

On alienation in the mathematics classroom

Luis Radford

Abstract: The goal of this article is to discuss a question that, curiously, has passed unaddressed in mathematics education research: the question of alienation in the mathematics classroom. In the first part, I bring out the conceptual structure of Marx's idea of alienation as it appears in Marx's *Economic and Philosophic Manuscripts* and discuss four different senses of alienation. In the second part, I argue that two of the most influential models that have informed mathematics education in the 20th century, namely the transmissive and the progressive models, are both alienating. In the last section I discuss the possibilities of overcoming alienation through a reconceptualization of mathematics teaching and learning based on a cultural-historical communitarian ethic of solidarity and critical stance.

1. Introduction

While the theme of alienation has a long history in philosophy (see, e.g. Lukács, 2012; Mészáros, 1972; Sayers, 2011; Schacht, 1970), it has not been prominently featured in mathematics education research. Were it not for a few exceptions (e.g., Baldino and Cabral, 2011; Brodie, Slonimsky, and Shalem, 2010; Williams, 2011, 2012), alienation, I would say, is not even obliquely addressed. I do not think that the reason is because alienation does not exist in contemporary mathematics classrooms. On the contrary: alienation in the mathematics classroom seems to be the rule, not the exception. Although differences may be noticeable from one country to another, *in practice*, to considerable extents, disciplines (and mathematics in particular) continue to be taught according to the precepts of the “transmissive” educational model (see, e.g., OECD, 2009, p. 99). The transmissive model is anything but new (Katz, 1971). For instance, Babylonian scribes in 2000 B.C. (Kramer, 1949) and students of the Renaissance Abacus Schools (Franci, 1988) were confronted with a transmissive pedagogy. But the transmissive model that we know today emerged in the late 19th and early 20th centuries in what came to be known as the “educational reform.” With its intellectual origins in behaviourism, the transmissive model was promoted by bureaucratic pedagogues who focused on implementing mass education to efficiently address the demands of industrial and business production (Tyack, 1974).

Although nowadays this model has acquired a sophisticated complexity – e.g., a digital capacity for monitoring school performance and student achievement at the local, regional, and international levels – its alienating nature has been kept intact. On the one hand, it reduces teachers to bureaucratic implementers of a prescriptive curriculum (Brown, 2011). On the other hand, it reduces students to passive, deficient receivers of knowledge (Freire, 2005). In this educational model, knowledge, indeed, is considered as a *commodity* that teachers deliver. To come to possess it, the students have to work hard through drill and repetition. And now that the market competition has reached an unprecedented fierce intensity, drill and repetition are no longer enough. Teachers are expected to teach creativity or to do whatever it takes to manufacture it. In a

recent article in *Macworld* magazine, Cipriani (2015, p. 71) notes: “In an increasingly competitive market, Bluetooth keyboard manufacturers have to seek out ways to differentiate themselves from the competition.” To cope with such market demands, curricula around the world, like the Ontario Curriculum, are adding *creativity* to the list of that which witnesses student skill development and knowledge acquisition (see, Ontario Ministry of Education, 2005).

The transmissive model puts forward a specific idea of human beings: it conceives of human beings as *private owners*. In the same manner as workers get a salary for their work, students get school marks for theirs; while the workers labour to acquire things, the students labour to acquire their own knowledge. In an interesting article published a few years ago, Lave and McDermott (2002) proceeded to read Marx’s famous section on Estranged Labour included in the *Economic and Philosophic Manuscripts*, written in Paris in 1844. They systematically replaced the word *labour* with the word *learning*. What they found was that Marx’s text, which was intended as a critique of political economy of the 19th century, turned out to be a contemporary critique of the school system:

Simply put, in critiquing the theories of political economy available in 1844, young Marx unwittingly wrote a quite devastating critique of the theories of learning in 2002. This is possible because education has been institutionalized under advanced capitalism as an integral part of the political economy (Lave and McDermott, 2002, pp. 21–22).

Why, then, has the theme of alienation passed generally unaddressed in mathematics education research? The question is a striking one given the fact that, *in practice*, the school operates like a “teaching factory,” much as if it were a “sausage factory,” as Marx already pointed out many years ago (1976, p. 644).

In principle, one might expect that proponents of the “child-centred” or “progressive” model would bring to the fore the question of alienation. However, this is not the case. The “progressive” model, which emerged about the same time as the transmissive educational model (Parker, 1990; Rugg & Shumaker, 1969), has focused on the student and the ideas of individuality, self-expression, freedom, and autonomy (Labaree, 2005). Since its inception up to today, proponents of progressivism have considered freedom and autonomy to provide the central condition for the students’ authentic learning. It is in this context that proponents of the “child-centred” or “progressive” model have often equated the principles of freedom and autonomy with the goals of education (Dearden, 1972, 1975; Morgan, 1996). Education, according to them, should not be about receiving truths through or from someone else. As Piaget put it towards the end of his life, “The goal of intellectual education is in learning to master the truth by oneself” (Piaget, 1973; p. 106). Education should rather be the creation of spaces for the student’s personal intellectual growth (Cobb, 1988).

Progressivists could not care less about alienation for they consider their program to be the actual antithesis of alienation: the progressive educational model is thought to be emancipatory in itself. For instance, In Neill’s (1992) famous book *Summerhill*, alienation is not even mentioned once. To worry about alienation would be futile, a waste of time. Really?

