MWS approach makes a commendable synthesis in order to present a comprehensi- ble and coherent perspective to the specific problem of the teaching and learning of mathematics where the epistemo- logical, psychological, semiotic, and instrumental domains become functionally entangled.

Now, stated as I just did, the synthesis offered by the MWS seems a trivial one—a theoretical synthesis in the “normal” development of didactic theories. However, I do not think that this is the case. Like all educational theo- ries, didactic theories are rooted in profound beliefs that shape the contours of the problems to be investigated; those beliefs also provide the basis to determine the manners in which the research should be tackled. To articulate an epis- temological view of teaching and learning with a psycho- logical and instrumental one—as the MWS approach does in its own cultural and historical context—is not a minor matter. It is certainly a daring and audacious step. It might be worthwhile recalling Brousseau's plenary talk at the 30th Conference of the International Group for the Psy- chology of Mathematics Education (PME 30). Brousseau reminds us that in the 1970s, when the question of envi- sioning scientific research on mathematics education arose, the best option seemed to be cognitive psychology. His conviction, however, was that cognitive psychology was not the route to follow:

> A study of the materials [contents] to be taught and of the conditions for teaching them, inspired by the ingenious experiments used in genetic epistemology to detect the mathematical behaviors of children, con- vinced me of the interest of an alternative route, one not envisioned in traditional academic frameworks. (Brousseau, 2006, p. 3)

The Théorie des situations didactiques that Brousseau developed in the following years is the crystallization of such a route. Psychology and its arsenal of theoretical constructs (e.g., perception, short- and long-term memory, imagination and the mental plane more generally) were not merely absent but deemed irrelevant to account for the "mathematical behaviors of the student." The student became envisioned in his/her general features: he/she was considered as an abstract entity. As Brousseau put it, "the subject of the didactic situation is [in the theory of didac- tical situations] a kind of theoretical subject" (Brousseau, in Salin, Clanché, & Sarrazy, 2005, pp. 23–24); that is, an epistemic subject. I will come back to this point in Sect. 3, when I comment on the articulation of the epistemologi- cal and psychological domains in the MWS approach. This short introduction is aimed at merely locating the MWS approach in its historical context so that we may better gauge the challenges that this synthetic project is willing to face and to better appreciate its scope and orientation.

In the next section I outline what I think are the central ideas of the MWS approach. I conclude the article with a sketch of what seems to me to be its accomplishments and challenges.

2 The mathematical working space

The central idea of the MWS is that mathematics is an activity where a mathematician carries out some actions to accomplish something. This activity is a mathematical work. The reference points are hence the mathematics, the mathematician, and how he/she does his/her work. The first question is, then, to try to describe as clearly as possible, mathematics as an activity. This description will serve later to delineate and understand the activity in which teachers and students engage in the classroom. In the introduction to this issue of ZDM, Kuzniak, Tanguay, and Elia (2016) acknowledge that in this perspective, "The student is no more viewed as a mere learner, carefully following the paths delineated by school exercises, but as a researcher exploring the realm of mathematics." They refer to previ- ous attempts such as the one revolving around the idea of "scientific debate." Yet, the debate seems unable to pinpoint some crucial aspects of the mathematician's activity that are to be taken as the model for the mathematical work. This work certainly involves debate, but is not reduced to it. This work is made up of discursive and non-discursive ele- ments. We find here a tension that surfaces again and again in discussions about how to theorize mathematics from an educational and didactic view point: is mathematics a discourse or an activity? (Radford, in press). The MWS approach positions itself within the latter view. The math- ematician's activity includes distinct phases, such as heu- ristic explorations, discovery, explanation, and justification. This view has been clearly summarized by an influential contemporary French mathematician—Jean-Pierre Kahane. In his plenary talk during the Canadian School Mathemat- ics Forum/Forum canadien sur l'enseignement des mathé- matiques held in Montreal in the spring of 2003, Kahane was trying to come to terms with the question of whether or not it is useful to teach mathematics today. Kahane observed that other disciplines rely heavily on mathemat- ics, and more precisely on the mathematics approach (i.e., la démarche mathématique). "Our fellow physicists, biolo- gists, computer scientists, economists," Kahane noted (2003, p. 8), "all tell us that what interests them mainly in mathematics training, is the mathematics approach." And he goes on to make explicit its specificity. He says:

> Mathematics has an original way of linking defini- tions, hypotheses, conclusions, theorems and proofs. Before being established by a proof, the validity of a mathematical statement can be guessed, illustrated, tested on some examples. But, ultimately, the validity of the statement is based on its proof.

