CERME9

Proceedings of the Ninth Congress of the European Society for Research in Mathematics Education

Editors: Konrad Krainer, Naďa Vondrová

Organized by: Charles University in Prague, Faculty of Education

erme

european society for research in mathematics education

Editors: Konrad Krainer, Naďa Vondrová

Publisher: Charles University in Prague, Faculty of Education and ERME

Place: Prague, Czech Republic

Year: 2015

ISBN: 978-80-7290-844-8

Solving equations: Gestures, (un)allowable hints, and the unsayable matter

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The goal of this paper is to contribute to the research on the introduction of solving linear equations. Subsumed in the "Comparing and Contrasting" category introduced in Prediger, Bikner-Ahsbahs and Arzarello's (2008) networking strategies, we contrast two episodes informed by two distinct theories and offer an insight into the teacher's role in introducing new knowledge in the classroom and the meaning-making narratives of hands-on didactic approaches to algebra. We examine the teachers' gestures and hints and what appears to be unsayable in the teacher-students' interaction.

Keywords: Linear equations, networking theories, teaching-learning, gestures.

INTRODUCTION

In the discussion of a particular classroom episode we found that our distinct research projects resorted to a very similar approach to introducing the process of solving linear equations. However, the equation's contexts, as well as the student-teacher interaction – in particular what the teachers were willing to say – were substantially different. The analysis of the episodes through the lenses of the theories that informed the design and implementation of the tasks as well as the interpretation of the data – the Gathering-Connecting-Structure-seeing (GCSt) model (Bikner-Ahsbahs & Halverscheid, 2014) and the Theory of Objectification (Radford, 2007) – led us to achieve a deeper understanding of the teaching-learning that is usually involved in a hands-on introduction to solving linear equations. In particular, resorting to a comparative analysis, which corresponds to what Prediger, Bikner-Ahsbahs, and Arzarello (2008) identify as "Contrasting and Comparing" theories, gave us new insights into the constraints and affordances with which teachers are endowed in their interaction with the students. Our comparative analysis also makes visible the limits of what is considered to be unsayable (i.e., that which would be *improper* to mention by the teacher) and how this unsayable shapes the contour of the space and kind of gestures teachers deploy in the interaction with the students. Last but not least, we reached a new awareness about the learning impact that the didactic context has as a potential horizon of narrative-based meaning production in the introductory steps in learning to solve linear equations.

RESEARCH BACKGROUNDS AND THE CONTEXT OF THIS RESEARCH

The authors of this paper both study the development of algebraic thinking. The first author is interested in algebraic structure sense (Hoch & Dreyfus, 2010) as a dynamic entity. The second author is interested in the social co-transformative sense-making processes through which the students gradually become critically acquainted with historically constituted cultural meanings and forms of reasoning and action.

In the course of a discussion about two classroom episodes dealing with students solving linear equations, one informed by research following the GCSt model and the other following the Theory of Objectification, an important distinction became apparent between the social-constructivist first theory that leaves lots of freedoms to the actors in the classroom on the one hand, and on the other hand the second theory that stresses the importance of the cultural nature and basis of the mathematical content – the idea that the algebra we teach in school is not a natural developmental outcome, but the outcome of a historical-cultural evolution. This important distinction turned out to set limits to what teachers can say in the classroom and thus defines what they cannot say – the unsayable. It also has an impact on the teacher's hints, gestures, and their meaning. This paper is an attempt to describe what we learned from comparing the same phenomenon – teaching-learning linear equations – and to formulate it in terms of the specificities of the target algebraic knowledge and the teacher's role in the introduction of a new concept.

THEORETICAL BACKGROUND

The Theory of Objectification (TO) considers knowledge as a historically developed cultural synthesis of actions and reflections (e.g., how to solve linear equations), which is concretized or realized in certain activities. In most cases, and especially in school, students do not enter this process on their own. Teachers and students engage in joint activity in order to make the cultural synthesis of actions and reflection noticeable to the students. In doing so, knowledge becomes an object of consciousness and thought. In the TO, the teacher's and students' joint activity or joint labour is referred to as "teaching-learning activity." The joint nature of teaching-learning does not mean that teachers and students play the same role. There is an asymmetrical division of labour that makes teaching-learning a tense process (Radford & Roth, 2011) filled with emotionality and fragility. Under this premise, the TO can be used as an instrument to thoroughly plan teaching-learning, however always with an awareness for the ever-developing relation between the actors.