The goal of this article is to discuss this question. It is also to explore new possibilities for overcoming alienation in the teaching and learning of mathematics. To try to meet these goals, I need to start from the beginning—that is to say, I need to discuss with some detail the concept of alienation. This is what I do in the next section. Then, I move to a discussion of the forms of alienation that underpin the progressive model. I note, *en passant*, that my interest is not merely theoretical. On the contrary: the progressive model continues to inspire mathematics education research and a critical analysis seems to be required. In the Synthesis Section I discuss the possibilities of overcoming alienation through a reconceptualization of mathematics teaching and learning based on a cultural-historical communitarian ethic of solidarity and critical stance.

2. The concept of alienation

It is impossible to discuss the concept of alienation without considering at the same time a set of interrelated theoretical constructs—such as the concept of the *individual* on whom alienation is predicated, and the *activities* that make the individual an alienated subject.

In the *Economic and Philosophic Manuscripts*, Marx develops a concept of alienation (*Entfremdung*) whose structure rests on:

- an anthropological concept of the individual;
- a specific concept of labour; and
- a precise relationship between the individuals and the objects they produce through labour.

Let me briefly dwell on these elements, starting by referring to Marx’s anthropological concept of the individual.

2.1. Humans as natural beings of need

Marx considers humans as *natural beings*. What this means is that humans are part of nature and, like other natural living beings, humans are (1) *beings of need* that (2) find their satisfaction in objects *outside* of themselves. Marx writes:

Hunger is a natural *need*; it therefore needs a *nature* outside itself, an *object* outside itself, in order to satisfy itself, to be stilled. Hunger is an acknowledged need of my body for an *object* existing outside it, indispensable to its integration and to the expression of its essential being (Marx, 1988; p. 154).

Marx's concept of humans means also that, like other living natural beings, humans are naturally provided with *vital impulses* to satisfy their need. Much as the plant that vitally turns to the sun to reach its light, humans are furnished "with *natural powers of life* . . . These forces exist in [them] as tendencies and abilities—as *impulses*" (Marx, 1988; p. 154).

The natural sentience of needs and the "natural powers of life" or "impulses" to satisfy the needs are not merely part of the pragmatic makeup of humans. In Marx's account – and Marx is following Spinoza (1989) on this idea – needs and the "impulses" to satisfy them acquire an *ontological* meaning. That is to say, they are part of what *is* to be a living natural being. This means that living natural beings are *objective*: they have an *object* of need that is outside themselves. Marx notes: "A being which does not have its nature outside itself is not a *natural* being . . . Its be-ing is not objective" (Marx, 1988; pp. 154– 155). Thus, the plant and the sun are natural objective beings: "The sun is the *object* of the plant – an indispensable object to it, confirming its life – just as the plant is an object of the sun, being an *expression* of the life-awakening power of the sun, of the sun's *objective* essential power" (p. 154).

Let me turn now to the manner in which the satisfaction of needs appears in Marx's account.

2.2. Human labour

The satisfaction of natural beings' needs – their life-activity – is not only a precarious basic activity through which needs are met. Although it is a survival activity, it is also the manner through which natural beings *express* themselves. This is what Marx means in the previous passage where he asserts that "The sun is the *object* of the plant . . . just as the plant is an object of the sun, being an *expression* of the life-awakening power of the sun, of the sun's *objective* essential power" (p. 154).

Human life-activity – that is to say, labour – acquires a fundamental meaning in Marx's philosophical system. Labour (*Arbeit*) is that through which humans express themselves—the fundamental ontological determination of "humanness," the "really human mode of existence" (Mészáros, 1972; p. 78).

This is why, for Marx, labour is most of all the *expression of a definite way of life*, the self-expression of the subject. As Marx put it in the *German Ideology*, labour

must not be considered simply as being the reproduction of the physical existence of the individuals. Rather, it is a definite form of activity of these individuals. A definite form of expressing their life, a definite *mode of life* on their part. As individuals express their life, so they are. What they are, therefore, coincides with their production, both with *what* they produce and with *how* they produce (Marx, 1998; p. 37; italics in the original).

Labour, in this general ontological sense, is *productive activity*: the production and reproduction of life, subsistence and human existence. In a remarkable passage of the *Economic and Philosophic Manuscripts*, Marx (2007, p. 121) asks: "What is life if not activity?"

Labour, hence, is something essentially tied to human nature. Because we are natural beings, we are creatures of needs. And because we are creatures of material, spiritual, aesthetic and other needs, we produce in order to fulfill those needs. But in producing to fulfill our needs we do not merely satisfy our survival requirements but produce human existence itself. This is what makes us *homo laborans*. To fail to grasp the crucial role that labour plays in Marx's philosophy would make unintelligible his concept of alienation. As Sayers contends,

Marx's theory of labour is not self-evident, nor is it based upon mere metaphors or images. It is a central element of a systematic philosophical theory of the relation of human beings to nature in which the concept of labour plays a fundamental role (Sayers, 2011; pp. 33–34).

2.3. The objects of human labour

Humans produce objects and also create new needs. If rats are doing the same things that they did thousands of years ago, individuals, by contrast, have produced, during the same time, a myriad of new objects, projects, and needs (Tulviste, 1991). By creating these objects and needs, which are objects of human life-activity, that is, objects of labour, humans create an *objective world*—a world outside themselves. Through the production of an *objective world*—the world of objects, and needs and wants, which is also the world of aesthetic experience and enjoyment—humans *inscribe* and *recognize* themselves in what they have thus produced—their work and reality. This process of inscribing and recognizing oneself in the human products of labour is what Marx terms *objectivation* (*Vergegenständlichung*, which should be distinguished from *Objektivierung*, objectification).