Hence, "A reasonable goal of mathematics education is that, at the end of their studies, students have a good idea of what a proof is" (p. 8).

The MWS approach spells out Kahane's description of the *démarche mathématique*—the mathematician's activ- ity—by distinguishing first the epistemological plane, which has to do with the specificities of the mathematical content and the tasks and problems that prompt the math- ematician's activity. The epistemological plane contains "a set of concrete and tangible objects; a set of such as drawing instruments or software; a theoretical system of reference based on definitions, properties and theorems" (Kuzniak, Tanguay, & Elia, 2016; see also Richard, Oller Marcén & Meavilla Seguí, 2016). But the mathematician's activity involves a cognitive dimension too, through which the mathematician interprets signs, imagines solutions, and produces results. As Vergnaud notes, no diagram, no non- linguistic symbolism, no algebra can fulfill its function without an interpretation of some sort, even if it remains internal or inner only (2001). In the MWS, the cognitive dimension is centered on the subject, "considered as a cog- nitive subject" (Kuzniak, Tanguay, & Elia, 2016).

A closer look at the mathematician's activity leads Kuzniak et al. to distinguish three axes: a semiotic, an instrumental, and a discursive axis. Since the mathemati- cian's activity is always evolving as he/she engages with mathematics, these axes are seen as *genetic*.

The semiotic genesis refers to the evolving semiotic activity that includes geometric figures, algebraic symbols, graphics, diagrams, photos, and so on. That is, anything with a representational quality: a capacity for standing to the individual in any respect.

The instrumental genesis refers to those parts of the mathematician's activity in which artefacts become sali- ent (see, e.g., Santos Trigo, Moreno Armella, and Cama- cho Machín, 2016). It includes material artefacts but also *techniques* of computation (e.g., classical constructions with ruler and compass and Euclidean division). The MWS idea of instrumental genesis comes close to what, in Activ- ity Theory, is called *operations*. They intend to account for the generalized aspect by which actions are carried out. Leont'ev says:

> Finally… thought is realized by some means, that is, with the help of determined conditions in the given instant — logical or mathematical. But any operations — regardless of whether they are outward-directed or inward, mental — represent in their genesis only the product of the development of corresponding actions in which are fixed, abstracted, and generalized the objective relationships characterizing objective condi- tions of action. They therefore have a relatively inde- pendent existence and are capable of being embod- ied in one material form or another — in the form of instruments, machines, multiplication tables, simple arithmetic, or complex calculator-computer appara- tus. (Leont'ev, 1978, p. 27)

The discursive genesis refers to the "deductive dis- course of proof" (Kuzniak, Tanguay, & Elia, 2016) specific to mathematics and its proper mathematically recognized organization on the basis of definitions, theorems, axioms, etc.

A diagram (see Fig. 1) provides a neat metaphor for the various interrelations within a MWS.

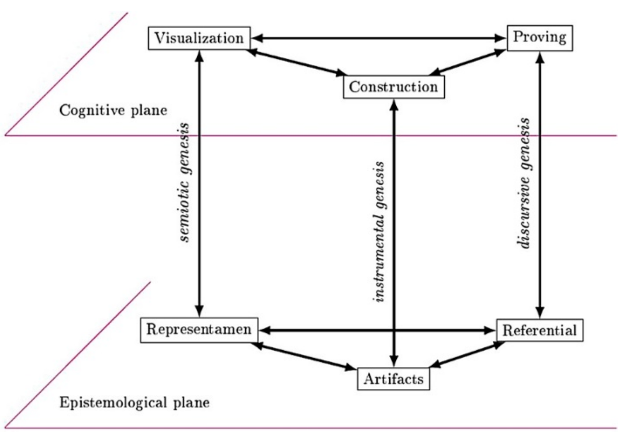


Fig. 1 The epistemological and cognitive plane and their three link- ing geneses

The previous ideas crystalize in a definition of MWS. A MWS is "a structure organized in a way that allows the analysis of the mathematical activity of individuals who are facing mathematical problems. In the case of school mathematics, these individuals are usually not experts but students, experienced or beginners" (Kuzniak, Tanguay, & Elia, 2016).