The Gathering-Connecting-Structure-seeing (GCSt) model (Bikner-Ahsbahs & Halverscheid, 2014) aims at describing the epistemic actions that are carried out in so called interest-dense situations. In these situations, the class or parts of it collectively participate in the name-giving epistemic actions: *Gathering* refers to the collection of bits of mathematical meaning in the given situation, e.g. empirical values. These are then *connected* with limited scope. In the example that may be a table or a graph. Based on the connections, the students may come to *see structures*, an event which is understood as constituting the construction of new knowledge. In the example, the students may see linearity as a feature of the examined function. Therefore, what qualifies as knowledge is not so much defined a priori, but rather by the observed behaviour of the students. This also implies a rather open task design and requires the teacher to be open towards the learning routes taken by the students.

METHODOLOGICAL CONSIDERATIONS

In the TO, learning is mediated by teaching-learning activities underpinned by a range of semiotic resources, such as signs (e.g., spoken and written language, diagrams), embodied actions (gestures, tactility, perception), and rhythm. Furthermore, the relationship between the involved individuals is seen as an important factor. In the GCSt model, the epistemic actions form the centre of the researchers' attention. As discussed above, no presuppositions are made about the nature of the actions, thus, depending on the context, they may cover the same semiotic resources that are of interest in the TO. As a result, to investigate learning, both the TO and the GCSt model privilege video analyses.

The analysis of the classroom episodes below is an instance of the "Comparing and Contrasting" category introduced in Prediger, Bikner-Ahsbahs, and Arzarello's (2008) networking strategies, seeking to conceptualize the role of the individuals, social interaction, and the specificities of the target knowledge.

DATA OVERVIEW

The data to be discussed here was originally collected in two projects with different foci and scopes. In the German project a grade 8 class (13–14-year-old students) in an integrated school in Bremen was filmed for about seven months in those lessons where algebraic structures were the target topic. The episode discussed here is from the very first of these lessons. In the Canadian project a Grade 2 class (7–8-year-old students) in Sudbury was followed for 5 years when the students were learning algebra. The episode discussed here is from the second day of the algebra lessons.

In both cases the solving of linear equations was introduced in a non-mathematical context that emulated the mathematical rules of linear equations. In the Canadian project, the students were presented a task that went as follows (the equation was illustrated by envelopes and cards on the blackboard see Figure 1):

Sylvain and Chantal have some hockey cards. Chantal has 3 cards and Sylvain has 2 cards. Her mother puts some cards in three envelopes making sure to put the same number of hockey cards in each envelope. She gives 1 envelope to Chantal and 2 to Sylvain. Now, both

Figure 1: The envelope equation can be seen on the blackboard in the background

children have the same amount of hockey cards. How many hockey cards are in an envelope?

In the German project, the students had matchbox equations on their tables that were introduced as puzzles. The students were told that on both sides of the equal sign there was the same total number of matches, some of them hidden in matchboxes. All of the matchboxes would contain the same amount of matches.

Abstracting from the two scenarios in both cases there were a) representations of unknown quantities with each of them representing the same quantity, and b) two sets composed of known and unknown quantities of objects with the same total quantity of objects in each set. Based on these two rules, linear equations may be presented in many more imagined contexts.

As implied by the GCSt model, the teacher of the class in Germany had instructions to give as little help as possible to allow the students develop their own ways of finding the correct solution. In contrast, the teacher in Canada had talked about the method of isolation on the previous day: the method that consists of removing same quantities from both sides of an equation in order to isolate the unknown.

For the purpose of this analysis, both transcripts were translated into English. Where the transcripts indicated important actions by the students or the teacher, stills were created from the video to accompany the transcript.

ANALYSIS OF THE EPISODES

The episode from the Canadian study is framed as a classroom discussion, while the episode from the German study shows a discussion solely between the teacher and two students who work on the task together. In both cases the students first followed an arithmetic trial-and-error approach and had already found and tested the correct solution to the equation. However, each teacher still wanted the students to get to the target algebraic approach.