Objectivation is a species' act. "The object of labour is, therefore, the *objectivation of man's species life*" (Marx, 2007; p. 123). Objectivation is the result of a process of *expression* (*Äusserung*) of the self, an achievement or realization.

To say that man (sic) is a *corporeal*, living, real, sensuous, objective being full of natural vigor is to say that he has *real, sensuous, objects* as the objects of his being or of his life, or that he can only *express* his life in real, sensuous objects (Marx, 1988; p. 154).

Since labour is itself that through which human objects are produced, labour becomes itself an object (Fischbach, 2007; p. 31).

Man (*sic*) makes his life-activity itself the object of his will and of his consciousness. He has conscious life-activity. It is not a determination with which he directly merges. Conscious life-activity directly distinguishes man from animal life-activity. It is just because of this that he is a species being (Marx, 1988; p. 76).

How, then, does alienation occur? In the *Economic and Philosophic Manuscripts*, alienation appears first as the alienation of the individuals from the product of their labour. Marx (1988, p. 71) notes that “the worker is related to the *product of his (sic) labour* as to an *alien* object.” The problem is not that the worker does no longer produce. The worker produces, but what she produces has become something alien. Marx goes on to say that:

The worker puts his (*sic*) life into the object; but now his life no longer belongs to him but to the object. . . Whatever the product of his labour is, he is not The *alienation* of the worker in his product means not only that his labour becomes an object, an *external* existence, but that it exists *outside him*, independently, as something alien to him, and that it becomes a power on its own confronting him; it means that the life which he has conferred on the object confronts him as something hostile and alien. (Marx, 1988; p. 72; emphasis in the original)

Alienation, hence, does not consist in that the individual’s work becomes an object, nor is it that the individual’s life becomes crystalized in the object. It is indeed in the nature of labour to produce something and, as previously mentioned, that the individual expresses and recognizes herself in it. Alienation consists instead in the precise fact that the produced object is no longer the *individual’s expression*. While producing, the individual is without objects in the anthropological sense mentioned above, that is, without objects in which she expresses and recognizes herself. What is alienating here is hence the loss of expressivity of life in the object. To put it succinctly, alienation is the loss of objectivation. The loss of objectivation – i.e., the loss of self-expression in the object – can only be alienating for a species, like ours, for which objectivity is part of its nature. Instead of expression, achievement, and self-realization, we have a product that becomes a *thing*.

So far we have discussed the first meaning of alienation as Marx articulates it in the *Economic and Philosophic Manuscripts*: the loss of expressivity—i.e., the loss of the self. There is a second meaning of alienation. The loss of expression in a product can only occur through the process of its production. “The loss of expressivity [estrangement] is manifested not only in the result but in the *act of production*—within the *producing activity* itself. . . The product is after all but the summary of the activity, of production” (Marx, 1988; pp. 73–74, amended; emphasis in the original). As a result, alienation can not only be predicated on individuals and things produced through labour but of labour itself: alienating labour is a labour that has turned into “thingnification,” that is, a process that fails to convey the expression of the individual’s own being in the outcome of the process. In other words, it is a process that has lost its objectivation nature. As Fischbach notes, “This is the fact of alienation: the activity of objectivation ends up, for the worker, to be without object,” while from the viewpoint of the activity, “the objectivation activity leads to non-objectivity. . . so that in the end what the alienated worker is deprived of is work itself as an object necessary for her own life” (Fischbach, 2007; p. 31; my translation). To sum up, the second meaning of alienation is *estranged labour*, that is, the alienation of the individual from her life-sustaining activity.

Marx pointed out a series of antinomies that derive from alienated labour:

What, then, constitutes the alienation of labour?

First, the fact that labour is external to the worker, i.e., it does not belong to his (*sic*) essential being; that in his work, therefore, he does not affirm himself but denies himself, does not feel content but unhappy, does not develop freely his physical and mental energy but mortifies his body and ruins his mind . . . He is at home when he is not working, and when he is working he is not at home. His labour is . . . therefore not the satisfaction of a need; it is merely a means to satisfy needs external to it (Marx, 1988; p. 74).

A third meaning of alienation is the alienation of the individual from the human species. Since labour is “the *objectivation of man’s (sic) species life*” (Marx, 2007; p. 123), the alienated individual is alienated from her species. Estranged labour thus turns “Man’s (*sic*) *species being*, both nature and his spiritual species property, into a being *alien* to him, into a *means* to his *individual existence*” (Marx, 1988; pp. 77–78). Production is no longer *generic*; it is no longer *species’ production*. It has lost its human dimension and turns into a mere means of individual subsistence. As a result, an ontological link between the subject and the human species is broken.

There is a fourth meaning of alienation: the alienation of the individual from other individuals. “An immediate consequence of the fact that man (*sic*) is estranged from the product of his labour, from his life-activity, from his species being is the estrangement of man from man” (Marx, 1988; p. 78). Individuals no longer cooperate in the production of human existence. Instead, in the capitalist framework that Marx endeavoured to understand, individuals *compete* with each other in the pursuit of self-interest and wealth.