The aim that the MWS seeks to achieve is clearly stated as follows:

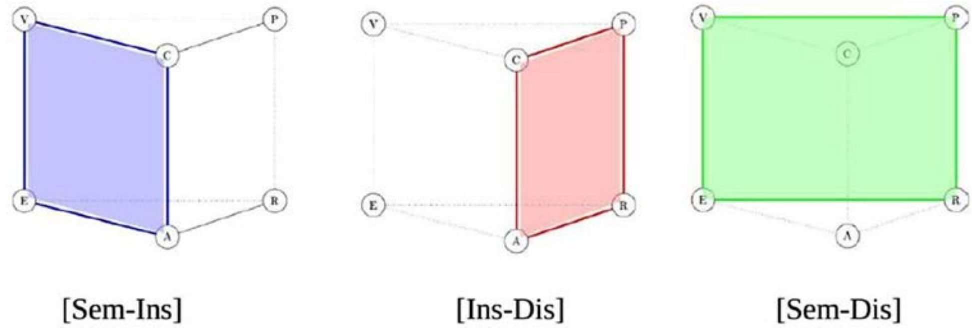
> Conceived as a developing and adaptable hosting structure for mathematical activities, the MWS and their study are aimed at supporting the analysis of how these different aspects interact, so to account for the way a given set of tasks and activities even- tually shapes (or fails to shape) a complex and rich mathematical work, from the learning standpoint of students as well as from the teaching standpoint of teachers. (Kuzniak, Tanguay, & Elia, 2016)

The operational nature of the MWS as a didactic approach rests on its capacity for tracking the mathemati- cal work of teachers and students as they engage in prob- lem solving and proving activity. Three different types of activity are identified: modeling, discovery, and validation. Modeling is associated with the left vertical plane joining the semiotic and the instrumental geneses. Discovery is associated with the right vertical plane joining the instru- mental and discursive geneses. Validation is associated with the vertical plane joining the semiotic and discursive geneses (see Fig. 2).

In one of the examples that Kuzniak, Nechache, and Drouhard discuss, Grade 10 students (15 years old) are confronted with the following problem about two identical wallets:

> The first [wallet] contains 3 banknotes of 10 euros and 5 banknotes of 20 euros. The second contains 2 banknotes of 10 euros and 4 banknotes of 20 euros. One wallet is chosen randomly and a banknote is

Fig. 2 The three *vertical planes* of the MWS (see Kuzniak, Nechache, & Drouhard; 2016)



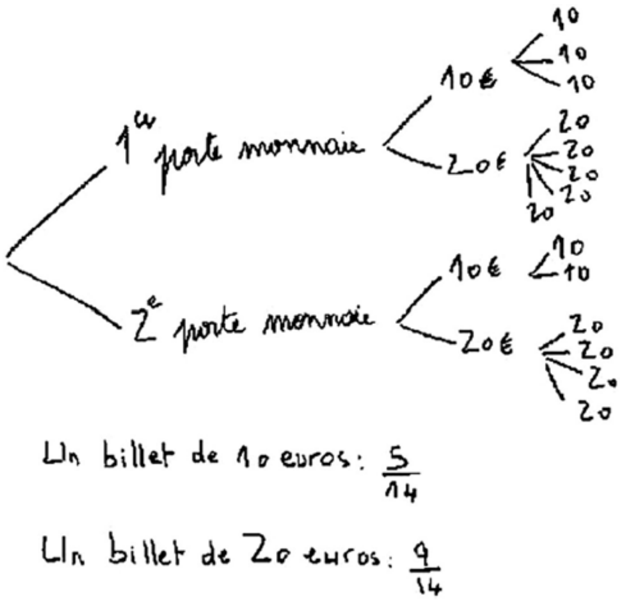


Fig. 2 The student's tree as drawn on the blackboard. From Kuzniak, Nechache, and Drouhard (2016)

drawn "blindly" from this wallet. What is the prob- ability of choosing one banknote of 10 euros? One banknote of 20 euros? (see Kuzniak, Nechache, & Drouhard; 2016)