As mentioned above, the teachers' instructions were very different in the two cases. The teacher in Canada engaged the class in a discussion about various methods to solve equations and was comfortable asking questions, submitting ideas and a new vocabulary. Thus, in the discussion below, which happened after the students suggested a trial-and-error method (see Radford, 2014), she suggests to use what the class has come to term the previous day the "isolating strategy," that is, the strategy based on removing equal terms from both sides of the equation. As we shall see, the teacher follows the still not fully linguistically articulated actions of Cheb and Cheb's pointing gestures, by moving the concrete envelopes and cards on the blackboard, making thereby apparent to the class:

94 Teacher: I'll go with the isolating strategy, Ok? Cheb? (*see Figure 1*)

95 Cheb: Umm... you remove one of Sylvain's envelopes and one of (*the teacher has already put the hand on the envelope, yet she stops to wait for the next part of C's utterance, turning her head towards C)* Chantal's envelopes

96 T: Is it important to remove the same thing from each side of the equal [sign]? (*she makes a two-hand gesture around the equal sign moving the hands to the bottom of the blackboard, where envelopes and cards have been moved, to indicate that removing action is happening in both sides of the equality*)

97 C: Yes. And you can remove the other envelope... Oh non! One of Sylvain's cards and one card from Chantal's (*the teacher removes one card from Chantal's, see Figure 2, left image*).

98 T: Aw! Again, one envelope, we remove one envelope (*see Figure 2, centre image, where the teacher points to the removed envelopes*), one card , [and] one card (*see Figure 2, right image, where the teacher touches the two removed cards*) ...

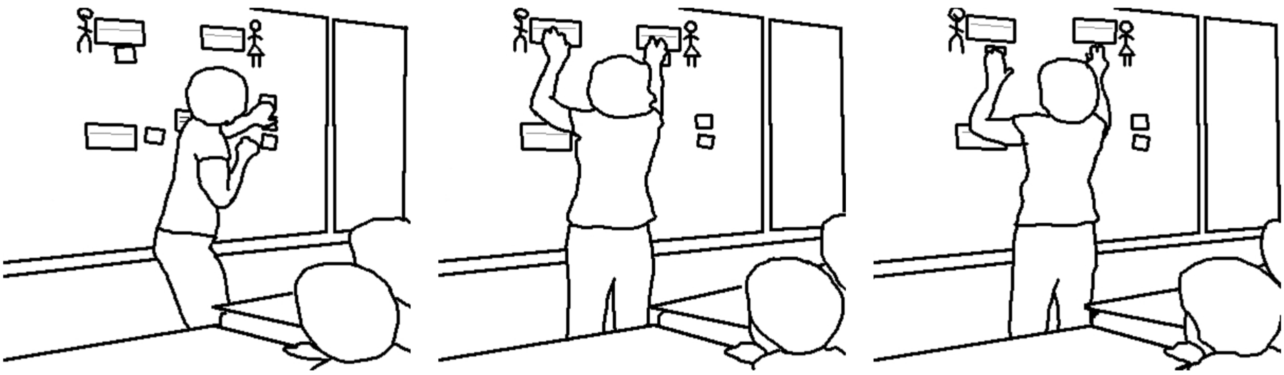

Figure 2: The teacher takes a card from the right side and moves it to the top of the blackboard (left image); she indicates the two envelopes (centre image) and then the two cards (right image) that have been removed

99 C: You remove one of Sylvain's cards and you remove one of Chantal's cards (*the teacher moves the cards towards the top of the blackboard*)

100 T: We remove another card of Chantal's cards. Then, that gives us...

101 C: The answer!

As we can see, the teacher moves the objects and performs the proposed actions. Where appropriate, she interrupts the flow of the discussion to emphasize for the whole class the algebraic conceptual meaning of the actions.

In contrast, the teacher in Germany has trouble doing so due to her professional self-concept (that was, at least in part, a result of the layout of the study she and her class were involved in). This leads to an interaction that is much longer than the one laid out above, and can thus only be presented in a condensed form. We particularly investigate how the teacher's struggle becomes apparent through her gesturing.