3. Progressive model of education

As mentioned in the Introduction, the progressive child-centred educational model of education arose about the same time as the transmissive program. But instead of orienting its pedagogical actions towards the societal needs of industrial production, its aim was to attend to the students’ needs and intellectual potentialities. As two advocates of the child-centred model put it, the aim is to concentrate “upon the development of personality, individuality.” (Rugg and Shumaker, 1969; p.

vii). As opposed to the transmissive model, the child-centred school focused on fostering “freedom to develop naturally” and on the “individual capacity for creative self-expression” (pp. 57–58).

The child-centred school grew out of the psychology of its own time, one that characterizes the individual in general and the student in particular as a rational agent, a kind of Cartesian cogitator that Canadian psychologist Jack Martin (2004), in a famous article, “The educational inadequacy of conceptions of self in educational psychology,” summarizes as follows:

an individual . . . constituted of componential mechanisms, processes, parts, and strategies . . . capable of simultaneous action and reflection on this action, much like a stereotypic scientist in close scrutiny and judgment of experimental phenomena of interest . . . [An individual] whose most vital resources are apparently available within its detached internality . . . a self that already knows its business, one that requires only a facilitative grooming to become more fully socialized and intellectually engaged (Martin, 2004; pp. 193–194, 197).

Such a concept of the student assumes that the origin of meaning, knowledge, and intentionality is located within, and must come from, the individual.

This rational and romantic conception of the modern individual that serves to articulate the early 20th century idea of the progressive student does not come out of the blue. It is a historical invention. Morris (1972) locates the first steps of this invention in the late Middle Ages. Traces of this historical invention are also found in the Renaissance when some merchants and bankers, emancipated from traditional feudal structures, started conceiving of themselves as owners and crafters of their own destiny. In his analysis of Flemish Renaissance paintings, the Bakhtinian specialist Tzvetan Todorov (2000) shows how the new idea of the individual led to a new genre in the visual arts – the *portrait* – where individuals bear specific, distinguishing traits (e.g., names, biographical data, personality characteristics). However, it was only in the 17th and 18th centuries that Descartes, Kant, and other philosophers articulated, in its clearest form, the modern idea of the individual as a sovereign, rational, autonomous subject. During the 19th and 20th centuries, the idea was gradually integrated into the educational context by the progressivist camp, leading to its chief idea that “knowledge is . . . [a] personal acquisition, obtained by learning from experience” (Darling & Nordenbo, 2002; p. 298).

“Progressivism” evolved differently in Germany, England, Russia, the United States, Canada, and other countries, stressing with various nuances the learner’s autonomy, and the role of investigation and play (Cremin, 1961; Holmes, 1991; Rohrs and Lenhart, 1995; Shirley, 1992). Bit by bit, from the aforementioned idea of knowledge as personal acquisition, emerged the idea of the student as someone who is not there to be taught but rather someone expected to think and learn through his/her own deeds. The progressive child-centred reform became crystalized in the idea that knowledge is something that each student has to construct by him/herself—as opposed to something that can be passed on or learned from others (von Glasersfeld, 1995; Thompson, 2014). Within this context, leading the students towards an idea that did not come from them was often understood as constraining the students’ freedom and autonomy: it was seen as coercing the students’ own solutions and imposing the teachers’ meanings upon them (Lerman, 1996; Radford, 2012).

In searching to promote the student’s freedom and autonomy, progressive educators – since the inception of the movement up to its contemporary Piagetian constructivist version – have built their pedagogy through a dichotomy between teachers and students. For instance, Rugg and Shumaker (1969) talk of “child initiative vs. teacher initiative” (p. 56). This dichotomy has offered progressivism the conceptual and methodological basis for pedagogical action. It serves to envision an allegedly emancipatory form of classroom knowledge production where students take control of, and ownership over, the ideas they produce. Explaining the concept of mathematics classroom discourse, Chapman says:

Discourse, as promoted in current reform perspectives of mathematics education, is not about classroom talk intended to convey exact meaning from teacher to student; instead, it is about communication that actively engages students *in a way that allows them to construct new meanings and understandings of mathematics for themselves*. (Chapman, 2009; p. 297; my emphasis).

In a similar vein, Lee (2006) also stresses the need to engage students in classroom discourse in ways that “They take *ownership* of *their* ideas and become able to *control* and use them” (Lee, 2006; pp. 7–8; my emphasis). Thornton and Reynolds (2006) complain that many of the TIMSS 1999 video classrooms (<http://www.timssvideo.com/timss-video-study>) feature “reproductive discourse, with the apparent goal of students being to guess what was in the teacher’s mind” (p. 275). They contrast these “teacher-centred” classrooms with a classroom they investigated. They remark: “students see themselves as active participants in learning, who have power over both the mathematics and the discursive practices of the classroom” (Vol. 5, p. 277). They go on to say: “Power is located with students” (p. 277). With power on the side of the students, the teacher’s authority has vanished and space has been created to ensure the students’ emancipation.