The authors discuss the part of the classroom activity in which a student goes to the blackboard and explains his solution. The student draws a tree (see Fig. 3). The tree is divided into two main parts, one indicating the first wallet (1er porte monnaie) and one for the second wallet (2^{e} porte monnaie), each one divided into two branches (10€ and 20€), subsequently divided into branches of 10 and 20. At the bottom, the student writes the answers: "One banknote of 10 euros: 5/14; one banknote of 20 euros: 9/14."

The tree appears first as the *translation* of a word- problem into a specific semiotic system that includes lines (branches) and results given in natural language, banknotes, and numbers. The student's mathematical work is seen to occur first in the semiotic and instrumental geneses plane. The tree is also used as a calculation device. We could say that the student moves from the left edge of the semiotic and instrumental plane to the right edge. But as the authors observe, there is no justification yet. The teacher inter- venes and asks for an explanation: why 5/14 and 9/14? Despite the teacher's efforts, the student does not produce the expected mathematical reasons and remains within the semiotic and instrumental geneses plane. He does not move to the semiotic and discursive geneses. The various planes of the MWS model allow us to understand where the activ- ity is taking place, the difficulties for the students to move between planes, and some of the challenges that teachers may face when trying to help the students move towards the semiotic-discursive plane. In Kuzniak, Nechache, and Drouhard's analysis, the teacher focused on the definition of a probabilistic space, lost sight of the weight of the tree branches, and failed to notice the student's error. There is a disarticulation between the epistemological plane and the semiotic-discursive plane.

But the MWS approach also helps us to appreciate the institutional forces that act upon the semiotic-discursive plane. It helps us reveal the kind of rationality that is being institutionalized through the mathematical work. Kuzniak, Nechache, and Drouhard notice that a theoreti- cal view of events in the world is surreptitiously intro- duced through specific assumptions of the probabilistic model. "The probabilistic model is that of equal probabil- ity. This model is not explicit, but the text makes refer- ence to it with the following terms: identical, randomly, blindly." With such a view comes a way of proving and ascertaining truth. The student answer is as follows: "Well, I took the two wallets like this. Then in the first wallet there are three banknotes of 10 and four banknotes of 20 euros." However, the teacher does not accept the stu- dent's answer. Of course, one cannot say that the student did not provide reasons for his calculations. The ques- tion is that the teacher wants *other* reasons, and, referring to the student's answer on the blackboard, asks: "Then why 5 over 14? Why 14?" In the Kuzniak, Nechache, and Drouhard analysis, the tree fulfils three functions: it "enables the representation of random experiments with several steps; it is also a calculation tool and, ultimately, it is institutionalized by the French national curriculum as a legitimate medium of proof." The student uses the tree as a mathematical representation of the word-problem, as a calculation tool, and even as a proof. But the proof does not correspond to the expected proof. It does not corre- spond to the rationality conveyed by the epistemic plane and that should emerge in the MWS.

By evidencing the kind of rationality that is expected in the mathematical work of teachers and students, the MWS shows very well the institutional choices and the ensuing constraints that surround any education project. Focus- ing on the French educational context, the MWS approach shows at the same time how these constraints affect in a profound way the pedagogical actions of the teachers and the students' learning (see also Montoya Delgadillo & Vivier 2016). The MWS approach invites us to inquire how the societal constraints operate in other educational systems, by reflecting the differences and similarities among other cultural contexts and theoretical orientations (see, e.g., Elia, Özel, Gagatsis, Panaoura, & Yetkiner Özel, 2016).