In the beginning of the interaction analysed here (line 10), the teacher gestures towards the two sides of the equation:

10. Teacher: (*briefly lays her hand on the right side, where Herbert is just finishing his counting – see Figure 3, left image*) Well what can one change here for example. so that it stays (*briefly holds both her hands above the two tables*) the same. (*multiply taps the tables with all of her fingers – see Figure 3, right image*) that must always stay the same that is very important.

A closer analysis of this first scene reveals that there are two gestures (see Figure 3). First, the teacher briefly and unspecifically lays her hand on the right side of the table and refers to this gesture by the word "here". Her idea is to focus on one side of the equation. However, at the same time the equation must stay an equation, which the teacher tries to stress by saying "so that it stays the same". The pronoun is concretised by the gestures shown in the right image: She means that the number of matches on both sides stays the same. This is too complex for the students, as neither the matches nor the sides of the equation are explicitly named as the relevant objects. Of course, this problem also applies to the aforementioned gesture. As a result, the students can make no sense of the teachers hint.

Figure 3: The two gestures in the first scene: Pointing at the right side of the equation (left) and highlighting the equality of the two sides (right)

After this first occurrence of gestures the teacher refrains from using any for almost two minutes. This is even more striking as she does in two instances use unspecific pronouns that would require clarification about what they refer to. The teacher's behaviour is probably due to the agreement that she should refrain from direct hints to the solution of the problem – an approach founded in the idea that the students should come to see structures on their own. However, during this time, she does talk about the two sides of the equation as the relevant objects. One could argue that from this explicit talk about the two sides it should indeed be clear for the students where they are expected to act, in terms of the GCSt model, the teacher helps with the gathering to make connecting and structure-seeing happen. But the structure is a new one, and it is hard to see without a break with the existing view.

In the scene that ends the comparatively long absence of gestures, the teacher uses gestures that point more directly at the two sides of the equation:

64 Sabine: (*moves her torso backwards, then to the front again, gestures with both arms*) The simplest would be if you simply tell us the solution. (*runs her fingers through her hair, Herbert laughs*)

65 T: Well it must (*points with her right hand first at the left and then at the right side, then turns it with the palm up*) always be the same but maybe- a bit (*makes a sudden upward movement with her right hand*) clearer. (*stands up straight again, crosses her arms*)

66 S: (*looks up at the teacher*) What does clearer mean? (*Sabine and the teacher look at each other*) (2sec)

However, she still uses the unspecific singular pronoun "it", again referring to the number of matches on the two sides. At the same time the word "it" stands for the whole situation that should become "a bit clearer". The students thus focus on what she means by "clearer". They demand a plan for action – this becomes visible already in the line before. In the last two utterances to be discussed here, the teacher reproduces the two gestures from the beginning, as can be seen in Figure 4. The teacher multiply taps on the right side (see Figure 4, left image). Here, for the first time, she adds a hint that taking away something might help, by claiming that "that are so many". However, the students still keep on aimlessly guessing what to do (88–91), meaning that they have no goal for their actions. Finally, the teacher additionally makes clear where the equality is to be preserved (see Figure 4, right image).

But only when the teacher gives her very concrete advice ("take away something"), Sabine is very quick at finding out what to take away. Here it proves that the groundwork laid before may not have been in vain, but can now be activated:

98 T: Okay what can one do on both sides so that it gets clearer you must always do the same. (*Herbert removes some dust from the table, the teacher looks at Sabine for some time, Sabine looks at the table, still holding the three boxes in her hand*) (3sec) (*gets up and walks away, whispering in Sabine's ear*) ‚take away something.

Figure 4: The two gestures re-enacted: Indicating the relevant side of the equation (left) and making clear that equality must be preserved (right). See Figure 3 for comparison

DISCUSSION

What are the students to learn and how can they learn it?

The isolation method is not the first method to which children resort when they are asked to solve an equation. Indeed, it is far from trivial to "isolate" the unknown to solve the equation. This is the method of analysis that the ancient Greeks devised. It is a deductive method, where relationships are deduced through a long chain of deductions, the last one being one in which you have the unknown equal to something. For students, it is at first much more reasonable to assume numbers and try and see if the assumption confirms the story (trial-and-error method), which is indeed what can be seen in both episodes.