Let me now take a closer look at the forms of classroom knowledge production advocated by the progressive reform movement. The reform’s view of mathematics classroom knowledge production revolves around the idea of students’ participation. Although this is certainly a commendable idea, we see that students’ participation is usually understood against the backdrop of a *dichotomy* between teachers and students. This dichotomy, the progressive pedagogues feel, is required in order to guarantee the expression and practice of the students’ intellectual freedom and autonomy. The students have to gain *control* over, and *ownership* of, knowledge and its mechanisms of production. In the Theory of Didactical Situations (Brousseau, 1997), teachers are advised not to show the students the answer. As Brousseau notes, if the teacher shows the student how to solve the problem, the student “does not make it her own” (Brousseau, 1997; p. 42. Since how to

solve the problem is not “her own,” in this line of thinking the student cannot be said to have achieved a genuine mathematical understanding.

Of course, I am not saying that teachers should show the answer to the students. I am highlighting the teacher/students dichotomy that serves to produce a progressive discourse and its pedagogy. Now, if we push our line of inquiry a bit further, we notice that in the contemporary progressive forms of knowledge production the students are conceived of as *private owners*. That is, the students are conceived of as subjects of a specific form of knowledge production that equates *doing* with *belonging*: what belongs to the students is what they do by themselves. What they do not do by themselves does not belong to them (Radford, 2014).

Within this context, *understanding* is featured as the epistemic equivalent of belonging: Understanding is the product of the students’ own cogitations and deeds. The students’ understanding is the product of their own labour—not the teachers’. How indeed – the question runs – could students understand something that they did not themselves produce? Along the lines of the student-as-private-owner paradigm, the students are predicated as constructing their own knowledge, and in their interaction with others they are considered as *negotiating* meanings and as *exchanging* ideas (Radford & Roth, 2011).

Let me briefly refer to two examples.

In a Grade 1 class, Ms. Smith, the teacher, deems appropriate to encourage her students to share their solutions. However, she is always afraid of making comments on the students’ solutions. As McClain and Cobb report, Ms. Smith “did not want to be an *authority* in the classroom. She made it clear that judging the worth of students’ contributions violated basic tenets of her nonimpositional educational philosophy” (McClain and Cobb, 2001, pp. 2247–2548; emphasis added).

In another Grade 1 class, Melissa was asked to solve subtraction problems. Melissa’s method was different from the teachers’ method. The teacher is afraid to induce Melissa into *his* method. After having given several problems to the student, the teacher decided to stop the line of enquiry. He “inferred that if he persisted he might merely train her to behave as *he* desired rather than encourage her to express *her* number concept in a novel way” (Cobb, 1988; p. 94; italics added).

These two examples show the tension between autonomy and authority that results from the assumed teacher vs. students dichotomy that runs along the progressive child-centred educational model: any comment, question, or action from the teacher is perceived as influencing the students’ *own* ideas—a teacher’s deed that jeopardizes the students’ learning autonomy. At first sight, it appears as if the student, as producer and titleholder of her own ideas, is not alienated. However, on closer examination – and the previous examples show it clearly – we see that what the student produces is not an *objective object*, that is, an object existing outside herself. Here the student is one with her products of labour. The student of the progressive model is the identity of subject and object—an *unmediated* being. The student becomes her producing process and product. Her production is a pure *subjective activity*. Her product is imprisoned in subjective labour. Her alienation is to be found in her non-objective being. It is worthwhile quoting Marx again on this point: “A being which does not have its nature outside itself is not a *natural* being . . . Its being is not objective” (Marx, 1988; pp. 154–155). It is in fact a being without objects. And without objects, the individual is deprived of objectivity. She cannot express herself in the world of the human species, in the species’ cultural-historical reality.

The alienation of the student of the child-centred progressive model consists in that her productive alienated activity “departs from its proper function of humanly mediating in the subject-object relationship between man (sic) and nature” (Mészáros, 1972; p. 82). Her alienated activity deprives her of any objectivity, starting with that of her own being. It is an activity that fails to put her in contact with the objective world, that is, her species’ world, and with other individuals. The student of the progressive model is hence alienated in the third and fourth senses discussed in the previous section. The product of the progressive model student is not labour as “*Lebensausserung*” (i.e., as manifestation of life) but as “*Lebensentausserung*” (i.e., as alienation of life). Her own object, the object that she has produced through her own activity, has to live a precarious and lonely existence. Its mode of existence is not real. It exists in a chimerical sense only. The idea I_A produced by an individual A cannot be compared to the idea I_B produced by an individual B. It is impossible to ascertain whether $I_A \propto I_B$ or $I_A \not\propto I_B$. What can be said is that A and B *seem* to have (or not to have) “nonconflicting mutual interpretations” (see Thompson, 2014; p. 99). As a result, what the student has produced can only be *taken-as-shared*, and not as genuinely socially shared. No student and no teacher can know what another student has produced. Here, production is a subjective matter. Teachers and students do not share a mathematical *reality*. They cannot. “[T]he teacher and students elaborate the taken-as-shared mathematical reality that constitutes the basis for their ongoing communication” (Cobb, Yackel, and Wood, 1992, p. 10). But this communication is merely assumed, hypothetical, illusory. It is the communication of incommunicable beings. At the end of the process, alienated from the species and other students, what is left to the individual student? “Nothing but a pure activity, that is to say, a pure abstract work force that does not have any objective conditions for its actualization” (Fischbach, 2007; p. 35). Alienation consists here “in the fact of not being able to understand and live differently than a purely active but absolutely powerless subject” (Fischbach, 2007; p. 38).