This short discussion allows us to see some of the gen- eral ideas that I find behind the MWS approach. It also allows us to show some aspects of its operational nature. In particular, it invites us to rethink the suitable didactic actions susceptible of bridging the gap between the stu- dents' reasons that emerge in their mathematical work and the institutionalized ones. It invites us to revisit once more the role of the teacher and the role of the students in know- ing and learning.

3 What are the accomplishments and the challenges that face the MWS approach?

As mentioned in Sect. 1, the MWS approach makes an audacious historical move in trying to include the epis- temological and cognitive planes that have remained to a large extent separated in the French tradition of *didactique des mathématiques*. The move of putting the epistemologi- cal and cognitive planes side by side is certainly innovative and promising. Yet, as we can guess, it presents difficulties of various sorts. One of them has to do with the conception of epistemology and the conception of cognition that such a project brings to the fore. There are not just one episte- mological theory and one cognitive theory. There are, for example, empiricist, idealist, and rationalist epistemologies that have ramifications within them. While Hume (1921) is a good representative of the empiricist epistemology, Kant (2003) and Descartes (1637) are good representatives of the idealist and rationalist ones, respectively.

Since epistemology has to do with the manners in which individuals come to know, the adoption of an epistemo- logical position and its ensuing theoretical assumptions is a matter full of practical consequences—particularly in the educational domain, where what is at stake is teaching and learning (Sierpinska and Lerman, 1996).

And in the same way as there are different epistemolo- gies, so are there different cognitive theories. Cognitive theories do not necessarily convey the same concept of cognition. Information processing theory inspired some cognitive theories in the 1980s where the mind became conceived of as a processor of information and the subject as a problem solver (Andler 2004; de Vega 1986; Kotovsky et Simon 1990). These theories analytically dissected the subject's mental functioning in sub-functions that allegedly operate without regard for the context. They portrayed a subject that in his/her cognitive endeavors is both a-histori- cal and a-cultural. Recent cognitive theories have turned to language to describe the individual's cognition (Friedrich, 1970; Harré & Gillett, 1994). Indeed, it is in language and its fabulous production of embodied metaphors that Lakoff and Núñez (2000) find the origins of mathematical con- cepts and mathematics cognition. Yet, generally speaking, in these cognitive traditions the subject of cognitive func- tioning and investigation remains an *individual* subject, as if thinking, visualizing, imagining, symbolizing, remem- bering, perceiving, etc. were individual-specific acts per se. To a great extent, cognitive psychology and empiri- cist, idealist, and rationalist epistemology have conceived of the subject in essentialist terms; that is, as an essential self-contained entity who produces ideas and meanings as he/she is confronted with problems to solve. And so, to a great extent, has mathematics education done as well. If the subject is considered *in abstracto*, in its "pure" relation- ship to knowledge, this entity is called an *epistemic subject*. Here the child is the abstract relationship that is disclosed between knowledge and a problem or set of problems to be solved. Piaget's genetic epistemology (1979) provides us with one of the clearest examples of the epistemic subject. If, by contrast, the subject is considered in his/her mental, psychological functioning, the subject is called a *psycho- logical subject*. Mathematics education cognitive research provides us with one of the clearest examples of the psy- chological subject (see, e.g., Anghileri, 1989 and Cobb, 1987). Here the psychological lenses are supposed to shed some light on the child's inner conceptualizations and ideas. In both cases, the production of knowledge is usu- ally considered to be something pertaining to *the* subject; that is, something that the epistemic or psychological sub- ject accomplishes on his/her own—even if it is acknowl- edged that what is accomplished (or not) is constrained by the social institution to which the subject belongs (as in the case of the probability example discussed above). In the end, in these accounts, the individual (e.g., the problem solver in psychological research or the classroom student in mathematics education research) remains as a sort of self- contained subject with virtually no decisive epistemic and cognitive connection to the historical and cultural context. Such a context appears only as a constraint by providing the epistemic plane with what Foucault (2003, p. 164) calls a “regime of truth.” It is hence not surprising that the edu- cational discourse still refers to *the* student’s learning and *his/her* own concepts, as if learning and the production of concepts were something purely individual. It is not sur- prising either that the relationship between teachers and students still remains so difficult to conceptualize (Cheval- lard, 1997; Radford, 2014) and that national and interna- tional assessments, and the report cards that the teachers send to the parents to track the student’s learning, are not collective but individual. The sophisticated educational machinery that has been put in place to surveil, deliver, and assess teaching and learning rests on the idea of thinking and learning as an individual-specific act per se.