In the examples presented here, the (linear) equation is supposed to emerge from an original context from where things and actions acquire an initial meaning (cards and envelopes in one case, matches and boxes in the other). This context is set in terms of a narrative that establishes an equality involving known and unknown numbers. Formally speaking, the two contexts explored here are similar. We can say that, in principle, the context offers the same potential in terms of algebraic meaning-making. Our analysis suggests, however, that the narrative (i.e., the linguistically connected account of events) is much more emphasized in the Canadian study. Cheb and the teacher talk much more in terms of cards and envelopes than the students and the teacher talk about matches and boxes in the German study. The potential significance of the context appears hence not to be equally exploited.

However, the exploitation of the significance of a meaningful context is not enough for the students to envision the algebraic isolation method. Indeed, to proceed to the simplification of the equation, the original narrative has to be disrupted by a (mathematical, in this case algebraic) sense that is already a real-life counter-sense. It is hardly natural to think about *removing* cards from the individuals in the story, while in fact the question is about the number of cards in an envelope. There is a shift from quantities as such to relations between quantities. The teacher and the students have to expand the narrative so that the removing actions and their results may acquire a new meaning. Hence there is a need for the teacher to interrupt the flow of actions and to make sure that the class finds a new mathematical meaning in what has been done to the equations, after the removal of same quantities. In the Canadian episode, the teacher shows a developed sense of the importance of this interruption and the special value of the algebraic solution so that she can refer to it at the appropriate moment. This developed sense is not natural. It was nurtured during the design of the classroom activity. The German episode ends with the teacher telling one of the students what to do on the two sides of the equation, which she has just identified as the place of action. The course of action appears improvised under the impression of the difficulties that arise rather than didactically planned.

Teacher intervention by gestures and its limitations

Until she decides to help the students in a more direct manner, the German teacher's attempts to guide her students consist mainly of gestures, as if the contextual actions required to simplify the equations had an ostensive meaning. However, gestures always work within the parameters of how teachers conceive of themselves in their teaching. They are directed to oneself and to others. It supposes that they work within the parameters of what we take a good teacher-students interaction to be, i.e., the meaning ascribed to interaction in the classroom. We see this tension in the manner in which the teacher in Germany refrains from telling the students. She tries to point to a solution solely by gestures.

However, to be understandable, to be meaningful as a hint to make connections (in the GCSt model) or as a guide into a cultural activity (in the TO), gestures need an explanation about what they refer to, and what it is that can be done. As has been pointed out, this is far from trivial, especially when we consider the framings of the problem: In both cases it is embedded in a narrative that makes the required actions meaningful in an abstract sense – an abstract sense that is opposed to the more quotidian sense where one may try to use trial-and-error methods.

The central hypothesis about the teacher's gestures in this episode is that they are too abstract. They require a deeper students' understanding than the one they have at this point. In particular, the teacher presumes that the students already see the same objects and relations between these objects as she does. It seems that the gestures are made from the standpoint of someone for whom the equation is already of a symbolic nature.

Maybe an understanding of the sides of the equation as the useful unit of analysis would suffice for the gestures to be fruitful, but the teacher does not even try to induce that explicitly. The teacher's gestures cannot find a kind of contextual narrative support to provide a rationale for the algebraic actions to find a meaning that may be accessible to the students.

CONCLUSIONS

The introduction of algebraic methods and ways of thinking is a crucial point in students' individual paths through mathematics – a point where many lose touch with the subject. The two episodes make clear that a teacher with an appropriate understanding of his or her role can help students substantially. Awareness for the novelty of algebraic methods is the essence of this understanding. It can help realize the decisive steps that the students have to take and to position other forms of help, such as gestures, in the teaching-learning situation. Without the consideration of context, otherwise helpful gestures are at risk to stay opaque to the students.

The result is surprising from the point of view where the learning should come from the students and is seen as an autonomous act of construction, as it is in the GCSt model. The episodes and their analysis presented here raises the question how one could even expect students to develop the complex deductive method of solving an equation without getting an introduction by an experienced person. In both episodes, the learning that happens in the end (or more precisely: that begins in the end, as the new knowledge will need to be consolidated) is based on an input from the teacher-students' interaction. To inform better teaching, it should be a goal of mathematics education researchers to better understand what this input is in different content fields, as we have tried here regarding the solving of linear equations.

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