4. Synthesis and concluding remarks

In this article I referred to two main educational models that have informed, to a considerable extent, the teaching and learning of mathematics during the 20th century—the transmissive and the progressive model. I have argued that the transmissive educational model reduces the teachers to bureaucratic agents who labour to implement a prescribed curriculum and reduces the students to passive learners. In the Introduction I contended that the educational practice derived from this model is alienating. The theoretical discussion of alienation presented in the second section lets us see that

the alienation in this model resides in the fact that the forms of classroom knowledge production (based on drill, repetition, memorization, etc.) alienate the students from the objects the students produce. These objects are not the students' *expressions*. The students cannot *recognize* themselves in what they have produced.

In the third section I examined the progressive educational model and argued that the educational practice it promotes is alienating too. What is, then, their difference? The difference is not to be found in the conception of students they convey. In both models the students are conceived of as private owners. The difference is rather the following. In the transmissive model the forms of knowledge production are imposed upon the students. As a result, the students do not recognize themselves in their object of labour. They are estranged from their labour and the objects of labour. In the progressive model the forms of knowledge production remain subjective. Naturally, this does not amount to saying that some students may experience individual enjoyment in solving mathematical problems. However, if – as in the example of Ms. Smith discussed above – the teacher refuses to invite the students and provide them with the possibilities to enter into contact with, and consider, *other* mathematical methods, the ensuing individual experiences may remain trapped in their subjectivist stance. The Melissa example illustrates this point: Melissa may very well enjoy solving subtraction problems through her own ideosyncratic methods. But left there, Melissa, in Marx's anthropological conception of the individual, is led to negate her most important human characteristic, namely the possibility of finding the object that fulfills her intellectual and emotional needs *outside of herself*: in the common world of history and culture.

We have seen that both educational models pursue different goals. While the transmissive model seeks to respond to social needs, the progressive model seeks to promote self-expression and the growth of the student's cognitive structures. While in the former the teacher assumes power and the students are relegated to a passive role, in the latter the students assume power and the teachers are relegated to an ancillary role (e.g., Chapman, 2009; Lee, 2006; Nguyen, 2012; Thornton and Reynolds, 2006). Although the locus of power and agency changes from one model to the other, the difference is not really structural. It is hence not surprising that both practices end up being equally alienating. Yet, the sense of alienation is not the same. If we refer to the four senses of alienation that we discussed in the second section, it becomes clear that the transmissive model of education produces practices that are alienating in the first two senses, while the progressive model produces practices that are alienating in the last two senses. In the end, it turns out that the progressive educational model is not the antithesis of traditional teaching but its dual model.

What could be done, then, to try to overcome alienation in the teaching and learning of mathematics? The alienation in the mathematics classroom is but a reflection of the alienating forms of production of society at large. There is indeed a deep relationship between school, work, and the social system as a whole. This relationship is such that, in practice, education has become an applied political branch of the current global capitalist economy and its forms of production. As Jones has put it,

The problem is that the school as an institution has developed to fit with, and to serve, the social system of which it forms a part.

What happens in school can be explained only by the reciprocal relations between school and what is going on outside it (Jones, 2011; p. 369).

Schools are driven by a cult of efficiency and corporatism (Callahan, 1970) that reduces the students to private owners and consumers. LaVallee (2014) notes that “‘public’ schools have not only had their educational practices and curriculum taken over by edu-businesses, but schools’ hidden curricula have also been likewise infiltrated by capitalism.” In this context, genuine learning and genuine work become commodified learning and commodified work (Jones notes “work” and “learning” in quotation marks to refer to the commodified versions of genuine work and learning). The characteristics of commodified learning (i.e., “learning”)

are not the result of defective design of the school system but the tried and tested, *inevitable*, and *ineradicable* means of producing the “outputs” that society requires of its “education” system. Ineradicable because *to change education means to change work*; to go from “learning” to learning involves going from “work” to work. To make education “whole again” – to restore its vital and creative powers – means to make work whole again. (Jones, 2011; p. 369).

As we can see from the previous remarks, the problem of alienation is not a specific educational problem. As such, its eventual solution is not merely educational. Yet, education should not be an engine of societal reproduction—quite the contrary. If there is a central place within social institutions to start making changes, this place is education, and in particular the school and the classroom. As Mészáros contends,

The capitalistically reified social relations of production do not perpetuate themselves *automatically*. They succeed in this only because the particular individuals “*interiorize*” the outside pressures: they adopt the overall perspectives of commodity-society as the unquestionable limits of their own aspirations (1972, p. 289).

Schools, and teachers in particular, have here a fundamental role to play in order to envision and put into practice new forms of human collaboration and forms of knowledge production that would better fit a genuine emancipatory critical agenda. Instead of being merely concerned with promoting individual achievement and traditional forms of academic success (the transmissive model) or modes of subjective expression and individual intellectual growth (the progressive model), teachers would be critically working with their students towards new radical senses of knowledge, self, society, and truly socially inclusive citizenship (Giroux, 1986).

How could this be achieved? What forms of classroom knowledge production can support such an emancipatory project?

In a reading note, Marx wrote:

Let us suppose that we had produced as human beings. In that event . . . in the individual expression of my own life I would have brought about the immediate expression of your life, and so in my individual activity I would have directly *confirmed* and *realized* my authentic nature, my *human, communal* nature. Our productions would be as many mirrors from which our natures would shine forth. (Quoted in Sayers, 2011; p. 99; emphasis in the original; see also Marx, 1968, p. 33).