Behind the manner in which we conceive and talk of the student and the student’s learning rests, of course, the daunting question of the relationship between the individu- als and their society and, ultimately, the very conception of the individual and the concept of society. In Western con- temporary societies, the concept of the individual revolves around the idea of the autonomous self. Our legal appara- tuses are built around this idea. However, the conception of the individual as an autonomous self is, as the German sociologist Elias notes, a historical phenomenon. Contrast- ing feudal and modern societal formations, Elias says:

> It may be easier to see in retrospect how closely this transition from a predominantly authoritarian mode of thinking to a more autonomous one… was bound up with the more comprehensive advance of individu- alization in the fifteenth, sixteenth and seventeenth centuries in Europe. It formed a parallel to the tran- sition from a more “external” conscience dependent on authorities to a more autonomous and “individual” one. (Elias, 1991, p. 97)

The transition from an authoritarian consciousness to one where individuals conceive of themselves as autono- mous beings does not come out of the blue. It is linked to the new forms of production and control over nature that arose with the emergence of market towns and the bour- geois world:

> One can see more clearly in retrospect how closely this new form of self-consciousness was linked to the growing commercialization and the formation of states, to the rise of rich court and urban classes and, not least, to the noticeably increasing power of human beings over non-human natural events. (Elias, 1991, pp. 97–98)

If, contrary to the traditional individualist conception of society and individuals, we conceive of society in general, and the mathematics classroom in particular, not as a set of individuals laboring in their own space bounded by contrac- tual relations of interaction, but as, ontologically speaking, something more fundamental—that is, as the *common histor- ical ground* of individualization—then learning and teaching could be envisioned as something more collective and social.

The theoretical position that I am outlining here would require us to understand human cognition as truly cultural (Luria, 1976, 1979; Shweder & LeVine, 1984). Such a position does not exclude or erase the subject as an indi- vidual and unique entity, but claims that our cognitive func- tioning—as singular and idiosyncratic as it may be—is also a cultural one. This theoretical position would also require us to understand the individual in general and the student in particular not as the origin of the production of knowl- edge, meaning, and intentionality, but as a participant fully engaged in the collective activity through which knowledge becomes disclosed as an object of understanding and cri- tique. As Mikhailov put it, “Understanding the real living, individual not as a ‘point of departure’ but as the result of all world history up to the present means individualising the social and understanding individuality as a social phe- nomenon” (Mikhailov, 1980, p. 149).

Let me come back now to the MWS approach. What are the epistemological and cognitive stances that the MWS approach conveys? How does this approach theoretically assume that things are known and learned? While it is clear that, in tune with other French didactic theories, knowledge is assumed to be of a cultural nature with a cultural mode of existence through social institutions, documents, and prac- tices, it is not really clear to me if learning is conceived of as an individual-specific deed or as collective phenomenon with repercussions and implications at the individual level. In the presentation of the MWS approach there is a dis- tinction between a *reference* MWS and a *personal* MWS. Kuzniak, Tanguay, and Elia (2016) argue that “In the set- ting of a given educational institution, giving access to effi- cient *personal* MWSs is the ultimate goal” (my emphasis). We are also told that the teacher too works within *his/her own* MWS—the teacher’s MWS. “In summary,” they tell us

> the mathematical work within school or class- room settings can be analyzed at three MWS levels: The mathematics as framed by the institution are described in the reference MWS. The latter must be adapted by the teacher into a suitable MWS, to allow its effective implementation in the classroom where each and every student will work in her/his personal MWS. (Kuzniak, Tanguay, & Elia, 2016)

Does it mean that we should conceive of learning as an individual-specific act and the classroom mathemati- cal work as the work of a set of individuals who, although laboring side by side, and even in interaction with each other, move nonetheless along their own space of concep- tualizations? I think that there is plenty of room to refine the theoretical understanding of the epistemological and cognitive planes that are at the basis of the MWS model. Such a refinement will certainly have an important impact on the methodologies to investigate the three geneses of the model.