Instead of being driven by self-interest, production–Marx is suggesting—should rather be driven towards a communal realization—the realization of the species through the realization of the individual. In the course of production, the individual produces herself as a human being and, in doing so, she produces *for* and *with* others. What is thus produced is “the direct embodiment of [her] individuality [and] simultaneously [her] own existence for the other” (Marx, 1988; p. 104). And since I cannot genuinely recognize the embodied existence of the other in the product thus produced without at the same time recognizing myself in that product, what is produced is, hence, the joint production of human existence.

Elsewhere (Radford, 2012, 2013, 2014) I have pleaded for a need to start with a communal ethical project that may orient the forms of classroom human collaboration. They may serve to redefine the forms of classroom knowledge production in order to go beyond the private-ownership paradigm discussed above. In this project, mathematical knowledge is not something to be possessed or personally constructed. It is rather something to visit, to attend, and to enjoy. It is rather a virtual space—a space of culturally and historically constituted potentialities: dynamic and always in transformation meanings that open up possibilities for new forms of action, reflection, and interpretation.

The ethical forms of human collaboration that I am advocating are driven by a general attitude towards the world and serve to configure the teachers’ and students’ *joint labour* in the classroom (Radford, 2014) that blurs the borders that separate the teachers from the students. Teachers and students labour *in concert as one*. The classroom appears as a public space of debates in which the students are encouraged to show openness towards others, responsibility, solidarity, care, and critical awareness. The classroom indeed appears as a space of encounters where teachers and students become what Freire called “presences in the world” (Freire, 2004, p. 98). That is to say, the classroom appears as a space of encounters, dissidence and subversion, where teachers and students become individuals who are more than in the world, individuals with a vested interest in one another and in their joint enterprise; individuals who intervene, transform, dream, apprehend, suffer, and hope *together*.

Acknowledgments

This article is a result of a research programs funded by the Social Sciences and Humanities Research Council of Canada (SSHRC/CRSH).

References

- Baldino, T., & Cabral, T. (2011). The productivity of students’ schoolwork. *Journal for Critical Education Policy Studies*, 9(2), 70–84.
- Brodie, K., Slonimsky, L., & Shalem, Y. (2010). Called to account. In U. Gellert, E. Jablonka, & C. Morgan (Eds.), *Proceedings of the 6th conference on mathematics education and society* (pp. 180–189). Berlin: Freie Universität Berlin.
- Brousseau, G. (1997). *Theory of didactical situations in mathematics*. Dordrecht: Kluwer.
- Brown, T. (2011). *Mathematics education and subjectivity*. NY: Springer.
- Callahan, R. (1970). *Education and the cult of efficiency*. Chicago: The University of Chicago Press.
- Chapman, O. (2009). Discourse to empower self in the learning of mathematics. *Proceedings of PME 33*, 2, 297–304.
- Cipriani, J. (2015). Kensington keyfolio with X3. *MacWorld*, 2015(August), 71–73.
- Cobb, P. (1988). The tension between theories of learning and instruction in mathematics education. *Educational Psychologist*, 23(2), 87–103.
- Cobb, P., Yackel, E., & Wood, T. (1992). A constructivist alternative to the representational view in mathematics education. *Journal for Research in Mathematics Education*, 23(1), 2–33.
- Cremin, L. (1961). *The transformation of the school. Progressivism in American education, 1876–1957*. New York: Vintage Books.
- Darling, J., & Nordenbo, S. (2002). Progressivism. In N. Blake, P. Smeyers, R. Smith, & P. Standish (Eds.), *The philosophy of education* (pp. 288–308). Oxford: Blackwell.
- Dearden, R. F. (1972). Autonomy and education. In R. F. Dearden, P. H. Hirst, & R. S. Peters (Eds.), *Education and the development of reason* London: Routledge & Kegan.
- Dearden, R. (1975). Autonomy as an educational ideal I. In S. Brown (Ed.), *Philosophers discuss education* (pp. 3–37). London: The MacMillan Press.
- Fischbach, F. (2007). *Pre'sentation [Introduction]*. In *Karl Marx: manuscrits e'conomico-philosophiques de 1844 [Economic and philosophic manuscripts of 1844]*. Paris: Vrin7–71 [F. Fischbach, Trans.].
- Franci, R. (1988). L'insegnamento della matematica in italia nel tre-quattrocento. *Archimedes*, 4, 182–194.
- Freire, P. (2004). *Pedagogy of indignation*. Boulder, Colorado: Paradigm Publishers.
- Freire, P. (2005). *The pedagogy of the oppressed*. New York: Continuum.
- Giroux, H. (1986). Authority, intellectuals, and the politics of practical learning. *Teachers College Record*, 88(1), 22–40.
- von Glasersfeld, E. (1995). *Radical constructivism*. London: The Falmer Press.
- Holmes, L. (1991). *The Kremlin and the schoolhouse*. Bloomington: Indiana University Press.
- Jones, P. (2011). The living and the dead in education: commentary on Julian Williams. *Mind, Culture, and Activity*, 18(4), 365–373.
- Katz, M. (1971). *Class, bureaucracy, and schools*. New York: Praeger Publishers.
- Kramer, S. (1949). Schooldays: a Sumerian composition relating to the education of a scribe. *Journal of the American Oriental Society*, 69(4), 199–215.
- Labaree, D. (2005). Progressivism, schools and schools of education: an American romance. *Paedagogica Historica*, 41(1–2), 275–288.
- LaVallee, T. (2014). Conquering the corporate colonial occupiers of public education: an intellectual application of guerrilla warfare theory to begin a revolution to win the revolution. *Paper presented at the 2014 AERA meeting, april 3–7, 2014*.
- Lave, J., & McDermott, R. (2002). Estranged learning. *Outlines*, 1, 19–48.
- Lee, C. (2006). *Language for learning mathematic*. London, UK: Open University Press.

- Lerman, S. (1996). Intersubjectivity in mathematics learning: a challenge to the radical constructivist paradigm? *Journal for Research in Mathematics Education*, 27(2), 133–150.
- Lukács, G. (2012). *Ontologie de l'être social. L'idéologie, l'aliénation [Ontology of the social being. Ideology, alienation]*. Paris: Delga.
- Martin, J. (2004). The educational inadequacy of conceptions of self in educational psychology. *Interchange: A Quarterly Review of Education*, 35, 185–208.
- Marx, K. (1968). In M. Rubel (Ed.), *Oeuvres. Économie II [Works. economy, vol. 2]* Paris: Pléiade.
- Marx, K. (1976). *Capital (vol. 1)*. (B. Fowkes, trans.). London: Penguin Books.
- Marx, K. (1988). *Economic and philosophic manuscripts of 1844*. (M. Milligan, trans.). New York: Prometheus Books [Original work published 1932].
- Marx, K. (1998). *The German ideology, including theses on Feuerbach and introduction to the critique of political economy*. New York: Prometheus.
- Marx, K. (2007). *Manuscripts économique-philosophiques de 1844 [Economic and philosophic manuscripts of 1844]*. (F. Fischbach, Trans.). Paris: Vrin [Original work published 1932].
- McClain, & Cobb, P. (2001). An analysis of development of socio mathematical norms in one first grade classroom. *Journal for Research in Mathematics Education*, 32(3), 236–266.
- Mészáros, I. (1972). *Marx's concept of alienation*. New York: Harper & Row.
- Morgan, J. (1996). A defence of autonomy as an educational ideal. *Journal of Philosophy of Education*, 30(2), 239–252.
- Morris, C. (1972). *The discovery of the individual, 1050–1200*. New York, NY: Harper & Row.
- Neill, A. S. (1992). *Summerhill school: a new view of childhood*. New York: St. Martin's Griffin [Original work published 1960].
- Nguyen, N. T. (2012). Let students take control! Fostering learner autonomy in language learning: an experiment. *2012 international conference on education and management innovation* (pp. 318–320).
- OECD (2009). Creating effective teaching and learning environments. www.oecd.org/edu/school/43023606.pdf.
- Ontario Ministry of Education (2005). *The Ontario curriculum. Mathematics*. Ottawa: Queen's Printer for Ontario.
- Parker, F. (1990). The Quincy method. *American Journal of Sociology*, 6(1), 114–120.
- Piaget, J. (1973). *To understand is to invent. The future of education*. New York: Grossman.
- Radford, L. (2012). Education and the illusions of emancipation. *Educational Studies in Mathematics*, 80(1), 101–118.
- Radford, L. (2013). Sumisión, alienación y (un poco de) esperanza: Hacia una visión cultural, histórica, ética y política de la enseñanza de las matemáticas [Submission, alienation and (a bit) of hope: Towards a cultural, historical, ethical and political vision of mathematics teaching]. *I CEMACYC*. <http://luisradford.ca>.
- Radford, L. (2014). De la teoría de la objetivación [On the theory of objectification]. *Revista Latinoamericana de Etnomatemática*, 7(2), 132–150.
- Radford, L., & Roth, W.-M. (2011). Intercorporeality and ethical commitment. *Educational Studies in Mathematics*, 77(2–3), 227–245.
- Rohrs, H., & Lenhart, V. (1995). *Progressive education across the continents: a handbook*. Berne: Peter Lang.
- Rugg, H., & Shumaker, A. (1969). *The child-centered school*. New York: World Book Company [Original work published 1928].
- Williams, J. S. (2011). Toward a political economic theory of education. *Mind, Culture, and Activity*, 18(3), 276–292.
- Williams, J. (2012). Use and exchange value in mathematics education: contemporary CHAT meets Bourdieu's sociology. *Educational Studies in Mathematics*, 80, 57–72.
- Sayers, S. (2011). *Marx & alienation. Essays on hegelian themes*. Hampshire, UK: Palgrave: Macmillan.
- Schacht, R. (1970). *Alienation*. New York: Doubleday.
- Shirley, D. (1992). *The politics of progressive education*. Cambridge, Harvard: University Press.
- Spinoza, B. (1989). *Ethics including the improvement of the understanding*. (R. Elwes, Trans.). Buffalo: Prometheus [Original work published 1667].
- Thompson, P. (2014). Constructivism in mathematics education. In S. Lerman (Ed.), *Encyclopedia of mathematics education* (pp. 96–100). New York: Springer.
- Thornton, S., & Reynolds, N. (2006). Analysing classroom interactions using critical discourse analysis. *Proceedings of PME 30*, 5, 273–280.
- Todorov, T. (2000). *Eloge de l'individu [Praise of the individual]*. Paris: Adam Biro.
- Tulviste, P. (1991). *The cultural-historical development of verbal thinking*. New York: Nova Science Publishers.
- Tyack, D. (1974). *The one best system*. Cambridge: Harvard University Press.