4 Synthesis and concluding remarks

The MWS approach certainly makes an interesting synthe- sis of French didactic theories and dares to bring together the epistemological and cognitive planes. It is a very coherent and well-organized approach, theoretically well grounded, with a profound aesthetic sense of unity—some- thing very characteristic of the French didactic traditions. Its starting point is the mathematician's work, which serves as a kind of model to envision what can be a classroom with an intense mathematical activity. To do so, the MWS approach suggests a flexible structure that links the episte- mological to the psychological plane through three planes defined by three different geneses: a semiotic, an instru- mental, and a discursive one. The operational nature of the MWS is remarkable, as shown by the various articles in this ZDM issue.

There are, of course, some challenges. In this article I examined two: the conception of epistemology and the con- ception of cognition to which the MWS approach resorts. I mentioned the longstanding problem that mathematics edu- cation faces in conceptualizing the epistemological and the psychological. I argued in particular that mathematics edu- cation still remains haunted by the influence of individual- ist epistemological and psychological accounts of knowl- edge and knowing.

Where does the MWS approach stand vis-à-vis this *problématique*? As mentioned before, I am not sure. On the one hand, I see that the MWS approach does take into consideration the cultural dimension through the manner in which the educational institution conveys and promotes certain forms of cultural rationality. On the other hand, I see that the MWS approach seems to take into consid- eration a social dimension that, although interesting, gives the impression of remaining confined to an "agglutinat- ing" conception of the classroom (one that, except for the emphasis on the epistemological, is not very different from the socioconstructivist conception of the social). That is, the classroom seems to appear as an "ensemble" of students who, although laboring side by side, and even in interaction with each other, move nonetheless along their own space of conceptualizations.

Concerning the first point, I think that cultures provide individuals with much more than regimes of truth. Cultures do constrain, but, as entities in flux and loci of tension and contradiction, they also provide affordances and new forms of action and conceptualization.

Concerning the second point, it might be the case that the MWS approach could expand and enrich its concep- tion of the social through a theoretical and practical stance that would consider the production of knowledge in the classroom as a deep *collective act*. It is here where I find Hitt, Saboya, and Cortés's (2016) article very enlightening. Hitt, Saboya, and Cortés discuss the creation of a specific MWS around the idea of promoting an arithmetic-alge- braic thinking in students. This working space integrates such aspects of a mathematics class as collaborative learn- ing, debate, and self-reflection that are much more than catalyzers of the student's learning. They are part and par- cel of what students learn and how they learn it, apparently making unnecessary the idea of a personal MWS. Maybe, instead of a personal MWS concept, something like a *col- lective* MWS concept could be added to the approach. Another possibility would be to keep the personal MWS concept, but as something closely related to a collective MWS (much in the same way that Leont'ev's (1978) con- cept of "personal sense" exists for a specific and unique individual not as something in and by itself, but as the individualization of a corresponding cultural concept of "meaning").

I mention this theoretical position as a possibility, for in the MWS approach there are several references to Activity Theory (Leont'ev, 1978). Activity Theory presents a strong concept of the social that has the merit, I believe, to con- sider the individual as an individual-in-society; hence, a unique individual who is unique while at the same time is an individual-with-others, the individualization of societal relations (Roth and Radford, 2011).

The insertion of an Activity Theory concept of the social would require the implementation of methodologies that allow one to understand teaching and learning as truly social collective phenomena occurring in MWSs. Such a choice may also require redefining the concept of the stu- dent as an epistemic and/or cognitive subject. Cognitive psychology has been successful in extracting from the sub- ject precisely what cannot be extracted from him/her—life, *concrete life*. Cognitive psychology has ended up with an abstract and lifeless subject, as abstract and lifeless as the epistemic subject of epistemology. Maybe we need to come back to the student as a concrete sentient subject, who suf- fers and finds fulfillment in learning—and in learning with others. After all, we do not teach to cognitive or epistemic subjects. We teach to human beings.

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